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On Some Classes of q-Difference Equations for the q-Generalized Tangent-Appell Polynomials

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Abstract. This article intends to investigate the q-recurrence relations and some new classes of q-difference equations for the q-generalized tangent-Appell polynomials. An analogous study of these results for the q-generalized tangent-Bernoulli polynomials is also presented. In addition, graphical representation and zeros of these polynomials are demonstrated using computer experiment.

Key Words and Phrases: q-calculus, q-generalized tangent Appell polynomials, generating function, q-difference equation.

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1. Introduction

The subject of q-calculus has established its importance in quantum mechanics, fluid mechanics, and combinatorics. It has a deep connection with Lie algebra and commutativity relations (see, for details, [13, 14, 12, 11]). The q-standard notations and definitions reviewed here are taken from [3].

The q-analogue of a number $n \in \mathbb{C}$ and factorial function are specified as

$$[n]_q = \frac{1-q^n}{1-q}, \; q \in \mathbb{C} \setminus \{1\}, \tag{1}$$

$$[n]_q! = \prod_{m=1}^{n} [m]_q = [1]_q[2]_q[3]_q \cdots [n]_q, \quad [0]_q! = 1, \quad n \in \mathbb{N}, \quad 0 < q < 1. \tag{2}$$

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is specified as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k = 0, 1, 2, \cdots, n; \quad n \in \mathbb{N}_0. \tag{3}$$

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The q-exponential functions are specified by

$$e_q(u) = \sum_{k=0}^{\infty} \frac{u^k}{[k]_q!}, \quad 0 < |q| < 1, \quad |u| < |1-q|^{-1}, \tag{4}$$

$$E_q(u) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{u^k}{[k]_q!}, \quad 0 < |q| < 1, \quad u \in \mathbb{C}, \tag{5}$$

and satisfy the following relation

$$e_q(t)E_q(-t) = E_q(t)e_q(-t) = 1. \tag{6}$$

The q-derivative D_q of functions $e_q(u)$ and $E_q(u)$ are given by

$$D_q e_q(ut) = t e_q(ut), \quad D_q E_q(ut) = t E_q(qut). \tag{7}$$

For any two arbitrary functions $f(u)$ and $g(u)$, the q-derivative operator D_q satisfies the following product and quotient relations [4]:

$$D_q(f(u)g(u)) = f(qu)D_q g(u) + g(u)D_q f(u) = f(u)D_q g(u) + g(qu)D_q f(u), \tag{8}$$

$$D_q\left(\frac{f(u)}{g(u)}\right) = \frac{g(qu)D_q f(u) - f(qu)D_q g(u)}{g(u)g(qu)} = \frac{g(u)D_q f(u) - f(u)D_q g(u)}{g(u)g(qu)}. \tag{9}$$

The tangent polynomials and numbers along with their q-analogue have enormous applications in physics, analytic number theory and other related areas. Several properties of these polynomials are studied and investigated by various mathematicians, (see [16, 15, 17]). Numerical experiments of the tangent polynomials have been the subject of extensive study in the past few years and much progress have been made both mathematically and computationally [8, 9, 10]. We recall the following definition of q-generalized tangent polynomials.

Definition 1. *The 2-variable q-generalized tangent polynomials (qGTP) $C_{n,m,q}(u,v)$ $\big(q \in \mathbb{C},\ 0 < |q| < 1,\ |mt| < \pi,\ m \in \mathbb{R}^+\big)$ is defined as [9]:*

$$\left(\frac{2}{e_q(mt)+1}\right) e_q(ut)E_q(vt) = \sum_{n=0}^{\infty} C_{n,m,q}(u,v)\frac{t^n}{[n]_q!}. \tag{10}$$

When $u = v = 0$, $C_{n,m,q}(0,0) := C_{n,m,q}$ are the corresponding q-generalized tangent numbers (qGTN) and are defined as:

$$C_{m,q}(t) = \left(\frac{2}{e_q(mt)+1}\right) = \sum_{n=0}^{\infty} C_{n,m,q}\frac{t^n}{[n]_q!}. \tag{11}$$

We have

$$\lim_{m\to 1} C_{n,m,q}(u,v) = \mathcal{E}_{n,q}(u,v), \quad \text{(2-variable } q\text{-Euler polynomials (}q\text{EP) [6])}$$
$$\lim_{m\to 2} C_{n,m,q}(u,v) = T_{n,q}(u,v), \quad \text{(2-variable } q\text{-tangent polynomials (}q\text{TP) [8])}$$

Definition 2. *The q-Appell polynomials* $A_{n,q}(u)$ $\big(q \in \mathbb{C},\ 0 < |q| < 1\big)$ *are defined by the following generating function [2, 7]:*

$$A_q(t)e_q(ut) = \sum_{n=0}^{\infty} A_{n,q}(u)\frac{t^n}{[n]_q!}, \tag{12}$$

where

$$A_q(t) = \sum_{n=0}^{\infty} A_{n,q}\frac{t^n}{[n]_q!}, \quad A_{0,q} = 1; \quad A_q(t) \neq 0 \tag{13}$$

is an analytic function at $t = 0$ *and* $A_{n,q} := A_{n,q}(0)$ *are q-Appell numbers.*

Different members of q-Appell family can be obtained by selecting appropriate function $A_q(t)$ in generating function (12). Some of its members along with their name, generating function and series definition are mentioned in Table 1.

Table 1: Certain Members Belonging to the q-Appell Family.

S. No.	$A_q(t)$	Name of q-Special Polynomials and its Associated Numbers	Generating Function of q-Special Polynomials and its Associated Numbers
I	$\frac{t}{e_q(t)-1}$	q-Bernoulli polynomials ($B_{n,q}(u)$); q-Bernoulli numbers ($B_{n,q}$) [1, 5]	$\frac{t}{e_q(t)-1}e_q(ut) = \sum_{n=0}^{\infty} B_{n,q}(u)\frac{t^n}{[n]_q!}$; $\frac{t}{e_q(t)-1} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{[n]_q!}$
II	$\frac{2}{e_q(t)+1}$	q-Euler polynomials ($\mathcal{E}_{n,q}(u)$); q-Euler numbers ($\mathcal{E}_{n,q}$) [5, 6]	$\frac{2}{e_q(t)+1}e_q(ut) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(u)\frac{t^n}{[n]_q!}$; $\frac{2}{e_q(t)+1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}\frac{t^n}{[n]_q!}$

By using monomiality principle, Yasmin et al [10] made hybrid of qGTP and q-Appell polynomials, which are defined as follows:

Definition 3. *The q-generalized tangent-Appell polynomials (qGTAP)* ${}_CA^{(m)}_{n,q}(u,v)$ $\big(q \in \mathbb{C},\ 0 < |q| < 1,\ |mt| < \pi,\ m \in \mathbb{R}^+\big)$ *are defined by means of the generating function [10]:*

$$\left(\frac{2}{e_q(mt)+1}\right) A_q(t)e_q(ut)E_q(vt) = \sum_{n=0}^{\infty} {}_CA^{(m)}_{n,q}(u,v)\frac{t^n}{[n]_q!}. \tag{14}$$

When $u = v = 0$, ${}_{C}A_{n,q}^{(m)}(0,0) := {}_{C}A_{n,q}^{(m)}$ are the corresponding q-generalized tangent-Appell numbers and are defined as:

$$\left(\frac{2}{e_q(mt)+1}\right) A_q(t) = \sum_{n=0}^{\infty} {}_{C}A_{n,q}^{(m)} \frac{t^n}{[n]_q!}. \tag{15}$$

Selecting suitable function $A_q(t)$ in generating function (14), several members belonging to the family of qGTAP ${}_{C}A_{n,q}^{(m)}(u,v)$ are obtained. Some of these members are listed in Table 2.

Table 2: Certain Members Belonging to the qGTAP ${}_{C}A_{n,q}^{(m)}(u,v)$.

S. No.	$A_q(t)$	Name of the Resultant Member	Generating Function of Resultant Polynomial and Resultant Number
I	$A_q(t) = \frac{t}{e_q(t)-1}$	q-generalized tangent -Bernoulli polynomials (qGTBP)	$\left(\frac{2}{e_q(mt)+1}\right)\left(\frac{t}{e_q(t)-1}\right) e_q(ut)E_q(vt) = \sum_{n=0}^{\infty} {}_{C}B_{n,q}^{(m)}(u,v)\frac{t^n}{[n]_q!}$; $\left(\frac{2}{e_q(mt)+1}\right)\left(\frac{t}{e_q(t)-1}\right) = \sum_{n=0}^{\infty} {}_{C}B_{n,q}^{(m)}\frac{t^n}{[n]_q!}$
II	$A_q(t) = \frac{2}{e_q(t)+1}$	q-generalized tangent -Euler polynomials (qGTEP)	$\left(\frac{2}{e_q(mt)+1}\right)\left(\frac{2}{e_q(t)+1}\right) e_q(ut)E_q(vt) = \sum_{n=0}^{\infty} {}_{C}\mathcal{E}_{n,q}^{(m)}(u,v)\frac{t^n}{[n]_q!}$; $\left(\frac{2}{e_q(mt)+1}\right)\left(\frac{2}{e_q(t)+1}\right) = \sum_{n=0}^{\infty} {}_{C}\mathcal{E}_{n,q}^{(m)}\frac{t^n}{[n]_q!}$

Remark 1. *As for $m = 2$ the qGTP $C_{n,m,q}(u,v)$ reduces to the qTP $T_{n,q}(u,v)$. So for the same choice of m, the results of qGTBP ${}_{C}B_{n,q}^{(m)}(u,v)$ and qGTEP ${}_{C}\mathcal{E}_{n,q}^{(m)}(u,v)$ (Table 2) reduces to the corresponding results of q-tangent Bernoulli and q-tangent Euler polynomials.*

Remark 2. *As for $m = 1$ the qGTP $C_{n,m,q}(u,v)$ reduces to the qEP $\mathcal{E}_{n,q}^{(m)}(u,v)$. So for the same choice of m, the results of qGTBP ${}_{C}B_{n,q}^{(m)}(u,v)$ and qGTEP ${}_{C}\mathcal{E}_{n,q}^{(m)}(u,v)$ (Table 2) reduces to the corresponding results of q-Euler Bernoulli and 2-iterated q-Euler polynomials.*

In this paper, the q-recurrence relations and some new classes of q-difference equations for the q-generalized tangent-Appell polynomials are derived in Section

2. An analogous study of these results for the q-generalized tangent-Bernoulli polynomials is presented in Section 3. In the last section, graphical representation, surface plot and zeros of these polynomials are demonstrated using computer experiment.

2. q-Recurrence Relations and q-Difference Equations

The q-recurrence relations and q-difference equations of the q-Appell and hybrid type q-Appell polynomials are helpful in finding the solutions to the developing problems originating in certain branches of science and engineering. In this section, we derive q-recurrence relations and q-difference equations for the qGTAP ${}_{C}A^{(m)}_{n,q}(u,v)$.

Lemma 1. *Assume that*

$$t\frac{D_{q,t}\left(C_{m,q}(t)\right)}{C_{m,q}(qt)}=\sum_{n=0}^{\infty}\gamma_n\frac{t^n}{[n]_q!} \tag{16}$$

and

$$\frac{C_{m,q}(t)}{C_{m,q}(qt)}=\sum_{n=0}^{\infty}\delta_n\frac{t^n}{[n]_q!}, \tag{17}$$

then

$$\gamma_0=0;\quad \gamma_n=-\frac{m}{2}[n]_qC_{n-1,m,q}(m,0),\quad for\ n\geq 1 \tag{18}$$

and

$$\delta_n=\frac{1}{2}\left(C_{n,m,q}(mq,0)+C_{n,m,q}\right), \tag{19}$$

where $C_{n,m,q}(u,v)$ and $C_{n,m,q}$ are the nth qGTP and qGTN given by equations (10) and (11) respectively.

Proof. Simplifying left hand side of equation (16) using equations (9) and (11), we get

$$t\frac{D_{qt}(C_{m,q}(t))}{C_{m,q}(qt)}=\frac{mte_q(mt)}{e_q(mt)+1}. \tag{20}$$

Now using generating function (10) in appropriate form gives

$$\sum_{n=0}^{\infty}\gamma_n\frac{t^n}{[n]_q!}=\frac{-m}{2}[n]_q\sum_{n=1}^{\infty}C_{n-1,m,q}(m,0)\frac{t^n}{[n]_q!}, \tag{21}$$

which on comparing the coefficients of t gives assertion (18).

Next, simplifying and rewriting left hand side of equation (17) by using equation (11), we obtain

$$\frac{C_{m,q}(t)}{C_{m,q}(qt)} = \frac{e_q(mqt)}{e_q(mt)+1} + \frac{1}{e_q(mt)+1}. \tag{22}$$

Now using generating function (10) in appropriate form gives

$$\sum_{n=0}^{\infty} \delta_n \frac{t^n}{[n]_q!} = \frac{1}{2}\sum_{n=0}^{\infty} \left(C_{n,m,q}(mq,0) + C_{n,m,q}\right) \frac{t^n}{[n]_q!}, \tag{23}$$

which on comparing the coefficients of t gives assertion (19). ◄

Using Lemma 1, we obtain following recurrence relation for qGTAP ${}_C A_{n,q}^{(m)}(u,v)$.

Theorem 1. *Assume that*

$$t\frac{D_{q,t}(A_q(t))}{A_q(qt)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!} \tag{24}$$

and

$$\frac{A_q(t)}{A_q(qt)} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!}. \tag{25}$$

Then the following linear homogeneous recurrence relation holds for the class of qGTAP ${}_C A_{n,q}^{(m)}(u,v)$

$$\begin{aligned} {}_C A_{n,q}^{(m)}(qu,v) &= uq^n {}_C A_{n-1,q}^{(m)}(u,v) + \frac{1}{[n]_q}\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{n-k}\left[\alpha_k + \frac{1}{2}\sum_{s=0}^{k-1}\begin{bmatrix} k \\ s \end{bmatrix}_q [k-s]_q\beta_s \right. \\ &\left. \big(v\left(C_{k-s-1,m,q}(mq,0) + C_{k-s-1,m,q}\right) - mC_{k-s-1,m,q}(m,0)\big)\right] {}_C A_{n-k,q}^{(m)}(u,v). \end{aligned} \tag{26}$$

Proof. Replacing u by qu in generating function (14) and then differentiating the resultant equation with respect to t using formulas (7) and (8). Afterwards multiplying by t and simplifying gives

$$\begin{aligned} \sum_{n=1}^{\infty} {}_C A_{n,q}^{(m)}(qu,v)\frac{t^n}{[n-1]_q!} &= C_{m,q}(qt)A_q(qt)e_q(qut)E_q(qvt)\left[qut + t\frac{D_{q,t}(A_q(t))}{A_q(qt)}\right. \\ &\left. + vt\frac{C_{m,q}(t)}{C_{m,q}(qt)}\frac{A_q(t)}{A_q(qt)} + t\frac{D_{q,t}\left(C_{m,q}(t)\right)}{C_{m,q}(qt)}\frac{A_q(t)}{A_q(qt)}\right]. \end{aligned} \tag{27}$$

Using generating function (14) (with t replaced by qt) and relations (16), (17), (24) and (25), equation (27) becomes

$$\sum_{n=1}^{\infty}[n]_q\ {}_CA_{n,q}^{(m)}(qu,v)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}q^n{}_CA_{n,q}^{(m)}(u,v)\frac{t^n}{[n]_q!}\Bigg[qut+\sum_{k=0}^{\infty}\alpha_k\frac{t^k}{[k]_q!}$$

$$+vt\sum_{s=0}^{\infty}\beta_s\frac{t^s}{[s]_q!}\sum_{k=0}^{\infty}\delta_k\frac{t^k}{[k]_q!}+\sum_{s=0}^{\infty}\beta_s\frac{t^s}{[s]_q!}\sum_{k=0}^{\infty}\gamma_k\frac{t^k}{[k]_q!}\Bigg]. \tag{28}$$

On simplifying equation (28) by using Cauchy product rule, we get

$$\sum_{n=1}^{\infty}[n]_q\ {}_CA_{n,q}^{(m)}(qu,v)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q q^{n-k}\Bigg[\alpha_k+\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}_q\beta_s\gamma_{k-s}\Bigg]$$

$${}_CA_{n-k,q}^{(m)}(u,v)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\Bigg[uq^{n+1}{}_CA_{n,q}^{(m)}(u,v)$$

$$+v\sum_{k=0}^{n}\sum_{s=0}^{k}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\s\end{bmatrix}_q q^{n-k}\beta_s\delta_{k-s}{}_CA_{n-k,q}^{(m)}(u,v)\Bigg]\frac{t^{n+1}}{[n]_q!}. \tag{29}$$

By using the fact that $\alpha_0=\gamma_0=0$ and rearranging limit of summation, the above equation can also be written as:

$$\sum_{n=1}^{\infty}{}_CA_{n,q}^{(m)}(qu,v)\frac{t^n}{[n]_q!}=\sum_{n=1}^{\infty}\Bigg[uq^n{}_CA_{n-1,q}^{(m)}(u,v)+\frac{1}{[n]_q}\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{n-k}$$

$$\Bigg[\alpha_k+\sum_{s=0}^{k-1}\begin{bmatrix}k\\s\end{bmatrix}_q\beta_s\big(\gamma_{k-s}+v[k-s]_q\delta_{k-s-1}\big)\Bigg]{}_CA_{n-k,q}^{(m)}(u,v)\Bigg]\frac{t^n}{[n]_q!}, \tag{30}$$

which on comparing the coefficients of t and putting values of γ and δ from Lemma 1 gives assertion (26). ◄

To obtain q-difference equation of qGTAP ${}_CA_{n,q}^{(m)}(u,v)$, we first prove following two lemmas.

Lemma 2. *The lowering operator of qGTAP ${}_CA_{n,q}^{(m)}(u,v)$ with respect to u and v for $n\geq 1$ is given by*

$$\frac{D_{q,u}}{[n]_q}\big\{{}_CA_{n,q}^{(m)}(u,v)\big\}={}_CA_{n-1,q}^{(m)}(u,v) \tag{31}$$

and

$$\frac{D_{q,v}}{[n]_q}\big\{{}_CA_{n,q}^{(m)}(u,v)\big\}={}_CA_{n-1,q}^{(m)}(u,qv), \tag{32}$$

respectively.

Proof. Differentiating generating function (14) with respect to u using (7), we get

$$\sum_{n=0}^{\infty} D_{q,u}\{{}_CA_{n,q}^{(m)}(u,v)\}\frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(mt)+1}\right)A_q(t)e_q(ut)E_q(vt), \tag{33}$$

which on using (14) on right hand side and then equating the coefficients of same powers of t yields assertion (31).

Next, differentiating generating function (14) with respect to v using (7), we get

$$\sum_{n=0}^{\infty} D_{q,u}\{{}_CA_{n,q}^{(m)}(u,v)\}\frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(mt)+1}\right)A_q(t)e_q(ut)E_q(qvt), \tag{34}$$

which on using (14) on right hand side and then equating the coefficients of same powers of t yields assertion (31). ◀

Lemma 3. *The k-times lowering operator of $qGTAP$ ${}_CA_{n,q}^{(m)}(u,v)$ with respect to u and v for $n \geq k$ is given by*

$$\frac{[n-k]_q!}{[n]_q!}D_{q,u}^k\{{}_CA_{n,q}^{(m)}(u,v)\} = {}_CA_{n-k,q}^{(m)}(u,v) \tag{35}$$

and

$$\frac{[n-k]_q!}{[n]_q!}D_{q,v}^k\{{}_CA_{n,q}^{(m)}(u,v)\} = q^{k(k-1)/2}{}_CA_{n-k,q}^{(m)}(u,q^kv), \tag{36}$$

respectively.

Proof. Differentiating generating function (14) k-times with respect to u using (7), we get

$$\sum_{n=0}^{\infty} D_{q,u}^k\{{}_CA_{n,q}^{(m)}(u,v)\}\frac{t^n}{[n]_q!} = \left(\frac{2t^k}{e_q(mt)+1}\right)A_q(t)e_q(ut)E_q(vt), \tag{37}$$

which on using (14) on right hand side and then equating the coefficients of same powers of t yields assertion (31).

Next, differentiating generating function (14) k-times with respect to v using (7), we get

$$\sum_{n=0}^{\infty} D_{q,u}^k\{{}_CA_{n,q}^{(m)}(u,v)\}\frac{t^n}{[n]_q!} = \left(\frac{q^{k(k-1)/2}2t^k}{e_q(mt)+1}\right)A_q(t)e_q(ut)E_q(q^kvt), \tag{38}$$

which on using (14) on right hand side and then equating the coefficients of same powers of t yields assertion (31). ◀

Using Lemma 2 and 3 in Theorem 1 we get the following result.

Theorem 2. *The $qGTAP$ ${}_CA^{(m)}_{n,q}(u,v)$, satisfy the following q-difference equations:*

$$\Big[uq^n D_{q,u}+\sum_{k=1}^{n}\frac{q^{n-k}}{[k]_q!}\Big(\alpha_k+\frac{1}{2}\sum_{s=0}^{k-1}\begin{bmatrix}k\\ s\end{bmatrix}_q[k-s]_q\beta_s\big(v\,(C_{k-s-1,m,q}(mq,0)+C_{k-s-1,m,q})$$
$$-\,mC_{k-s-1,m,q}(m,0)\big)\Big)D^k_{q,u}\Big]\,{}_CA^{(m)}_{n,q}(u,v)-[n]_q\,{}_CA^{(m)}_{n,q}(qu,v)=0 \quad (39)$$

and

$$uq^n D_{q,v}\ {}_CA^{(m)}_{n,q}\left(u,\frac{v}{q}\right)+\sum_{k=1}^{n}\frac{q^{n-\frac{k(k+1)}{2}}}{[k]_q!}\left(\alpha_k+\frac{1}{2}\sum_{s=0}^{k-1}\begin{bmatrix}k\\ s\end{bmatrix}_q[k-s]_q\beta_s\right.$$
$$\left.\big(v(C_{k-s-1,m,q}(mq,0)+C_{k-s-1,m,q})-mC_{k-s-1,m,q}(m,0)\big)\right)D^k_{q,v}\ {}_CA^{(m)}_{n,q}\left(u,\frac{v}{q^k}\right)$$
$$-\,[n]_q\ {}_CA^{(m)}_{n,q}(qu,v)=0. \quad (40)$$

In the next section, an analogous study of these results for certain polynomials belonging to the class of qGTAP ${}_CA^{(m)}_{n,q}(u,v)$ is carried out.

3. Examples

In order to give applications of Theorems 1 and 2, we consider the following example:

Taking $A_q(t)=\frac{t}{e_q(t)-1}$ in generating function (14), leads to obtain q-generalized tangent Bernoulli polynomials (qGTBP) denoted by ${}_CB^{(m)}_{n,q}(u,v)$ defined in Table 2 (I).

To obtain its q-recurrence relations and q-difference equations, we will first consider the following lemma.

Lemma 4. *Assume that*

$$t\frac{D_{q,t}(B_q(t))}{B_q(qt)}=\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{[n]_q!} \quad (41)$$

and

$$\frac{B_q(t)}{B_q(qt)}=\sum_{n=0}^{\infty}\beta_n\frac{t^n}{[n]_q!}, \quad (42)$$

then

$$\alpha_0=0;\quad \alpha_n=\frac{-1}{q}B_{n,q}(1),\quad \textit{for } n\geq 1 \quad (43)$$

and

$$\beta_n = \frac{B_{n+1,q}(q) - B_{n+1,q}}{q\ [n+1]_q}, \tag{44}$$

where $B_q(t) = \frac{t}{e_q(t)-1}$ *is* $A_q(t)$ *for Bernoulli polynomials given in Table 1 (I).*

Proof. Simplifying left hand side of equation (41) by putting $B_q(t) = \frac{t}{e_q(t)-1}$ and using relation (9), we get

$$t\frac{D_{qt}(B_q(t))}{B_q(qt)} = \frac{e_q(t) - 1 - te_q(t)}{q(e_q(t)-1)}. \tag{45}$$

Now using generating function of q-Bernoulli polynomials (Table 1 (I)) in appropriate form gives

$$\sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!} = \frac{1}{q} - \frac{1}{q}\sum_{n=0}^{\infty} B_{n,q}(1)\frac{t^n}{[n]_q!}, \tag{46}$$

which on comparing the coefficients of t gives assertion (43).

Next, simplifying and rearranging left hand side of equation (42) by putting $B_q(t) = \frac{t}{e_q(t)-1}$, we obtain

$$\frac{B_q(t)}{B_q(qt)} = \frac{1}{qt}\left(\frac{te_q(qt)}{e_q(t)-1} - \frac{t}{e_q(t)-1}\right). \tag{47}$$

Now using generating function of q-Bernoulli polynomials (Table 1 (I)) in appropriate form and then simplifying, we obtain

$$\sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!} = \frac{1}{q}\sum_{n=0}^{\infty}\left(\frac{B_{n+1,q}(q) - B_{n+1,q}}{[n+1]_q}\right)\frac{t^n}{[n]_q!}, \tag{48}$$

which on comparing the coefficients of t gives assertion (44). ◄

Using Lemma 4 in Theorems 1 and 2, following results for qGTBP ${}_C B_{n,q}^{(m)}(u,v)$ are obtained.

Theorem 3. *The following linear homogeneous recurrence relation holds for the class of qGTBP* ${}_C B_{n,q}^{(m)}(u,v)$*:*

$$\begin{aligned} {}_C B_{n,q}^{(m)}(qu,v) = uq^n {}_C B_{n-1,q}^{(m)}(u,v) + \frac{1}{[n]_q}\sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{n-k}\Bigg[\frac{-1}{q}B_{k,q}(1) + \frac{1}{2}\sum_{s=0}^{k-1}\begin{bmatrix} k \\ s \end{bmatrix}_q \\ \frac{B_{s+1,q}(q) - B_{s+1,q}}{q\ [s+1]_q}[k-s]_q\big(v\left(C_{k-s-1,m,q}(mq,0) + C_{k-s-1,m,q}\right) \\ - mC_{k-s-1,m,q}(m,0)\big)\Bigg]{}_C B_{n-k,q}^{(m)}(u,v). \end{aligned} \tag{49}$$

Theorem 4. *The qGTBP* ${}_C B_{n,q}^{(m)}(u,v)$ *satisfy the following q-difference equations:*

$$\Bigg[uq^n D_{q,u} + \sum_{k=1}^{n} \frac{q^{n-k}}{[k]_q!}\left(\frac{-1}{q} B_{k,q}(1) + \frac{1}{2}\sum_{s=0}^{k-1} \begin{bmatrix} k \\ s \end{bmatrix}_q [k-s]_q \frac{B_{s+1,q}(q) - B_{s+1,q}}{q\ [s+1]_q}\right.$$
$$\left.\big(v(C_{k-s-1,m,q}(mq,0) + C_{k-s-1,m,q}) - mC_{k-s-1,m,q}(m,0)\big)\right) D_{q,u}^{k}\Bigg]$$
$${}_C B_{n,q}^{(m)}(u,v) - [n]_q {}_C B_{n,q}^{(m)}(qu,v) = 0 \quad (50)$$

and

$$uq^n D_{q,v}\ {}_C B_{n,q}^{(m)}\left(u, \frac{v}{q}\right) + \sum_{k=1}^{n} \frac{q^{n-\frac{k(k+1)}{2}}}{[k]_q!}\left(\frac{-1}{q} B_{k,q}(1) + \frac{1}{2}\sum_{s=0}^{k-1} \begin{bmatrix} k \\ s \end{bmatrix}_q [k-s]_q\right.$$
$$\left.\frac{B_{s+1,q}(q) - B_{s+1,q}}{q\ [s+1]_q}\big(v(C_{k-s-1,m,q}(mq,0) + C_{k-s-1,m,q}) - mC_{k-s-1,m,q}(m,0)\big)\right)$$
$$D_{q,v}^{k}\ {}_C B_{n,q}^{(m)}\left(u, \frac{v}{q^k}\right) - [n]_q\ {}_C B_{n,q}^{(m)}(qu,v) = 0. \quad (51)$$

Similarly, we can find corresponding results for other q-special polynomials belonging to the family of qGTAP.

4. Graphical Representation and Zeros

In the previous section, difference equations of qGTBP ${}_C B_{n,q}^{(m)}(u,v)$ are obtained. In this section, we present the graphical and computational aspects related to these polynomials. The software "Mathematica" is used to show the behaviour of the qGTBP ${}_C B_{n,q}^{(m)}(u,v)$ by means of graph, 3D surface plot and plotting of zeros. In Figure 1 (left) and Figure 1 (right), graph for the even values $n = 0, 2, 4 \cdots, 20$ and odd values $n = 1, 3, 5, \cdots, 19$ are shown respectively for $v = 1$, $m = 2$ and $q = \frac{1}{2}$.

By using numerical investigation and computer experiments, we find the real and complex zeros and observe the phenomenon of distribution of the zeros. In order to make the above discussion more clear, we draw the graphs showing shapes with scattered real zeros of qGTBP ${}_C B_{n,q}^{(m)}(u,v)$. In Figure 2 (left) and Figure 2 (right), graph for the even value $n = 30$ and an odd value $n = 31$ along with their real zeros are shown respectively for $v = 1$, $m = 2$ and $q = \frac{1}{2}$.

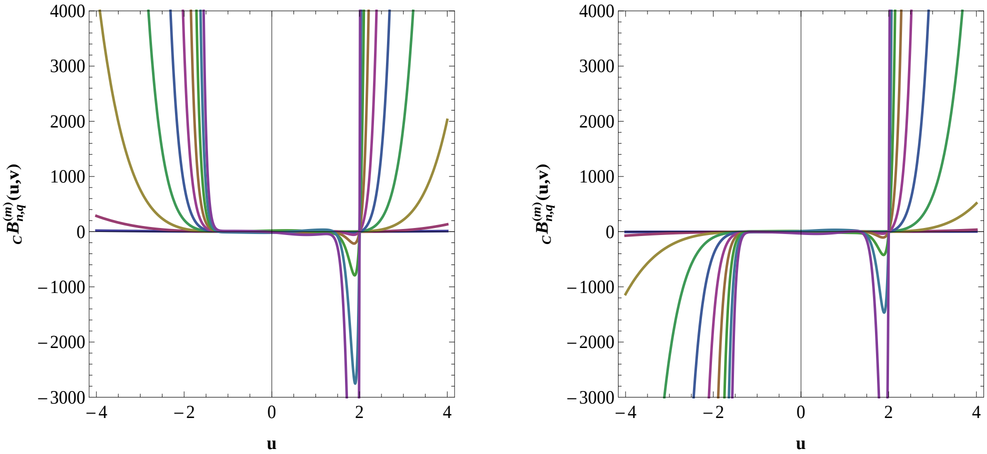

Figure 1: Curves of qGTBP ${}_CB_{n,q}^{(m)}(u,v)$.

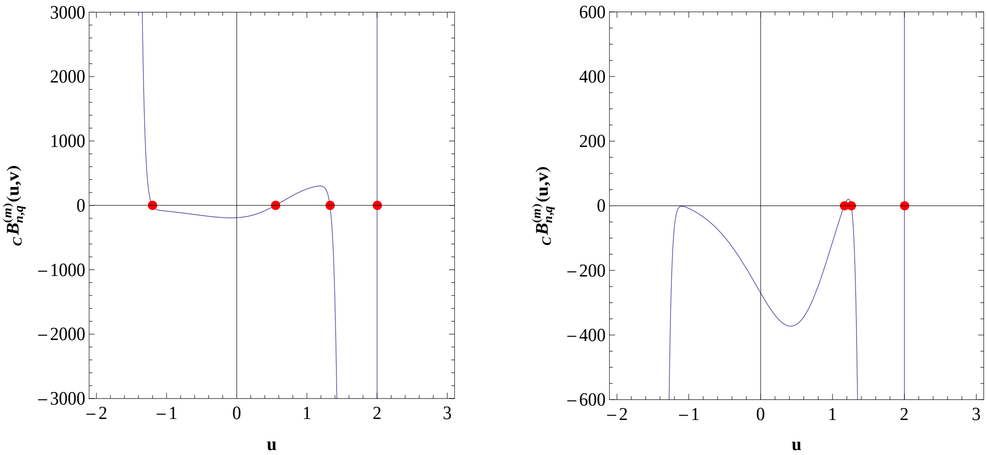

Figure 2: Graph along with real zeros of qGTBP ${}_CB_{n,q}^{(m)}(u,v)$.

Using computers it has been checked for several values of n that for $b \in \mathbb{R}$ and $u \in \mathbb{C}$, zeros of ${}_CB_{n,q}^{(m)}(u,b)$ has $Im(u) = 0$ reflection symmetry. However, zeros of ${}_CB_{n,q}^{(m)}(u,b)$ has not $Re(u) = a$ reflection symmetry (see Figure 3). But, it still remains unknown whether this is true or not for all values n. In Figure 3 (left) and Figure 3 (right), zeros for the even value $n = 30$ and an odd value $n = 31$ are shown respectively for $v = 1$, $m = 2$ and $q = \frac{1}{2}$.

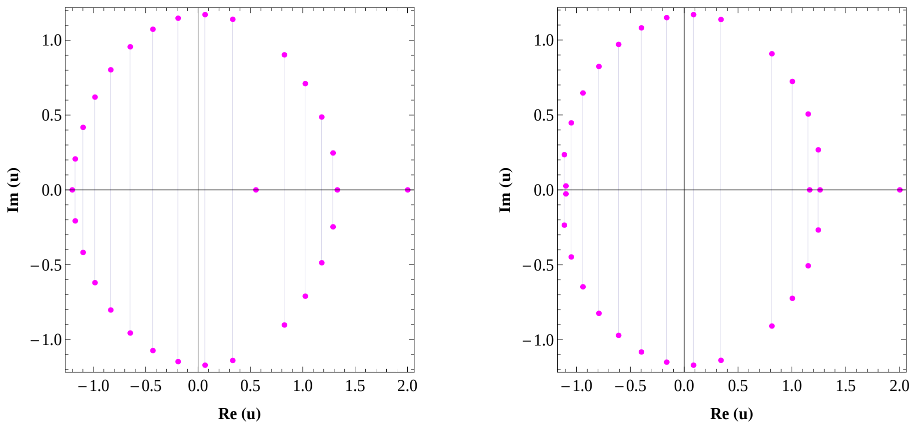

Figure 3: Zeros of qGTBP ${}_{C}B_{n,q}^{(m)}(u,v)$ has $Im(u) = 0$ reflection symmetry.

A 3D surface plot displays a 3-dimensional view of the surface defined by a function of two variables such that $z = f(u,v)$. The 3D surface plots are more informative and better for analysis. The predictor variables are displayed on the u and v axes, while the response variable z is represented by a smooth surface (3D surface plot) or a grid. It help to visualize the response surface and hence provide a more clear concept. In Figure 4 (left) and Figure 4 (right), surface plot for the even value $n = 30$ and an odd value $n = 31$ are shown respectively for $v = 1$, $m = 2$ and $q = \frac{1}{2}$.

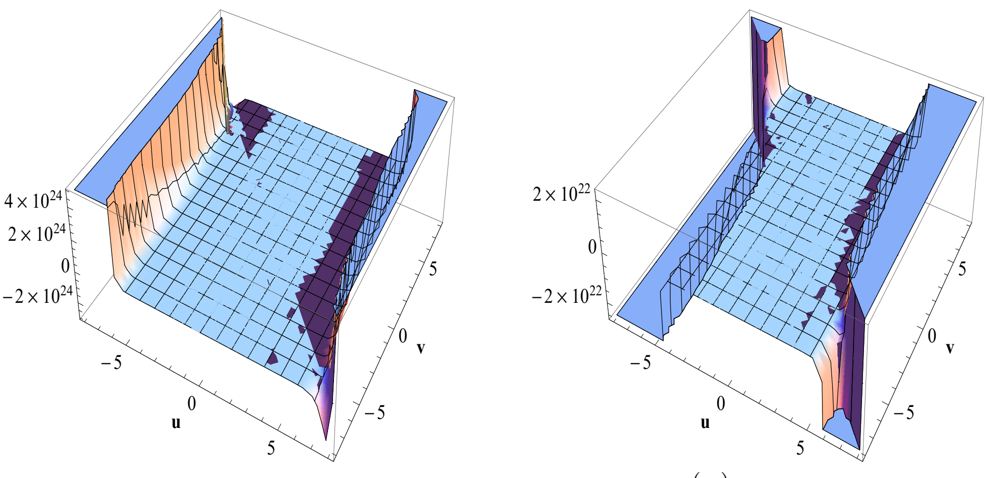

Figure 4: Surface plot of qGTBP ${}_{C}B_{n,q}^{(m)}(u,v)$.

The figures presented in this research work gives an unrestricted capability to create visual mathematical investigations of the behaviour of qGTBP ${}_{C}B_{n,q}^{(m)}(u,v)$. We expect that the research in this direction will be a new approach using numerical computations for the study of the member polynomials of qGTAP ${}_{C}A_{n,q}^{(m)}(u,v)$. The approach presented in this paper is general and opens new possibilities to deal with other hybrid families of q-special polynomials.

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