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On a Generalized Difference Sequence Space

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Abstract. In this work using the generalized difference operator Δ_m^n, we generalize the sequence space $m(\phi)$ to sequence space $m(\phi, p, \beta)(\Delta_m^n)$, give some topological properties about this space and show that the space $m(\phi, p, \beta)(\Delta_m^n)$ is a $BK-$space by a suitable norm. The results obtained generalizes some known results.

Key Words and Phrases: difference sequence, BK-space, symmetric space, normal space.

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1. Introduction

Some sequence spaces so called the difference sequence spaces $\Delta(X)$ first was presented by Kizmaz in 1981 [15] and then many mathematicians studied on these kind of sequences and obtained some generalized difference sequence spaces. Et and Çolak [8] have established these kind of spaces $\Delta^n(X)$ as follows.

Given a sequence space X and a number $n \in \mathbb{N}$, the space $\Delta^n(X)$ is defined as

$$\Delta^n(X) = \{x = (x_k) : (\Delta^n x_k) \in X\},$$

where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$ and so that $\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}$ for every $k \in \mathbb{N}$. Et and Çolak [8] showed that $\Delta^n(c_0)$, $\Delta^n(c)$ and $\Delta^n(\ell_\infty)$ are $BK-$spaces with the norm

$$\|x\|_{\Delta_1} = \sum_{i=1}^{n} |x_i| + \|\Delta^n x\|_\infty,$$

where the notations c_0, c and ℓ_∞ symbolize the spaces of null, convergent and bounded sequences, respectively and w symbolize the space of all sequences.

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Then, using a new operator $\Delta_m^n, (m, n \in \mathbb{N})$ Tripathy et al. ([5],[6],[24]) have defined another and new type difference sequence space $\Delta_m^n(X)$ as

$$\Delta_m^n(X) = \{x = (x_k) : (\Delta_m^n x_k) \in X\},$$

where $\Delta_m^0 x = x$, $\Delta_m^1 x = (x_k - x_{k+m})$, $\Delta_m^n x = (\Delta_m^n x_k) = \left(\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m}\right)$ and so $\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^n \binom{n}{v} x_{k+mv}$, and give some topological properties about this space and show that the space $\Delta_m^n(X)$ is a $BK-$space by the norm

$$\|x\|_{\Delta 2} = \sum_{i=1}^{mn} |x_i| + \|\Delta_m^n x\|_\infty$$

for $X = c_0, c$ and ℓ_∞. In recent times, these kind of sequences have been examined in many studies such as ([1], [2], [3], [7], [9],[10], [11], [12], [13], [14], [16], [21], [22]) and in many others.

2. Main Results

We devote this section to construct and examine a space of sequences. The notation $m(\phi, p, \beta)(\Delta_m^n)$ will be used to indicate the class we are talking about. Then some containment relations and topological properties of the space will be given. The obtained results are more general than those of Et et al. ([4],[11]), Sargent [20] and, Tripathy and Sen [23] .

Assume that (ϕ_n) is a sequence such that $\phi_n \leq \phi_{n+1}$, $n\phi_{n+1} \leq (n+1)\phi_n$ and $\phi_n > 0$ for every $n \in \mathbb{N}$ and we will use the notation Φ to indicate the class of this type of sequences (ϕ_n).

The spaces

$$m(\phi) = \left\{ x = (x_k) \in w : \sup_{s \geq 1,\, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\},$$

for $p > 0$:

$$m(\phi, p) = \left\{ x = (x_k) \in w : \sup_{s \geq 1,\, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k|^p < \infty \right\}$$

were introduced by Sargent [20], Tripathy and Sen [23], respectively, and then they were studied by Mursaleen et al. ([17],[18],[19]), where φ_s is the class of subsets of $\mathbb{N}$, which has no more than s elements.

Let $m, n \in \mathbb{N}, 0 < \beta \leq 1$ and $p > 0$. We define

$$m(\phi, p, \beta)(\Delta_m^n) = \left\{ x = (x_k) \in w : \sup_{s \geq 1,\, \sigma \in \varphi_s} \frac{1}{\phi_s^\beta} \sum_{k \in \sigma} |\Delta_m^n x_k|^p < \infty \right\}.$$

Clearly we see that $m(\phi, p, \beta)\left(\Delta_m^0\right) = m(\phi, p, \beta)$ and $m(\phi, 1, 1)\left(\Delta_m^0\right) = m(\phi)$ in this definition. We shall write $m(\phi, p, \beta)(\Delta^n)$ in place of $m(\phi, p, \beta)(\Delta_m^n)$ for $m = 1$ and we shall write $m(\phi, \beta)(\Delta_m^n)$ in place of $m(\phi, p, \beta)(\Delta_m^n)$ for $p = 1$. The sequence space $m(\phi, p, \beta)(\Delta_m^n)$ contains some unbounded sequences for $m, n \geq 1, 0 < \beta \leq 1$ and $p > 0$. For example the sequence $(x_k) = (k^n)$ is an element of $m(\phi, p, \beta)(\Delta_m^n)$ for $m = 1, \beta = 1$ but is not an element of ℓ_∞.

Remark 1. *i) If $n = 0$ and $\beta = 1$, then $m(\phi, p, \beta)(\Delta_m^n)$ reduces to $m(\phi, p)$ which was defined by Tripathy and Sen in [23].*

ii) If $n = 0, p = 1$ and $\beta = 1$, then $m(\phi, p, \beta)(\Delta_m^n)$ reduces to $m(\phi)$ which was defined by Sargent in [20].

iii) If $m = 1$, then $m(\phi, p, \beta)(\Delta_m^n)$ reduces to $m(\phi, p, \beta)(\Delta^n)$ which was defined by Et and Karakaya in [11].

iv) If $m = 1$ and $\beta = 1$, then $m(\phi, p, \beta)(\Delta_m^n)$ reduces to $m(\phi, p)(\Delta^n)$ which was defined by Çolak and Et in [4].

Theorem 1. *$m(\phi, p, \beta)(\Delta_m^n)$ is a Banach space with the norm*

$$\|x\|_{\Delta_3} = \sum_{i=1}^{r} |x_i| + \sup_{s \geq 1,\, \sigma \in \varphi_s} \frac{1}{\phi_s^\beta} \left(\sum_{k \in \sigma} |\Delta_m^n x_k|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{1}$$

and a complete $p-$normed space by

$$\|x\|_{\Delta_4} = \sum_{i=1}^{r} |x_i|^p + \sup_{s \geq 1,\, \sigma \in \varphi_s} \frac{1}{\phi_s^\beta} \sum_{k \in \sigma} |\Delta_m^n x_k|^p, \quad 0 < p < 1, \tag{2}$$

where $r = mn$ for $m \geq 1, n \geq 1$.

Proof. To see that $m(\phi, p, \beta)(\Delta_m^n)$ is a normed space with norm (1) is straightforward for finite $p \geq 1$. Assume that $\left(x^l\right)$ is a Cauchy sequence, where $x^l = (x_k^l)_{k=1}^\infty = \left(x_1^l, x_2^l, ...\right) \in m(\phi, p, \beta)(\Delta_m^n)$ for each $l \in \mathbb{N}$. Then for any $\varepsilon > 0$ there exists a number $n_0 \in \mathbb{N}$ such that

$$\left\|x^l - x^t\right\|_{\Delta_3} = \sum_{i=1}^{r} \left|x_i^l - x_i^t\right| + \sup_{s \geq 1,\, \sigma \in \varphi_s} \frac{1}{\phi_s^\beta} \left(\sum_{k \in \sigma} \left|\Delta_m^n \left(x_k^l - x_k^t\right)\right|^p \right)^{\frac{1}{p}} < \varepsilon \tag{3}$$

for every $l, t > n_0$. Hence

$$\left|x_i^l - x_i^t\right| < \varepsilon$$

for all $i = 1, 2, ..., r$ and

$$\frac{1}{\phi_s^\beta}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k^t\right)\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for $s \geq 1, \sigma \in \varphi_s$ and so

$$\left|\Delta_m^n\left(x_k^l - x_k^t\right)\right| \leq \varepsilon$$

for all $l, t > n_0$. On the other hand we have

$$\begin{aligned}\left|x_{k+nm}^l - x_{k+nm}^t\right| &\leq \left|\sum_{v=0}^n (-1)^v \binom{n}{v}\left(x_{k+mv}^l - x_{k+mv}^t\right)\right| + \left|\binom{n}{0}\left(x_k^l - x_k^t\right)\right| ... \\ &+ \left|(-1)^v \binom{n}{n-1}\left(x_{k+m(n-1)}^l - x_{k+m(n-1)}^t\right)\right|.\end{aligned}$$

Hence for each $k \in \mathbb{N}$ we obtain

$$\left|x_k^l - x_k^t\right| \to 0$$

as $l, t \to \infty$. This means that $(x_k^l)_{l=1}^\infty = \left(x_k^1, x_k^2, ...\right)$ is a Cauchy with complex terms and the completeness of $\mathbb{C}$ gives the convergence of that sequence.

$$\lim_l x_k^l = x_k$$

say, for each $k \in \mathbb{N}$. Let us define $x = (x_k)$. From (3) we have

$$\sum_{i=1}^r \left|x_i^l - x_i^t\right| < \varepsilon$$

and

$$\frac{1}{\phi_s^\beta}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k^t\right)\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for $s \geq 1, \sigma \in \varphi_s$ and for all $l, t > n_0$. Hence we have

$$\lim_t \sum_{i=1}^r \left|x^l - x_i^t\right| = \sum_{i=1}^r \left|x_i^l - x_i\right| < \varepsilon$$

and

$$\lim_t \frac{1}{\phi_s^\beta}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k^t\right)\right|^p\right)^{\frac{1}{p}} = \frac{1}{\phi_s^\beta}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k\right)\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for $s \geq 1, \sigma \in \varphi_s$ and for each $k \in \mathbb{N}$ and for all $l > n_0$. Hence we get

$$\left\|x^l - x\right\|_{\Delta_3} = \sum_{i=1}^r \left|x_i^l - x_i\right| + \sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s^\beta}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k\right)\right|^p\right)^{\frac{1}{p}} < \varepsilon$$

for all $l \geq n_0$. This shows that $\left(x^l\right) \to x$ as $l \to \infty$. Hence $x^l - x = \left(x_k^l - x_k\right)_k \in m(\phi, p, \beta)(\Delta_m^n)$. Since $x^l - x, x^l \in m(\phi, p, \beta)(\Delta_m^n)$ and $m(\phi, p)(\Delta_m^n)$ is a linear space, we have $x = x^l - \left(x^l - x\right) \in m(\phi, p, \beta)(\Delta_m^n)$. Hence $m(\phi, p, \beta)(\Delta_m^n)$ is complete.

Taking $0 < p < 1$, one may prove that the space $m(\phi, p, \beta)(\Delta_m^n)$ is $p-$ normed by (2). ◄

Theorem 2. *$m(\phi, p, \beta)(\Delta_m^n)$ is a $BK-$space.*

Proof. We know that $m(\phi, p, \beta)(\Delta_m^n)$ is a Banach space by Theorem 1. Now let $\left\|x^l - x\right\|_{\Delta_3} \to 0$ $(l \to \infty)$ and $\varepsilon > 0$ be given. Then there exists a $n_0 \in \mathbb{N}$ such that

$$\left\|x^l - x\right\|_{\Delta_3} < \varepsilon$$

for all $l > n_0$. Hence we have

$$\sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s^\beta}\left(\sum_{k\in\sigma}\left|\Delta_m^n\left(x_k^l - x_k\right)\right|^p\right)^{\frac{1}{p}} < \varepsilon,$$

and so

$$\left|x_k^l - x_k\right| < \varepsilon\phi_1^\beta,$$

for all $l > n_0$ and for each $k \in \mathbb{N}$. Consequently this means that $m(\phi, p, \beta)(\Delta_m^n)$ is a Banach space with continuous coordinates (that is, $\left\|x^l - x\right\|_{\Delta_3} \to 0$ implies $\left|x_k^l - x_k\right| \to 0$, for each $k \in \mathbb{N}$, as $l \to \infty$) and this completes the proof. ◄

Theorem 3. *Although $m(\phi, p, \beta)$ is solid and monotone, the space $m(\phi, p, \beta)(\Delta_m^n)$ is not solid, is not monotone, is not sequence algebra and is not symmetric, for $m, n \geq 1, 0 < \beta \leq 1$ and $p > 0$.*

Proof. Let $x \in m(\phi, p, \beta)$ be given and $y = (y_n)$ be a sequence with $|x_n| \le |y_n|$ for each $n \in \mathbb{N}$. Then we get

$$\sup_{s\ge1,\, \sigma\in\varphi_s} \frac{1}{\phi^\beta} \sum_{n\in\sigma} |x_n|^p \le \sup_{s\ge1,\, \sigma\in\varphi_s} \frac{1}{\phi_s^\beta} \sum_{n\in\sigma} |y_n|^p$$

Hence $m(\phi, p, \beta)$ is solid and hence monotone.

For the proof of the other parts of the Theorem, we may use the examples given below. ◄

Example 1. *$m(\phi, p, \beta)(\Delta_m^n)$ is not a sequence algebra. Indeed $x, y \in m(\phi, p, \beta)$ (Δ_m^n), but $xy \notin m(\phi, p, \beta)$ (Δ_m^n) for $x = \left(k^{n-2}\right)$, $y = \left(k^{n-2}\right)$, where $m = 1$ and $\beta = 1$.*

Example 2. *$m(\phi, p, \beta)(\Delta_m^n)$ is not solid too. Indeed $x \in m(\phi, p, \beta)(\Delta_m^n)$, but $(\alpha_k x_k) \notin m(\phi, p, \beta)(\Delta_m^n)$ if $x = \left(k^{n-1}\right)$ and $(\alpha_k) = \left((-1)^k\right)$ for $m = 1$ and $\beta = 1$.*

Example 3. *We have that $u = (u_k) \in m(\phi, p, \beta)(\Delta_m^n)$ if $(u_k) = \left(k^{n-1}\right)$, $m = 1$ and $\beta = 1$. Let (v_k) be a rearrangement of (u_k) which is defined as follows:*

$$(v_k) = (u_1, u_2, u_4, u_3, u_9, u_5, u_{16}, u_6, u_{25}, u_7, u_{36}, u_8, u_{49}, u_{10}, \ldots).$$

Then $v \notin m(\phi, p, \beta)(\Delta_m^n)$. Hence $m(\phi, p, \beta)(\Delta_m^n)$ is not symmetric.

The next result is an outcome of Theorem 3.

Corollary 1. *$m(\phi, p, \beta)(\Delta_m^n)$ is not perfect, for $m, n \ge 1, 0 < \beta \le 1$ and $p > 0$.*

Theorem 4. *$m(\phi, \beta)(\Delta_m^n) \subset m(\phi, p, \beta)(\Delta_m^n)$ for each $m, n \ge 1, 0 < \beta \le 1$ and $p \ge 1$.*

Proof. Omitted. ◄

Theorem 5. *Let $0 < \beta \le \gamma \le 1$ and $p \ge 1$. Then $m(\phi, p, \beta)(\Delta_m^n) \subset m(\psi, p, \gamma)(\Delta_m^n)$ iff $\sup_{s\ge1} \left(\frac{\phi_s^\beta}{\psi_s^\gamma}\right) < \infty$.*

Proof. Assume that $\sup_{s\ge1} \left(\frac{\phi_s^\beta}{\psi_s^\gamma}\right) < \infty$. Then we have $\phi_s^\beta \le K\psi_s^\gamma$ for some $K > 0$ and for each s. If $x \in m(\phi, p, \beta)(\Delta_m^n)$, then

$$\sup_{s\ge1,\, \sigma\in\varphi_s} \frac{1}{\phi_s^\beta} \left(\sum_{k\in\sigma} |\Delta_m^n x_k|^p\right)^{\frac{1}{p}} < \infty.$$

So we have

$$\sup_{s\geq 1,\,\sigma\in\varphi_s}\frac{1}{\psi_s^{\gamma}}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}}<K\sup_{s\geq 1}\sup_{\sigma\in\varphi_s}\frac{1}{\phi_s^{\beta}}\left(\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p\right)^{\frac{1}{p}}<\infty.$$

Hence $x\in m\left(\psi,p,\gamma\right)\left(\Delta_m^n\right)$.

Conversely assume that $m\left(\phi,p,\beta\right)\left(\Delta_m^n\right)\subset m\left(\psi,p,\gamma\right)\left(\Delta_m^n\right)$ and suppose that $\sup_{s\geq 1}\left(\dfrac{\phi_s^{\beta}}{\psi_s^{\gamma}}\right)=\infty$. Then under this supposition we can establish a sequence (s_i) of positive integers that provides $\lim\limits_{i}\left(\dfrac{\phi_{s_i}^{\beta}}{\psi_{s_i}^{\gamma}}\right)=\infty$. Given $K\in\mathbb{R}^{+}$, there exists $i_0\in\mathbb{N}$ with $\dfrac{\phi_{s_i}^{\beta}}{\psi_{s_i}^{\gamma}}>K$ whenever $s_i\geq i_0$. This yields that $\phi_{s_i}^{\beta}>K\psi_{s_i}^{\gamma}$ and so $\dfrac{1}{\psi_{s_i}^{\gamma}}>\dfrac{K}{\phi_{s_i}^{\beta}}$. Then we can write

$$\frac{1}{\psi_{s_i}^{\gamma}}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p>\frac{K}{\phi_{s_i}^{\beta}}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p$$

for all $s_i\geq i_0$. Taking supremum on both sides over $s_i\geq i_0$ and $\sigma\in\varphi_s$ we get

$$\sup_{s_i\geq i_0,\,\sigma\in\varphi_s}\frac{1}{\psi_{s_i}^{\gamma}}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p>\sup_{s_i\geq i_0,\,\sigma\in\varphi_s}\frac{K}{\phi_{s_i}^{\beta}}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p \tag{4}$$

for $x=(x_k)\in m\left(\phi,p,\beta\right)\left(\Delta_m^n\right)$. Since (4) holds and $K\in\mathbb{R}^{+}$, we have

$$\sup_{s_i\geq i_0,\,\sigma\in\varphi_s}\frac{1}{\psi_{s_i}^{\gamma}}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p=\infty$$

whenever $x\in m\left(\phi,p,\beta\right)\left(\Delta_m^n\right)$ with

$$0<\sup_{s\geq 1,\,\sigma\in\varphi_s}\frac{1}{\phi_s^{\beta}}\sum_{k\in\sigma}\left|\Delta_m^n x_k\right|^p<\infty.$$

Hence $x\notin m\left(\psi,p,\gamma\right)\left(\Delta_m^n\right)$. This contradicts to $m\left(\phi,p,\beta\right)\left(\Delta_m^n\right)\subset m\left(\psi,p,\gamma\right)\left(\Delta_m^n\right)$.
◄

The next result is an outcome of Theorem 5.

Corollary 2. *Let* $0<\beta\leq\gamma\leq 1$ *and* $p\geq 1$*, then* *i)* $m(\phi,p,\beta)(\Delta_m^n) = m(\psi,p,\gamma)(\Delta_m^n)$ *iff* $0 < \inf_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\psi_s^\gamma}\right) < \sup_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\psi_s^\gamma}\right) < \infty$,

ii) $m(\phi,p,\beta)(\Delta_m^n) = m(\psi,p,\beta)(\Delta_m^n)$ *iff* $0 < \inf_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\psi_s^\beta}\right) < \sup_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\psi_s^\beta}\right) < \infty$,

iii) $m(\phi,p,\beta)(\Delta_m^n) = m(\phi,p,\gamma)(\Delta_m^n)$ *iff* $0 < \inf_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\phi_s^\gamma}\right) < \sup_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\phi_s^\gamma}\right) < \infty$,

iv) $m(\phi,p,\beta)(\Delta_m^n) = m(\psi,p)(\Delta_m^n)$ *iff* $0 < \inf_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\psi_s}\right) < \sup_{s\geq 1}\left(\dfrac{\phi_s^\beta}{\psi_s}\right) < \infty$,

v) $m(\phi,p)(\Delta_m^n) = m(\psi,p)(\Delta_m^n)$ *iff* $0 < \inf_{s\geq 1}\left(\dfrac{\phi_s}{\psi_s}\right) < \sup_{s\geq 1}\left(\dfrac{\phi_s}{\psi_s}\right) < \infty$.

Theorem 6. *$m(\phi,p,\beta)\left(\Delta_m^{n-1}\right) \subset m(\phi,p,\beta)(\Delta_m^n)$ and the inclusion is strict, for $m,n \geq 1, 0 < \beta \leq 1$ and $p \geq 1$.*

Proof. As is known the inequality $|u+v|^p \leq 2^p(|u|^p + |v|^p)$ is satisfied for any numbers u, v and $1 \leq p < \infty$. Hence, if $x \in m(\phi,p,\beta)\left(\Delta_m^{n-1}\right)$, then for $1 \leq p < \infty$

$$\frac{1}{\phi_s^\beta}\sum_{k\in\sigma}|\Delta_m^n x_k|^p \leq 2^p\left(\frac{1}{\phi_s^\beta}\sum_{k\in\sigma}\left|\Delta_m^{n-1}x_k\right|^p + \frac{1}{\phi_s^\beta}\sum_{k\in\sigma}\left|\Delta_m^{n-1}x_{k+1}\right|^p\right)$$

and thus $x \in m(\phi,p,\beta)(\Delta_m^n)$. ◄

For the strictness of the inclusion we may use the example given below.

Example 4. *If $\phi_n = 1$, for all $n \in \mathbb{N}, m = 1, \beta = 1$ and $x = \left(k^{n-1}\right)$, then $x \in \ell_p(\Delta_m^n) \backslash \ell_p\left(\Delta_m^{n-1}\right)$. (Actually, if $x = \left(k^{n-1}\right)$ then $\Delta^{n-1}(x) = \left((-)^{n-1}(n-1)!\right)$ and $\Delta^n(x) = 0$).*

Theorem 7. *We have $\ell_p(\Delta_m^n) \subset m(\phi,p,\beta)(\Delta_m^n) \subset \ell_\infty(\Delta_m^n)$.*

Proof. The first inclusion is clear. Now if $x \in m(\phi,p,\beta)(\Delta_m^n)$, then

$$\sup_{s\geq 1,\, \sigma\in\varphi_s}\frac{1}{\phi_s^\beta}\left(\sum_{k\in\sigma}|\Delta_m^n x_k|^p\right)^{\frac{1}{p}} < \infty$$

and so $|\Delta_m^n x_k| < K\phi_1^\beta$, for each $k \in \mathbb{N}$ and at least a positive number K. Thus $x \in \ell_\infty(\Delta_m^n)$. ◄

Theorem 8. *$m(\phi,p,\beta)(\Delta_m^n) \subset m(\phi,q,\beta)(\Delta_m^n)$ if $q > p > 0$.*

Proof. It follows by using the inequality

$$\left(\sum_{k=1}^{n} |\Delta_m^n x_k|^q\right)^{\frac{1}{q}} \le \left(\sum_{k=1}^{n} |\Delta_m^n x_k|^p\right)^{\frac{1}{p}}$$

which is satisfied under condition $q > p > 0$. ◀

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