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On the Family of Improved Super-Halley Method for Unbounded Operators

N. Gupta*, J.P. Jaiswal

Abstract. This article centered the semilocal convergence of the improved super-Halley method to know the solution of a nonlinear equation that undergoes the two different sets of assumptions. To soften the classical convergence conditions is the first set of assumptions. The second set is based mainly on a modification of the condition required to the third derivative of the operator involved. In particular, instead of requiring that the third derivative is bounded, we demand that it is centered. We exhibit the existence of the solution with its uniqueness in the convergence theorem and subsequently find an expression for a priori error bound. Two numerical illustrations have been carried out to validate the theory.

Key Words and Phrases: Banach space, recurrence relations, semilocal convergence, ω-continuity condition, super-Halley method.

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1. Introduction

In scientific and engineering application, we confront with many complexities in locating the root of the following equation

$$Q(i) = 0. \tag{1}$$

Here Q is assumed to be the continuous and differentiable function and is defined on a real-valued domain. There are numerous problems in the field of medical sciences, natural sciences, physical sciences, engineering etc., that can be converted into the nonlinear equations. Many real-life based applications like equilibrium theory, the theory of elasticity, optimization problems etc. generally reduces into the nonlinear equations. Likewise, integral equations, initial value

*Corresponding author.

problems, differential equations etc. are used to formulate the real-world problems mathematically and it required to solve the system of equations to get their solutions.

The presence of high-speed computing machines or various software has gained more advantages as it is used to solve different types of nonlinear equations faster and also with more accuracy. Most of the problems cannot be solved analytically, hence iterative methods used to get the approximate solution.

This study deals with finding of the solution of the equation (1) via. iterative method and for this the most trendy and broadly quadratically convergent method i.e. Newton's method [1] is used, which helps to locate an approximate root say α, and the method is given as

$$i_{i+1} = i_i - \frac{q(i_i)}{q'(i_i)}, \quad i = 0, 1, 2, \cdots, \tag{2}$$

where $q'(i_i) \neq 0$. Generally, the local convergence and semilocal convergence analysis have been studied either by using the recurrence relation technique or majorizing sequences.

The semilocal convergence provides the information around the solution, while the local convergence provides the knowledge around the initial point. Many researchers have developed various iterative methods and analyzed their convergence. In Banach spaces, the semilocal convergence of Newton's method has been studied by Kantorovich [2]. Primarily, Kantorovich provided the method based on the majorant principle and later on described the method based on the recurrence relation technique. Rall [3] and other authors have studied on the enhancement of the outcomes related to the recurrence relation technique. To analyze the convergence of higher-order schemes like [4, 5], many authors have been studied various methods under different continuity condition which can be seen in the refs. [6, 7, 8, 9, 10, 11].

Many authors have realized the need of higher-order iterative methods and therefore developed many schemes to achieve better efficiency. Its importance can be seen where the rapid convergence is needed, like in a stringent system of equations.

The article has mainly two goals. The first goal is to get a finite bound of the norm of the third-order Fréchet derivative and to find a convergence ball for the case when third-order Fréchet derivative is not satisfying the continuity condition is another goal.

The paper is arranged as follows: Section 2 includes the method with some preliminary results. Some norms involved in the study by using the recurrence relation have been calculated in section 3. Section 4 and 5 contains semilocal convergence analysis of the considered method under different sets of hypotheses. The theoretical study is validated through numerical examples in section 6. Finally, the conclusion is given in Section 7.

2. Preliminary Results

We use the below-mentioned notations throughout the paper:
$Z_1, Z_2 \equiv$ Banach spaces, $\Omega \equiv$ a non-empty open subset of Z_1; $\Omega_0 \subseteq \Omega$ denotes a non-empty convex subset, $L(i,r) = \{j \in Z_1 : \|j-i\| < r\}$, $\overline{L(i,r)} = \{j \in Z_1 : \|j-i\| \leq r\}$. Here, we analyze the following improved super-Halley method suggested in the ref. [12]

$$\begin{aligned} k_n &= i_n - [I + \frac{1}{2}D_Q(i_n) + \frac{1}{2}D_Q(i_n)^2 + D_Q(i_n)^\theta \Phi(D_Q(i_n))]\Gamma_n Q(i_n), \\ i_{n+1} &= k_n - [I - \Gamma_n Q''(v_n)(k_n - i_n) + \delta[\Gamma_n Q''(v_n)(k_n - i_n)]^2]\Gamma_n Q(k_n), \quad (3) \end{aligned}$$

where I denotes the identity operator, $\theta \geq 3$ and $\delta \in [-1,1]$ are two constants, $D_Q(i_n) = [\Gamma_n Q''(i_n - \frac{1}{3}\Gamma_n Q(i_n))\Gamma_n Q(i_n)]$, $\Gamma_n = [Q'(i_n)]^{-1}$, $v_n = i_n - \frac{1}{3}\Gamma_n Q(i_n)$. Here, the operator Φ satisfies $\|\Phi(D_Q(i_n))\| \leq \Psi(\|D_Q(i_n)\|)$ where $\Psi(\rho) \geq 0$ is a non-decreasing real function and for $\rho \in (0,s)$, it is bounded. In the ref. [12] the following hypotheses have been assumed:
$(M1)\|\Gamma_0 Q(i_0)\| \leq \zeta$,
$(M2)\|\Gamma_0\| \leq \xi$,
$(M3)\|Q''(i)\| \leq S$, $i \in \Omega$,
$(M4)\|Q'''(i)\| \leq T$, $i \in \Omega$,
$(M5)\|Q'''(i) - Q'''(j)\| \leq \omega(\|i-j\|), \quad \forall i,j \in \Omega$,
where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ refers to be a non-decreasing, continuous function. For $i > 0$, $\omega(i) \geq 0$ and for $\epsilon \in [0,1]$ and $z \in [0,+\infty)$, $\omega(\epsilon z) \leq \phi(\epsilon)\omega(z)$ holds, where $\phi : [0,1] \to \mathbb{R}_+$ is also a non-decreasing, continuous function. Furthermore, there are some nonlinear functions such that the norm of their third-order Fréchet derivative is unbounded in the given domain but can be bounded at some initial point of the given domain. This can be seen in the refs. [13, 14, 15]. For example, look at a function q which is defined in the interval $(-2,2)$

$$q(i) = \begin{cases} i^3 ln(i^2) - 6i^2 - 3i + 8, & i \in (-2,0) \cup (0,2), \\ 0, & i = 0. \end{cases} \quad (4)$$

Here, the fact that the $q'''(i)$ is unbounded in the interval $(-2,2)$ but bounded at $i = 1$, is true. Hence, to avoid the unboundedness, we modify the assumption $(M4)$ by the following continuity condition as:
$(N1)\|Q'''(i_0)\| \leq A, \quad i_0 \in \Omega_0 = L(i_0, \epsilon)$,

where i_0 be an initial approximation and $\epsilon > 0$. For now, choose $\epsilon = \frac{\zeta}{\tilde{a}_0}$, where $\tilde{a}_0$ will be described later and the coherence of choice of such ϵ will be proved. Also, consider the local ω-continuity in place of the assumption $(M5)$ as
$(N2)\|Q'''(i) - Q'''(j)\| \leq \omega(\|i - j\|) \quad \forall\, i, j \in \Omega_0$.
A number of authors have contemplated some of the hypotheses instead of assuming the complete set of conditions. This idea can be found in the refs. [16, 17, 18]. Consider the mixed Hammerstein type of nonlinear integral equation

$$i(s) = 1 + \int_0^1 K(s,\rho)\left(\frac{1}{2}i(\rho)^{\frac{5}{2}} + \frac{7}{16}i(\rho)^3\right)d\rho, \quad s \in [0,1], \tag{5}$$

where $i \in [0,1], \rho \in [0,1], K(s,\rho)$ is the Green's function which is defined as

$$K(s,\rho) = \begin{cases} (1-s)\rho & \rho \leq s, \\ s(1-\rho) & s \leq \rho. \end{cases}$$

It is an example which fulfils the concept of softening the classical convergence condition.

Begin with a nonlinear operator $Q : \Omega \subseteq Z_1 \to Z_2$ and let the assumption $(M1) - (M3)$ are satisfied. Designate the following auxiliary scalar functions and these will be used frequently in the later development. Out of which, g, h and ψ functions are from the ref. [12] and ϖ and φ have been recalculated here:

$$\varpi(\rho) = g(\rho) + \frac{1}{2}\rho\left[1 + \rho g(\rho) + |\delta|(\rho g(\rho))^2\right]\left[1 + \rho + 2\rho^{\theta-1}\Psi(\rho) + g(\rho)^2\right], \tag{6}$$

$$h(\rho) = \frac{1}{1 - \rho\varpi(\rho)}, \tag{7}$$

$$\begin{aligned} \varphi(\rho) &= \Big[\rho g(\rho)\left(1 + |\delta|\rho g(\rho)\right) + \rho g(\rho)\left(1 + \rho g(\rho) + |\delta|(\rho g(\rho))^2\right) \\ &+ \frac{\rho}{2}\left(1 + \rho g(\rho) + |\delta|(\rho g(\rho))^2\right)^2\psi(\rho)\Big]\psi(\rho), \end{aligned} \tag{8}$$

where

$$\psi(\rho) = \frac{1}{2}\rho\left[1 + \rho + 2\rho^{\theta-1}\Psi(\rho) + g(\rho)^2\right], \tag{9}$$

and

$$g(\rho) = 1 + \frac{\rho}{2} + \frac{\rho^2}{2} + \rho^\theta\Psi(\rho). \tag{10}$$

Next is to analyze the properties of the above-defined functions. Let $f(\rho) = \varpi(\rho)\rho - 1$ such that $f(0) = -1 < 0$ and $f(\frac{1}{2}) \approx \frac{3299}{8192} > 0$, then the function $f(\rho)$ has at least one real positive root in the interval $(0, \frac{1}{2})$, say s. Here, define the following lemmas and these will be used to prove the main theorem(s) later.

Lemma 1. *Let the functions* ϖ, h *and* φ *be given in* (6), (7) *and* (8), *respectively and* $s > 0$ *be the smallest real root of the equation* $\varpi(\rho)\rho - 1 = 0$, *then*
(a) $\varpi(\rho) > 1$ *and* $h(\rho) > 1$ *are increasing functions for* $\rho \in (0, s)$,
(b) for $\rho \in (0, s)$, $\varphi(\rho)$ *is also an increasing function.*

Proof. The proof is easy by using the equations (6) – (8). ◄

Define $\zeta_0 = \zeta, \xi_0 = \xi, a_0 = S\xi\zeta$ and $\partial_0 = h(a_0)\varphi(a_0)$. Further, designate the sequences as follows:

$$\zeta_{n+1} = \partial_n \zeta_n, \tag{11}$$

$$\xi_{n+1} = h(a_n)\xi_n, \tag{12}$$

$$a_{n+1} = S\xi_{n+1}\zeta_{n+1} = h(a_n)\partial_n a_n, \tag{13}$$

$$\partial_{n+1} = h(a_{n+1})\varphi(a_{n+1}), \tag{14}$$

where $n \geq 0$. Some of the essential characteristics of the above-defined sequences are mentioned in the following lemma.

Lemma 2. *If* $a_0 < s$, *where* $s \geq 0$ *is the smallest real root of* $\varpi(\rho)\rho - 1 = 0$ *and the condition* $h(a_0)\partial_0 < 1$ *holds, then*
(a) $h(a_n) > 1$ *and* $\partial_n < 1$ *for* $n \geq 0$,
(b) $\{\zeta_n\}$, $\{a_n\}$ *and* $\{\partial_n\}$ *are decreasing sequences,*
(c) $\varpi(a_n)a_n < 1$ *and* $h(a_n)\partial_n < 1$ *for* $n \geq 0$.

Proof. By mathematical induction, this lemma can be verified. ◄

Lemma 3. *Let the expressions of* ϖ, h *and* φ *be mentioned in the relations* (6), (7) *and* (8), *respectively. Assume that* $\alpha \in (0, 1)$, *then* $\varpi(\alpha\rho) < \varpi(\rho)$, $h(\alpha\rho) < h(\rho)$ *and* $\varphi(\alpha\rho) < \alpha^2\varphi(\rho)$, *for* $\rho \in (0, s)$.

Proof. For $\rho \in (0, s)$, $\alpha \in (0, 1)$ and by using the equations (6), (7) and (8), this lemma can be verified. ◄

3. Recurrence Relations

We designate some of the norms which is mentioned in the ref. [12] for the considered method (3) and some of them have been calculated. For $n = 0$, the existence of Γ_0 provides the existence of v_0, j_0 such that

$$\|j_0 - i_0\| \leq \zeta_0, \quad \|v_0 - i_0\| \leq \frac{1}{3}\zeta_0, \tag{15}$$

i.e. j_0 and $v_0 \in L(i_0, R\zeta)$, where $R = \frac{\varpi(a_0)}{1-\partial_0}$. Moreover,

$$D_Q(i_0) = \Gamma_0 Q''\left(i_0 - \frac{1}{3}\Gamma_0 Q(i_0)\right)\Gamma_0 Q(i_0).$$

On taking max-norm, we get

$$\|D_Q(i_0)\| \leq \|\Gamma_0\|\left\|Q''\left(i_0 - \frac{1}{3}\Gamma_0 Q(i_0)\right)\right\|\|\Gamma_0 Q(i_0)\| \leq \|\Gamma_0\|\|Q''(v_0)\|\|\Gamma_0 Q(i_0)\| \leq a_0.$$

From the second sub-step of the scheme (3),

$$k_0 - i_0 = -\left[I + \frac{1}{2}D_Q(i_0) + \frac{1}{2}D_Q(i_0)^2 + D_Q(i_0)^\theta \Phi(D_Q(i_0))\right]\Gamma_0 Q(i_0).$$

On using max-norm, we have

$$\begin{aligned} \|k_0 - i_0\| &\leq \left[1 + \frac{1}{2}\|D_Q(i_0)\| + \frac{1}{2}\|D_Q(i_0)\|^2 + \|D_Q(i_0)\|^\theta \Psi(\|D_Q(i_0)\|)\right]\|\Gamma_0 Q(i_0)\| \\ &\leq \left[1 + \frac{1}{2}a_0 + \frac{1}{2}a_0^2 + a_0^\theta \Psi(a_0)\right]\zeta_0 < g(a_0)\zeta_0. \end{aligned} \tag{16}$$

It is similar to obtain

$$\begin{aligned} \|k_0 - j_0\| &\leq \left[\frac{1}{2}\|D_Q(i_0)\| + \frac{1}{2}\|D_Q(i_0)\|^2 + \|D_Q(i_0)\|^\theta \Psi(\|D_Q(i_0)\|)\right]\|\Gamma_0 Q(i_0)\| \\ &\leq \left[\frac{1}{2}a_0 + \frac{1}{2}a_0^2 + a_0^\theta \Psi(a_0)\right]\zeta_0 < [g(a_0) - 1]\zeta_0. \end{aligned} \tag{17}$$

Furthermore,

$$\begin{aligned} \|i_1 - k_0\| &\leq \left\|-[I - \Gamma_0 Q''(v_0)(k_0 - i_0) + \delta[\Gamma_0 Q''(v_0)(k_0 - i_0)]^2]\Gamma_0 Q(k_0)\right\| \\ &\leq [1 + a_0 g(a_0) + |\delta|(a_0 g(a_0))^2]\xi_0 \|Q(k_0)\|. \end{aligned} \tag{18}$$

From Taylor's formula, we have

$$Q(k_0) = Q(i_0) + Q'(i_0)(k_0 - i_0) + \int_0^1 Q''(i_0 + t(k_0 - i_0))(k_0 - i_0)^2(1 - t)dt. \tag{19}$$

On taking norm, we have

$$\|Q(k_0)\| \leq \psi(a_0)\frac{\zeta}{\xi}. \tag{20}$$

Consequently,

$$\|i_1 - i_0\| \leq \|i_1 - k_0\| + \|k_0 - i_0\| \leq \varpi(a_0)\zeta_0, \tag{21}$$

which shows that $i_1 \in L(i_0, R\zeta)$ because of the assumption $\partial_0 < \frac{1}{h(a_0)} < 1$. Notice that $a_0 < s$, hence $\varpi(a_0) < \varpi(s)$, then

$$\|I - \Gamma_0 Q'(i_1)\| \leq a_0 \varpi(a_0) < 1. \tag{22}$$

Thus $\Gamma_1 = [Q'(i_1)]^{-1}$ exists and with the help of Banach lemma, we may write

$$\|\Gamma_1\| \leq \frac{\xi_0}{1 - a_0 \varpi(a_0)} = \xi_1. \tag{23}$$

Now, again by applying Taylor's expansion,

$$Q(i_1) = Q(k_0) + Q'(k_0)(i_1 - k_0) + \int_0^1 Q''(k_0 + t(i_1 - k_0))(1-t)dt(i_1 - k_0)^2, \tag{24}$$

and

$$Q'(k_0) = Q'(i_0) + \int_0^1 Q''(i_0 + t(k_0 - i_0))dt(k_0 - i_0). \tag{25}$$

On using the equation (25) in the equation (24) we get

$$\begin{aligned} Q(i_1) &= Q(k_0) + Q'(i_0)(i_1 - k_0) + \int_0^1 Q''(i_0 + t(k_0 - i_0))dt(k_0 - i_0)(i_1 - k_0) \\ &+ \int_0^1 Q''(k_0 + t(i_1 - k_0))(i_1 - k_0)^2(1 - t)dt. \end{aligned} \tag{26}$$

Using the last step of the method (3), the above expression can be rewritten as

$$\begin{aligned} Q(i_1) &= [Q''(v_0)(k_0 - i_0) - \delta\Gamma_0[Q''(v_0)(k_0 - i_0)]^2]\Gamma_0 Q(k_0) \\ &+ \int_0^1 Q''(i_0 + t(k_0 - i_0))dt(k_0 - i_0)(i_1 - k_0) \\ &+ \int_0^1 Q''(k_0 + t(i_1 - k_0))(i_1 - k_0)^2(1 - t)dt. \end{aligned}$$

And thus

$$\|Q(i_1)\| \leq \varphi(a_0)\frac{\zeta}{\xi}, \tag{27}$$

and

$$\|j_1 - i_1\| \leq h(a_0)\varphi(a_0)\zeta_0 = \zeta_1. \tag{28}$$

Also, because $\varpi(a_0) > 1$ and by triangle inequality, we find

$$\|j_1 - i_0\| \leq R\zeta, \quad \|v_1 - i_0\| \leq R\zeta, \tag{29}$$

which implies $j_1, v_1 \in L(i_0, R\zeta)$. Further,

$$S\|\Gamma_1\|\|\Gamma_1 Q(i_1)\| \leq h^2(a_0)\varphi(a_0)a_0 = a_1. \tag{30}$$

Besides, we can have the following significant lemmas.

Lemma 4. *Let all the hypotheses of the Lemma 2 hold and the inferences* $(M1)-(M3)$ *are true then for all* $n \geq 0$ *following conditions holds:*

$$\begin{aligned}
&(i)\Gamma_n = [Q'(i_n)]^{-1} \text{ exists and } \|\Gamma_n\| \leq \xi_n,\\
&(ii)\|\Gamma_n Q(i_n)\| \leq h(a_{n-1})\varphi(a_{n-1})\|\Gamma_{n-1}Q(i_{n-1})\| \leq \partial_{n-1}\zeta_{n-1} = \zeta_n,\\
&(iii)S\|\Gamma_n\|\|\Gamma_n Q(i_n)\| \leq a_n,\\
&(iv)\|k_n - i_n\| \leq g(a_n)\zeta_n,\\
&(v)\|i_{n+1} - i_n\| \leq \varpi(a_n)\zeta_n,\\
&(vi)\|i_{n+1} - i_0\| \leq R\zeta, \text{ where } R = \frac{\varpi(a_0)}{1-\partial_0}. \qquad (31)
\end{aligned}$$

We can prove the above lemma by using the next lemma.

Lemma 5. *Under the hypotheses of Lemma 2, let* $\gamma = h(a_0)\partial_0$ *and* $\lambda = \frac{1}{h(a_0)}$, *we have*

$$\prod_{i=0}^{n} \partial_i \leq \lambda^{n+1}\gamma^{\frac{3^{n+1}-1}{2}}, \qquad (32)$$

$$\sum_{i=n}^{n+m} \zeta_i \leq \zeta\lambda^n\gamma^{\frac{3^n-1}{2}}\left(\frac{1-\lambda^{m+1}\gamma^{\frac{3^n(3^m+1)}{2}}}{1-\lambda\gamma^{3^n}}\right), \quad n \geq 0,\ m \geq 1. \qquad (33)$$

Proof. We will show it by applying the mathematical induction. For $n = 0$, on using the Lemma 3 and the equation (13), $a_1 = \gamma a_0$ and therefore,

$$\partial_1 = h(\gamma a_0)\varphi(\gamma a_0) < \gamma^2\partial_0 < \lambda\gamma^{3^1}.$$

Let it holds when $n = k$, then

$$\partial_k \leq \lambda\gamma^{3^k}, \quad k \geq 1.$$

Next claim is to show that it also holds for $n = k+1$. Thus

$$\partial_{k+1} < h(\gamma a_k)\varphi(\gamma a_k) < \lambda\gamma^{3^{k+1}}.$$

Therefore, $\partial_n \leq \lambda\gamma^{3^n}$ holds for all $n \geq 0$. On using the above relation, we can have

$$\prod_{i=0}^{k} \partial_i \leq \prod_{i=0}^{k} \lambda\gamma^{3^i} = \lambda^{k+1}\prod_{i=0}^{k}\gamma^{3^i} = \lambda^{k+1}\gamma^{\frac{3^{k+1}-1}{2}}, \quad k \geq 0.$$

By using the above-derived inequality in the equation (11),

$$\zeta_n = \partial_{n-1}\zeta_{n-1} = \partial_{n-1}\partial_{n-2}\zeta_{n-2} = \cdots = \zeta_0\prod_{i=0}^{n-1}\partial_i \leq \zeta\lambda^n\gamma^{\frac{3^n-1}{2}}, \quad n \geq 0.$$

Clearly $0 < \lambda < 1$ and $0 < \gamma < 1$, then $\zeta_n \to 0$ as $n \to \infty$. Take

$$c = \sum_{i=k}^{k+m} \lambda^i \gamma^{\frac{3^i}{2}}, \quad k \geq 0, \quad m \geq 1.$$

Rewrite the above equation as follows:

$$\begin{aligned} c &\leq \lambda^k \gamma^{\frac{3^k}{2}} + \lambda \gamma^{3^k} \sum_{i=k}^{k+m-1} \lambda^i \gamma^{\frac{3^i}{2}} \\ &\leq \lambda^k \gamma^{\frac{3^k}{2}} + \lambda \gamma^{3^k} \left(c - \lambda^{k+m} \gamma^{\frac{3^{k+m}}{2}} \right) \\ &< \lambda^k \gamma^{\frac{3^k}{2}} \left(\frac{1 - \lambda^{m+1} \gamma^{\frac{3^k(3^m+1)}{2}}}{1 - \lambda \gamma^{3^k}} \right). \end{aligned} \tag{34}$$

Further,

$$\sum_{i=k}^{k+m} \zeta_i \leq \sum_{i=k}^{k+m} \zeta \lambda^i \gamma^{\frac{3^{i-1}}{2}} \leq \zeta \lambda^k \gamma^{\frac{3^{k-1}}{2}} \left(\frac{1 - \lambda^{m+1} \gamma^{\frac{3^k(3^m+1)}{2}}}{1 - \lambda \gamma^{3^k}} \right).$$

Now, we quickly prove the Lemma 4. By employing the mathematical induction, one can verify $(i) - (v)$ for $n \geq 0$. So, for $n \geq 1$, on using the equation (31) and above result,

$$\|i_{n+1} - i_0\| \leq \sum_{i=0}^{n} \|i_{i+1} - i_i\| < R\zeta. \quad ◀$$

Lemma 6. [15] *Let* $R = \frac{\varpi(a_0)}{1 - \partial_0}$ *and* $h(a_0)\partial_0 < 1$ *and* $a_0 < s$ *and* $s \geq 0$ *denotes the smallest real root of* $\varpi(\rho)\rho - 1 = 0$, *then* $R < \frac{1}{a_0}$.

4. Semilocal Convergence When Q''' Condition is Omitted

The aim of this section is to prove the convergence of the method (3) under the $(M1) - (M3)$ assumptions. Moreover, we will get a convergence ball that contains the approximate solution uniquely and an expression of a priori error bound is derived.

Theorem 1. *Let* Z_1 *and* Z_2 *be two Banach spaces and the nonlinear operator* $Q : \Omega \subseteq Z_1 \to Z_2$, *be a continuously second-order Fréchet differentiable on a* *nonempty open convex subset* Ω. *Let the hypotheses* $(M1)-(M3)$ *hold and* $i_0 \in \Omega$. *Let* $a_0 = S\xi\zeta$ *and* $\partial_0 = h(a_0)\varphi(a_0)$ *satisfy the conditions* $a_0 < s$ *and* $h(a_0)\partial_0 < 1$, *where* $s \geq 0$ *is the smallest real root of* $\varpi(\rho)\rho - 1 = 0$ *and the functions* ϖ, h *and* φ *are mentioned in the equations* (6), (7) *and* (8), *respectively. Also, suppose* $\overline{L(i_0, R\zeta)} \subseteq \Omega$, *where* $R = \frac{\varpi(a_0)}{1-\partial_0}$. *Then starting with* i_0, *the iterative sequence* $\{i_n\}$ *creating from the algorithm* (3) *which converges to zero* i^* *of* $Q(i) = 0$ *with* $i_n, i^* \in \overline{L(i_0, R\zeta)}$ *and* i^* *is an exclusive zero of* $Q(i) = 0$ *in* $L(i_0, \frac{2}{S\xi} - R\zeta) \cap \Omega$. *Further, let* $\gamma = h(a_0)\partial_0$ *and* $\lambda = \frac{1}{h(a_0)}$, *then its error bound is defined as*

$$\|i_n - i^*\| \leq \varpi(a_0)\zeta\lambda^n\gamma^{\frac{3^n-1}{2}}\left(\frac{1}{1-\lambda\gamma^{3^n}}\right), \tag{35}$$

Proof. The sequence $\{i_n\}$ is well established in $\overline{L(i_0, R\zeta)}$. Now

$$\|i_{k+m} - i_k\| \leq \sum_{i=k}^{k+m-1} \|i_{i+1} - i_i\| \leq \varpi(a_0)\zeta\lambda^k\gamma^{\frac{3^k-1}{2}}\left(\frac{1-\lambda^m\gamma^{\frac{3^k(3^{m-1}+1)}{2}}}{1-\lambda\gamma^{3^k}}\right), \tag{36}$$

which proves that $\{i_k\}$ is a Cauchy sequence. So, there exists i^* satisfying

$$\lim_{k\to\infty} i_k = i^*.$$

Let $k = 0$, $m \to \infty$ in the equation (36), we obtain

$$\|i^* - i_0\| \leq R\zeta, \tag{37}$$

which implies that $i^* \in \overline{L(i_0, R\zeta)}$. Next claim is to show $Q(i^*) = 0$. Since

$$\|\Gamma_0\|\ \|Q(i_n)\| \leq \|\Gamma_n\|\ \|Q(i_n)\|, \tag{38}$$

then by tending $n \to \infty$ and using the continuity of Q in Ω, one can conclude that $Q(i^*) = 0$. Lastly, for the uniqueness of the solution, let $i^{**} \in L(i_0, \frac{2}{S\xi} - R\zeta) \cap \Omega$ be another solution of $Q(i)$. By using Taylor's theorem, we get

$$0 = Q(i^{**}) - Q(i^*) = \int_0^1 Q'((1-t)i^* + ti^{**})dt(i^{**} - i^*) = P(i^{**} - i^*).$$

Also,

$$\begin{aligned} &\|\Gamma_0\| \left\| \int_0^1 [Q'((1-t)i^* + ti^{**}) - Q'(i_0)]dt \right\| \\ \leq\ & S\xi \int_0^1 [(1-t)\|i^* - i_0\| + t\|i^{**} - i_0\|]dt \leq \frac{S\xi}{2}\left[R\zeta + \frac{2}{S\xi} - R\zeta\right] < 1, \end{aligned}$$

which implies P is invertible and it leads to the conclusion that $i^{**} = i^*$. ◄

5. Semilocal Convergence When Q''' is Bounded on Initial Approximation

In this segment, the convergence theorem relies upon the weaker hypotheses $(M1)-(M3),(N1)$ and $(N2)$. Here, define the following sequences as

$$\tilde{\zeta}_{n+1}=\tilde{\partial}_n\tilde{\zeta}_n, \tag{39}$$

$$\tilde{\xi}_{n+1}=h(\tilde{a}_n)\tilde{\xi}_n, \tag{40}$$

$$\tilde{a}_{n+1}=S\tilde{\xi}_{n+1}\tilde{\zeta}_{n+1}=h(\tilde{a}_n)\tilde{\partial}_n\tilde{a}_n, \tag{41}$$

$$\tilde{b}_{n+1}=T\tilde{\xi}_{n+1}\tilde{\zeta}^2{}_{n+1}=h(\tilde{a}_n)\tilde{\partial}^2{}_n\tilde{b}_n, \tag{42}$$

$$\tilde{c}_{n+1}=\tilde{\xi}_{n+1}\tilde{\zeta}^2{}_{n+1}\omega(\tilde{\zeta}_{n+1})\leq h(\tilde{a}_n)\phi(\tilde{\partial}_n)\tilde{\partial}_n^2\tilde{c}_n, \tag{43}$$

$$\tilde{\partial}_{n+1}=h(\tilde{a}_{n+1})\varphi'(\tilde{a}_{n+1},\tilde{b}_{n+1},\tilde{c}_{n+1}), \tag{44}$$

where $n\geq 0$ and $T=A+\omega\left(\frac{\zeta}{\tilde{a}_0}\right)$. Here, designate $\tilde{\zeta}_0=\zeta,\tilde{\xi}_0=\xi,\tilde{a}_0=S\xi\zeta,\tilde{b}_0=T\xi\zeta^2,\tilde{c}_0=\xi\zeta^2\omega(\zeta)$ and $\tilde{\partial}_0=h(\tilde{a}_0)\varphi'(\tilde{a}_0,\tilde{b}_0,\tilde{c}_0)$. From Lemma 4 and Lemma 6, it is known that

$$\|i_n-i_0\|<R\zeta<\frac{\zeta}{\tilde{a}_0}.$$

Thus, $i_n\in L(i_0,\frac{\zeta}{\tilde{a}_0})$. Similarly, for $\rho\in[0,1],\ n\geq 1$,

$$\begin{aligned}\|i_n+t(v_n-i_n)-i_0\| &\leq \|i_n-i_0\|+\|v_n-i_n\|\\ &\leq \sum_{i=0}^{n-1}\|i_{i+1}-i_i\|+\frac{1}{3}\tilde{\zeta}_n\leq\varpi(\tilde{a}_0)\sum_{i=0}^{n}\tilde{\zeta}_i\leq R\zeta<\frac{\zeta}{\tilde{a}_0}.\end{aligned}$$

Thus $\{i_n+t(v_n-i_n)\}\in L(i_0,\frac{\zeta}{\tilde{a}_0})$ and,

$$\begin{aligned}\|i_n+st(k_n-i_n)-i_0\| &\leq \|i_n-i_0\|+\|k_n-i_n\|\\ &\leq \sum_{i=0}^{n-1}\|i_{i+1}-i_i\|+\|k_n-i_n\|\leq\varpi(\tilde{a}_0)\sum_{i=0}^{n}\tilde{\zeta}_i\leq R\zeta<\frac{\zeta}{\tilde{a}_0}.\end{aligned}$$

Therefore $\{i_n+st(k_n-i_n)\}\in L(i_0,\frac{\zeta}{\tilde{a}_0})$. This gives the fact that the choice for $\epsilon=\frac{\zeta}{\tilde{a}_0}$ is relevant. Let there exists a root $\tilde{a}_0\in(0,s)$ of the following equation

$$i=\left[A+\omega\left(\frac{\zeta}{i}\right)\right]\xi\zeta^2.$$

Obviously, $\tilde{b}_0 = T\xi\zeta^2$, where $T = A + \omega\left(\frac{\zeta}{\tilde{a}_0}\right)$. It would be remembered that for all $i \in L(i_0, \frac{\zeta}{\tilde{a}_0})$, we have

$$\|Q'''(i)\| \leq \|Q'''(i_0)\| + \|Q'''(i) - Q'''(i_0)\| \leq A + \omega(\|i - i_0\|) \leq A + \omega\left(\frac{\zeta}{\tilde{a}_0}\right) = T.$$

Here comprise three auxiliary scalar functions from the ref. [12]

$$\begin{aligned} \varphi'(\rho, u, v) &= \left[\frac{1}{3}g(\rho)u + (1+|\delta|)(g(\rho)\rho)^2 + \frac{1}{2}g(\rho)^2 u + |\delta|(g(\rho)t)^3\right]\tilde{\psi}(\rho, u, v) \\ &+ \frac{1}{2}\rho\left[1 + \rho g(\rho) + |\delta|(\rho g(\rho))^2\right]^2 \tilde{\psi}(\rho, u, v)^2, \end{aligned} \tag{45}$$

where

$$\begin{aligned} \tilde{\psi}(\rho, u, v) &= \rho^\theta \Psi(\rho) + \rho^3\left[\frac{1}{2} + \rho^{\theta-2}\Psi(\rho)\right] + \frac{1}{2}\rho(g(\rho) - 1)^2 + \frac{1}{6}\rho u \\ &+ \frac{1}{2}u(g(\rho) - 1) + \left(\frac{1}{6}J_1 + \frac{1}{2}J_2\right)v, \end{aligned} \tag{46}$$

$$g(\rho) = 1 + \frac{\rho}{2} + \frac{\rho^2}{2} + \rho^\theta\Psi(\rho), \quad J_1 = \int_0^1 \phi\left(\frac{t}{3}\right)dt \text{ and } \quad J_2 = \int_0^1 \phi(\rho)(1-t)^2 dt. \tag{47}$$

On applying the characteristics of the mathematical induction and from the assumptions $(M1)-(M3), (N1)$ and $(N2)$ following relations hold for all $n \geq 0$:

$(i)\Gamma_n = [Q'(i_n)]^{-1}$ exists and $\|\Gamma_n\| \leq \tilde{\xi}_n$,
$(ii)\|\Gamma_n Q(i_n)\| \leq \tilde{\zeta}_n$,
$(iii)S\|\Gamma_n\|\|\Gamma_n Q(i_n)\| \leq \tilde{a}_n$,
$(iv)T\|\Gamma_n\|\|\Gamma_n Q(i_n)\| \leq \tilde{b}_n$,
$(v)\|\Gamma_n\|\|\Gamma_n Q(i_n)\|^2\omega(\|\Gamma_n Q(i_n)\|) \leq \tilde{c}_n$,
$(vi)\|k_n - i_n\| \leq g(\tilde{a}_n)\|\Gamma_n Q(i_n)\| \leq g(\tilde{a}_n)\tilde{\zeta}_n$
$(vii)\|i_{n+1} - i_n\| \leq \varpi(\tilde{a}_n)\tilde{\zeta}_n$,

Another main theorem which depends upon the weaker assumptions stated as:

Theorem 2. *Let* $Q : \Omega \subseteq Z_1 \to Z_2$, *be a continuously third-order Fréchet differentiable on a nonempty open convex subset* $\Omega_0 \subseteq \Omega$. *Let the assumptions* $(M1)-(M3), (N1)$ *and* $(N2)$ *hold and* $i_0 \in \Omega_0$. *Assume that* $\tilde{a}_0 = S\xi\zeta, \tilde{b}_0 = T\xi\zeta^2, \tilde{c}_0 = \xi\zeta^2\omega(\zeta)$ *and* $\tilde{\partial}_0 = h(\tilde{a}_0)\varphi'(\tilde{a}_0, \tilde{b}_0, \tilde{c}_0)$ *satisfy the conditions* $\tilde{a}_0 < s$ *and*

$h(\tilde{a}_0)\tilde{\partial}_0 < 1$, *where* $s \geq 0$ *is the smallest real root of* $\varpi(\rho)\rho - 1 = 0$ *and the functions* g, h *and* φ' *are given in the equations* (6), (7) *and* (45) *respectively. Also, suppose* $\overline{L(i_0, \tilde{R}\zeta)} \subseteq \Omega_0$ *where* $\tilde{R} = \frac{\varpi(\tilde{a}_0)}{1-\tilde{\partial}_0}$. *Then starting with* i_0, *the iterative sequence* $\{i_n\}$ *creating from the method* (3) *converges to zero* i^* *of* $Q(i) = 0$ *with* $i_n, i^* \in \overline{L(i_0, \tilde{R}\zeta)}$ *and* i^* *is an exclusive zero of* $Q(i) = 0$ *in* $L(i_0, \frac{2}{S\xi} - \tilde{R}\zeta) \cap \Omega$. *In addition, its error bound is defined as*

$$\|i_n - i^*\| \leq \varpi(\tilde{a}_0)\zeta\tilde{\lambda}^n\tilde{\gamma}^{\frac{5^n-1}{4}}\left(\frac{1}{1-\tilde{\lambda}\tilde{\gamma}^{5^n}}\right), \tag{48}$$

where $\tilde{\gamma} = h(\tilde{a}_0)\tilde{\partial}_0$ *and* $\tilde{\lambda} = \frac{1}{h(\tilde{a}_0)}$.

Proof. Similar to the proof given in Theorem (1). ◄

6. Numerical Testing

Example 1. [18] Consider a nonlinear integral equation

$$i(s) = 1 + \int_0^1 K(s,\rho)\left(\frac{1}{2}i(\rho)^{\frac{5}{2}} + \frac{7}{16}i(\rho)^3\right)d\rho, \quad s \in [0,1], \tag{49}$$

where $i \in [0,1], \rho \in [0,1]$ and K denotes the Green's function.

Solving the equation (49) is similar to get the solution for $Q(i) = 0$, where $Q : \Omega \subseteq C[0,1] \to C[0,1]$.

$$[Q(i)](s) = i(s) - 1 - \int_0^1 K(s,\rho)\left(\frac{1}{2}i(\rho)^{\frac{5}{2}} + \frac{7}{16}i(\rho)^3\right)d\rho, \quad s \in [0,1].$$

Here, let $\Omega = L(0,2)$. The Fréchet derivatives of Q are as follows

$$\begin{aligned} Q'(i)j(s) &= j(s) - \int_0^1 K(s,\rho)\left(\frac{5}{4}i(\rho)^{\frac{3}{2}} + \frac{21}{16}i(\rho)^2\right)j(\rho)d\rho, \quad j \in \Omega, \\ Q''(i)jk(s) &= -\int_0^1 K(s,\rho)\left(\frac{15}{8}i(\rho)^{\frac{1}{2}} + \frac{21}{8}i(\rho)\right)j(\rho)k(\rho)d\rho, \quad j,k \in \Omega. \end{aligned}$$

On choosing initial approximation $i_0 = 1$,

$$\|\Gamma_0\| = \frac{128}{87} = \xi, \quad \|\Gamma_0 Q(i_0)\| \leq \frac{15}{87} = \zeta, \quad \|Q''(i)\| \leq \frac{15\sqrt{2}}{64} + \frac{21}{32} = S.$$

So $a_0 \approx 0.2505$. Since $a_0\varpi(a_0) = 0.4014 < 1$, $h(a_0)\partial_0 = 0.768 < 1$. So the hypotheses of Theorem (1) are fulfilled and hence the approximate solution will exists in $L(1, 0.5112)$ and will be unique in $L(1, 0.8651) \cap \Omega$. Hence, based on our results, one can deduce that the existence ball of the solution is superior to that of Wang in [18] and the uniqueness ball is inferior. ◄

Example 2. [15] Consider a function

$$q(i) = \begin{cases} i^3 ln(i^2) - 6i^2 - 3i + 8 & i \in (-2, 0) \cup (0, 2), \\ 0 & i = 0. \end{cases} \tag{50}$$

Let $L(0, 2) = \Omega$ and $i_0 = 1$ be an initial point. The Fréchet derivatives of q are as follows:

$$\begin{aligned} q'(i) &= 3i^2 ln(i^2) + 2i^2 - 12i - 3, \\ q''(i) &= 6iln(i^2) + 10i - 12, \\ q'''(i) &= 6ln(i^2) + 22. \end{aligned}$$

Clearly, q''' is not bounded in the given Ω and do not satisfies the condition $(M4)$ but satisfy the assumption $(N1)$ and thus,

$$\|\Gamma_0\| \leq \frac{1}{13} = \xi, \ \|\Gamma_0 q(i_0)\| \leq \frac{1}{13} = \zeta, \ \|q''(i)\| \leq 12ln(4)+32 = S, \ \|q'''(i_0)\| = 22,$$

$$\|q'''(i) - q'''(j)\| \leq \frac{12}{1 - \frac{13}{32+12log(4)}}|i - j|, \quad \forall\, i, j \in L\left(1, \frac{13}{32 + 12ln(4)}\right).$$

Here $\omega(z) = \frac{12}{1-\frac{13}{32+12log(4)}}z$ and $\varphi(\epsilon) = 1$. Here $a_0 \approx 0.2878$ and since $a_0\varpi(a_0) = 0.50363 < 1$, $h(a_0)\partial_0 = 0.02215 < 1$. So the assumptions of Theorem (2) are satisfied. And therefore, the approximate solution lies in $L(1, 0.13611)$ which will be unique in the ball $L(1, 0.39848) \cap \Omega$.

7. Conclusion

In this article, we have analyzed the semilocal convergence of an improved super-Halley method in the Banach spaces. The analysis has been done using the recurrence relation technique by assuming the two different set of hypotheses. In the first set, the classical convergence assumptions have been relaxed to prove the convergence of the approximate solution along with its existence and uniqueness and an expression of a priori error bound has been derived. In another approach, the norm of the third-order Fréchet derivative is assumed at an initial point to avoid the unboundedness of the function on a given domain. Moreover, it fulfils the local ω-continuity condition. The theoretical study is also corroborated by two numerical tests.

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Neha Gupta
Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal, M.P. India-462003
E-mail: neha.gupta.mh@gmail.com

J. P. Jaiswal
Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal, M.P. India-462003
Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur, C.G., India- 495009.
E-mail: asstprofjpmanit@gmail.com

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