Azerbaijan Journal of Mathematics
Special Issue dedicated to the 67th birth anniversary of Prof. M. Mursaleen, *2021*
ISSN 2218-6816

Extremal Decomposition of the Complex Plane for n-Radial System of Points

I. Denega

Abstract. The paper is devoted to an extremal problem of geometric function theory of complex variable related with estimate of functional defined on system of mutually non-overlapping domains. An improved method is proposed for solving problems on extremal decomposition of the complex plane. The main results of the paper generalize and strengthening all known results in this problem for n-radial systems of points. In particular, the problem on extremal decomposition of the complex plane is solved for two free poles.

Key Words and Phrases: inner radius of the domain, Non-overlapping domains, Quadratic differential, Green's function, Conformal automorphism of the complex plane.

2010 Mathematics Subject Classifications: 30C75

1. Preliminaries

Recently, extremal problems on non-overlapping domains with free poles on the complex plane are of great interest [1–17]. The theory of quadratic differentials is one of the important elements in the study of this extremal problems. And the basic structural theorem of Jenkins [1], which gives a complete description of the global structure of trajectories of a positive quadratic differential on a finite Riemann surface, is one of the key results of this theory.

In 1934 Lavrent'ev [2] solved the problem about product of conformal radii of two mutually non-overlapping simply connected domains: let a_1 and a_2 be some fixed points on the complex plane $\mathbb{C}$, B_k, $a_k \in B_k$, $k \in \{1, 2\}$ be any non-overlapping simply connected domains in $\overline{\mathbb{C}}$. Then for functions $w = f_k(0)$, $k \in \{1, 2\}$, which are regular in the circle $|z| < 1$ and univalently mapping it to the domain B_k such that $f_k(0) = a_k$, we have inequality

$$|f_1'(0)| \cdot |f_2'(0)| \leqslant |a_1 - a_2|^2. \tag{1}$$

Equality in the inequality (1) is attended for the half-plane B_1, B_2 and points a_1, a_2, which are symmetrical about their common boundary.

Lavrent'ev used this result to some aerodynamics problems. It follows from the proof of this theorem, as a corollary, the well-known statement of Koebe-Bieberbach in the theory of univalent functions.

Later, Lavrent'ev's result was generalized to the case of meromorphic functions. Then, for the domains $B_1 \subset \overline{\mathbb{C}}$ and $B_2 \subset \overline{\mathbb{C}}$ equality in (1) is attended if and only if domains B_1 and B_2 have the following form

$$B_1 = \left\{ w \in \overline{\mathbb{C}} : \left| \frac{w - a_1}{w - a_2} \right| < \rho \right\}, \quad B_2 = \left\{ w \in \overline{\mathbb{C}} : \left| \frac{w - a_1}{w - a_2} \right| > \rho \right\}$$

or vice versa. An example of such configuration of the domains may be the case where one domain is bounded by some circle and the other is unrestricted, that is, a complement to the first domain. It should be noted that for the case of meromorphic functions, the family of extremals has continual power.

In 1952 Kolbina [3] summarized Lavrentyev's result by taking functions in fixed positive degrees: for any finite different points a_1, a_2, maximum of the value

$$|f_1'(0)|^{\alpha} \cdot |f_2'(0)|^{\beta},$$

$(\alpha > 0,\ \beta > 0)$ with respect to all possible systems of functions $f_k(z), f_k(0) = a_k\, (k \in \{1, 2\})$ which are regular in the circle $|z| < 1$ and univalently mapping $|z| < 1$ onto the domains B_1, B_2 such that $a_1 \in B_1 \subset \mathbb{C}$, $a_2 \in B_2 \subset \mathbb{C}$, is attained for functions, reflecting $|z| < 1$ onto the domains that are circular domains of the following quadratic differential

$$Q(w)dw^2 = -\frac{(a_2 - a_1)[w - a_1 - \alpha(a_2 - a_1)/(\alpha - \beta)]}{(w - a_1)^2(w - a_2)^2}\, dw^2.$$

In 1969 Goluzin [4] obtained an accurate evaluation for the case of three domains

$$\prod_{k=1}^{3} |f_k'(0)| \leqslant \frac{64}{81\sqrt{3}} \cdot |a_1 - a_2| \cdot |a_1 - a_3| \cdot |a_2 - a_3|.$$

Further, in 1997 Kuz'mina [5] showed that the problem of the evaluation for the case of four domains is reduced to the smallest capacity problems in certain continuum family and received the exact inequality

$$\prod_{k=1}^{4} |f_k'(0)| \leqslant \frac{9}{4^{\frac{8}{3}}} \left(|a_1 - a_2| \cdot |a_1 - a_3| \cdot |a_2 - a_3| \cdot |a_1 - a_4| \cdot |a_2 - a_4| \cdot |a_3 - a_4| \right)^{\frac{2}{3}}.$$

For $n \geqslant 5$ full solution of the problem is not obtained at this time.

Based on these elementary estimates a number of new estimates for functions realizing a conformal mapping of a circle onto domains with certain special properties are obtained. Estimates of this type are fundamental to solving some metric problems arising when considering the correspondence of boundaries under a conformal mapping. Also, the results can be applied to coverage theorems, distortion theorems, estimates of coefficients of univalent functions and in the study of the number of critical points in parabolic basins.

Until 1974, in problems on extremal decomposition of the complex plane a system of points a_k, $k=\overline{1,n}$, of the complex plane were fixed. Tamrazov [6] first attracted the attention of experts to the study of extremal problems associated with quadratic differentials with non-fixed poles. And he solved a significant extremal problem of the geometric function theory of complex variable with five free simple poles.

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}}=\mathbb{C}\bigcup\{\infty\}$ be its one point compactification, $\mathbb{R}^+=(0,\infty)$. Let $r(B,a)$ be the inner radius of the domain $B\subset\overline{\mathbb{C}}$ with respect to a point $a\in B$. For inner radius of the domain B the following relation holds

$$g_B(z,a)=\ln\frac{1}{|z-a|}+\ln r(B,a)+o(1),\quad z\to a,$$

where $g_B(z,a)$ is the generalized Green function of the domain B.

The system of points $A_n:=\left\{a_k\in\mathbb{C}:\ k=\overline{1,n}\right\}$ is called n-radial, if $|a_k|\in\mathbb{R}^+$ for $k=\overline{1,n}$ and

$$0=\arg a_1<\arg a_2<\ldots<\arg a_n<2\pi.$$

Denote $\alpha_k:=\frac{1}{\pi}\arg\frac{a_{k+1}}{a_k}$, $\alpha_{n+1}:=\alpha_1$, $k=\overline{1,n}$, $\sum\limits_{k=1}^{n}\alpha_k=2$.

In this paper for prove the main results we use the following statements.

It is well known [13], that the functional

$$\frac{r^{\alpha}(B_0,a_0)\cdot r^{\beta}(B_1,a_1)\cdot r^{\gamma}(B_2,a_2)}{|a_0-a_1|^{\alpha+\beta-\gamma}\cdot|a_0-a_2|^{\alpha-\beta+\gamma}\cdot|a_1-a_2|^{-\alpha+\beta+\gamma}},$$

where α, β, $\gamma\in\mathbb{R}^+$, $\{B_k\}_{k=0}^{2}$ is any system of mutually non-overlapping domains such that $a_k\in B_k\subset\overline{\mathbb{C}}$, $k\in\{0,1,2\}$, is invariant with respect to all conformal automorphisms of the complex plane $\overline{\mathbb{C}}$.

Since Kolbina [13] investigated this functional only for simply connected domains, it is not difficult to show that the statement is true also for multiply connected domains.

Let

$$w = T(z) = \frac{az+b}{cz+d}, \quad |T| = ad - bc \neq 0,$$

be fractional-linear function which conformally mapping $\overline{\mathbb{C}}_z$ onto $\overline{\mathbb{C}}_w$. In view of the conformal invariance of the Green function, we obtain

$$g_{B_k}(z, a_k) = g_{B_k^+}(w, a_k^+).$$

Therefore,

$$g_{B_k^+}(w, a_k^+) = g_{B_k^+}\left(\frac{az+b}{cz+d}, \frac{aa_k+b}{ca_k+d}\right) =$$

$$= \ln \frac{1}{\left|\dfrac{az+b}{cz+d} - \dfrac{aa_k+b}{ca_k+d}\right|} + \ln r(B_k^+, a_k^+) + o(1).$$

As a result of simple transformations, we get

$$g_{B_k^+}(w, a_k^+) = \ln \frac{1}{|z-a_k|} + \ln \frac{|ca_k+d|^2}{|ad-bc|} r(B_k^+, a_k^+) + o(1).$$

Hence,

$$r\left(B_k^+, a_k^+\right) = r(B_k, a_k) \cdot \frac{|ad-bc|}{(ca_k+d)^2}.$$

Then, we easily obtain

$$\frac{r^\alpha(B_0^+, a_0^+) r^\beta(B_1^+, a_1^+) r^\gamma(B_2^+, a_2^+)}{|T(a_0)-T(a_1)|^{\alpha+\beta-\gamma} |T(a_0)-T(a_2)|^{\alpha-\beta+\gamma} |T(a_1)-T(a_2)|^{-\alpha+\beta+\gamma}} =$$

$$= \frac{\dfrac{|T|^{\alpha+\beta+\gamma} r^\alpha(B_0, a_0) r^\beta(B_1, a_1) r^\gamma(B_2, a_2)}{(ca_0+d)^{2\alpha}(ca_1+d)^{2\beta}(ca_2+d)^{2\gamma}}}{\dfrac{|T|^{\alpha+\beta+\gamma} |a_0-a_1|^{\alpha+\beta-\gamma} |a_0-a_2|^{\alpha-\beta+\gamma} |a_1-a_2|^{-\alpha+\beta+\gamma}}{(ca_0+d)^{2\alpha}(ca_1+d)^{2\beta}(ca_2+d)^{2\gamma}}} =$$

$$= \frac{r^\alpha(B_0, a_0) r^\beta(B_1, a_1) r^\gamma(B_2, a_2)}{|a_0-a_1|^{\alpha+\beta-\gamma} |a_0-a_2|^{\alpha-\beta+\gamma} |a_1-a_2|^{-\alpha+\beta+\gamma}}.$$

In 1994 Dubinin [7] obtained next result for three domains: for any three pairwise non-overlapping domains B_0, B_1, B_2 of the complex plane such that $a_k \in B_k$, $k = \overline{0,2}$, $a_0 = 0$, $a_k = (-1)^k i$, we have

$$r^{\sigma^2}(B_0, 0)\, r(B_1, a_1)\, r(B_2, a_2) \leqslant$$

$$\leqslant 2^{\sigma^2+6} \cdot \sigma^{\sigma^2} \cdot (2-\sigma)^{-\frac{1}{2}(2-\sigma)^2} \cdot (2+\sigma)^{-\frac{1}{2}(2+\sigma)^2}, \quad \sigma \in (0,2),$$

equality in this inequality is achieved when points 0, $-i$, i and domains B_0, B_1, B_2, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = \frac{(4-\sigma^2)w^2 - \sigma^2}{w^2(w^2+1)^2} dw^2.$$

2. Main results

Consider the following problem.

Problem. For all values of the parameter $\gamma \in (0, n]$ to find a maximum of the functional

$$Y_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where $n \in \mathbb{N}$, $n \geqslant 2$, $A_n = \{a_k\}_{k=1}^{n}$ is an arbitrary n-radial system of points, $a_0 = 0$, $\{B_k\}_{k=0}^{n}$ is the system of mutually non-overlapping domains such that $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{0, n}$, and to describe all extremals.

The problem of finding maximum of the functional $Y_n(\gamma)$ for all $\gamma \in (0, n]$ was formulated in [7, 8] in the list of unsolved problems for the case when points a_k, $k = \overline{1, n}$, lie on the unit circle. In work [7], the above-formulated problem was solved for $\gamma = 1$ and all values of the natural parameter $n \geqslant 2$, when $|a_k| = 1$, $k = \overline{1, n}$. Namely, it was shown that the following inequality holds

$$r(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant r\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k = \overline{0, n}$, are the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-1)w^n + 1}{w^2(w^n-1)^2}\, dw^2.$$

In work [10], Kovalev got its solution for definite sufficiently strict limitations on the geometry of arrangement of the systems of points on a unit circle, namely, for systems of points for which the following inequalities hold

$$|a_k| = 1, \quad 0 < \alpha_k \leqslant 2/\sqrt{\gamma}, \quad k = \overline{1, n}, \quad n \geqslant 5.$$

In work [11], it was shown that the result by Kovalev is also true for $n = 4$. The solution of this problem for $\gamma \in (0, 1]$ was given in work [9]. Some partial cases of this problem were studied, for example, in [6–17]. And note, that studying maximum of the functional $Y_n(\gamma)$ in the works [7, 8, 10, 12] in particular cases for certain values of γ it was shown that the quantity

$$Y_n^0(\gamma) = r^\gamma\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k = \overline{0, n}$, $d_0 = 0$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2,$$

takes the form

$$Y_n^0(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

Thus, the following statement is true.

Theorem 1. *Let $\gamma \in (0, 2]$. Then, for any 2-radial system of different points $A_2 = \{a_1, a_2\} \in \mathbb{C}\backslash\{0\}$ and for any mutually non-overlapping domains B_0, B_1, B_2, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_1 \in B_1 \subset \overline{\mathbb{C}}$, $a_2 \in B_2 \subset \overline{\mathbb{C}}$, the following inequality holds*

$$r^\gamma\left(B_0, 0\right) r\left(B_1, a_1\right) r\left(B_2, a_2\right) \leqslant$$

$$\leqslant \frac{4\gamma^{\frac{\gamma}{2}}}{\left(1-\frac{\gamma}{4}\right)^{2+\frac{\gamma}{2}}} \left(\frac{1-\frac{\sqrt{\gamma}}{2}}{1+\frac{\sqrt{\gamma}}{2}}\right)^{2\sqrt{\gamma}} |a_1 a_2|^\gamma \left(\frac{1}{2}|a_1 - a_2|\right)^{2-\gamma}.$$

The sign of equality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(4-\gamma)w^2 + \gamma}{w^2(w^2-1)^2} dw^2. \tag{2}$$

Proof. Let $a_0 = 0$, $A_2 = \{a_1, a_2\} \in \mathbb{C}\backslash\{0\}$ is any 2-radial system of different points, B_0, B_1, B_2 are pairwise non-overlapping domains in $\overline{\mathbb{C}}$ such that $0 \in B_0$, $a_1 \in B_1$, $a_2 \in B_2$. Then, there is unique conformal automorphism of the complex plane $\overline{\mathbb{C}}$

$$\widetilde{\omega} = T(\omega),$$

which transforms three given points a_0, a_1, a_2 into the points $T(a_0)=0$, $T(a_1)=1$, $T(a_2)=-1$ (see, for example, [13]). Then, due to the conformal invariance of the functional $Y_2(\gamma)$ the following equality holds

$$\frac{r^{\gamma}(B_0,0)r(B_1,a_1)r(B_2,a_2)}{|a_1|^{\gamma}\cdot|a_2|^{\gamma}\cdot|a_1-a_2|^{2-\gamma}}=\frac{r^{\gamma}(\widetilde{B_0},0)r(\widetilde{B_1},1)r(\widetilde{B_2},-1)}{2^{2-\gamma}},$$

where $\widetilde{Y_2(\gamma)}=r^{\gamma}(\widetilde{B_0},0)r(\widetilde{B_1},1)r(\widetilde{B_2},-1)$, $\widetilde{B_0}=T(B_0)$, $\widetilde{B_1}=T(B_1)$, $\widetilde{B_2}=T(B_2)$. It follows that

$$\frac{Y_2(\gamma)}{|a_1a_2|^{\gamma}\cdot|a_1-a_2|^{2-\gamma}}=\frac{\widetilde{Y_2(\gamma)}}{2^{2-\gamma}},$$

and, thus,

$$Y_2(\gamma)=\widetilde{Y_2(\gamma)}\,|a_1a_2|^{\gamma}\left(\frac{1}{2}|a_1-a_2|\right)^{2-\gamma}.$$

Further, using the Dubinin result for three mutually non-overlapping domains [7], the following inequality holds

$$Y_2(\gamma)\leqslant Y_2^0(\gamma)\,|a_1a_2|^{\gamma}\left(\frac{1}{2}|a_1-a_2|\right)^{2-\gamma},$$

where

$$Y_2^0(\gamma)=\frac{4\gamma^{\frac{\gamma}{2}}}{\left(1-\frac{\gamma}{4}\right)^{2+\frac{\gamma}{2}}}\left(\frac{1-\frac{\sqrt{\gamma}}{2}}{1+\frac{\sqrt{\gamma}}{2}}\right)^{2\sqrt{\gamma}}.$$

Theorem 1 is proved.

Remark 1. *From Theorem 1 we obtain complete solution of the above posed Problem for $n=2$.*

Corollary 1. *Let $\gamma\in(0,2]$. Then, for any 2-radial system of different points $A_2=\{a_1,a_2\}\in\mathbb{C}\backslash\{0\}$ such that*

$$|a_1a_2|\leqslant 1,\quad \left(\frac{1}{2}|a_1-a_2|\right)^{2-\gamma}\leqslant 1,$$

and for any mutually non-overlapping domains B_0, B_1, B_2, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $a_1\in B_1\subset\overline{\mathbb{C}}$, $a_2\in B_2\subset\overline{\mathbb{C}}$, inequality

$$r^{\gamma}\left(B_0,0\right)r\left(B_1,a_1\right)r\left(B_2,a_2\right)\leqslant\frac{4\gamma^{\frac{\gamma}{2}}}{\left(1-\frac{\gamma}{4}\right)^{2+\frac{\gamma}{2}}}\left(\frac{1-\frac{\sqrt{\gamma}}{2}}{1+\frac{\sqrt{\gamma}}{2}}\right)^{2\sqrt{\gamma}}$$

is true. The sign of equality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential (2).

Corollary 2. *Let $\gamma \in (0, 2]$. Then for any different points a_1 and a_2 of the unit circle and any pairwise non-overlapping domains B_0, B_1, B_2, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_1 \in B_1 \subset \overline{\mathbb{C}}$, $a_2 \in B_2 \subset \overline{\mathbb{C}}$, the following inequality holds*

$$r^{\gamma}(B_0, 0)\, r(B_1, a_1)\, r(B_2, a_2) \leqslant \frac{4\gamma^{\frac{\gamma}{2}}}{\left(1 - \frac{\gamma}{4}\right)^{2+\frac{\gamma}{2}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{2}}{1 + \frac{\sqrt{\gamma}}{2}}\right)^{2\sqrt{\gamma}} \left(\frac{1}{2}|a_1 - a_2|\right)^{2-\gamma}.$$

The sign of equality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential (2).

Using Theorem 1 we have the following result.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geqslant 3$, $\gamma \in (0, n]$. Then, for any fixed n-radial system of points $A_n = \{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ and for any collection of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, the inequality*

$$Y_n(\gamma) \leqslant \left(Y_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}} \left(\prod_{k=1}^{n} |a_k a_{k+1}|\right)^{\frac{\gamma}{n}} \left(\prod_{k=1}^{n} \frac{1}{2}|a_k - a_{k+1}|\right)^{1-\frac{\gamma}{n}} \tag{3}$$

is valid.

Proof. Taking into account Theorem 1 we obtain the following upper estimate for the maximum of the functional $Y_n(\gamma)$ on any n-radial system of points. It is easy to see that

$$(Y_n(\gamma))^2 = \left(r^{\frac{2\gamma}{n}}(B_0, 0)\right)^n \prod_{k=1}^{n} r^2(B_k, a_k) = \left[r^{\frac{2\gamma}{n}}(B_0, 0)\, r(B_1, a_1)\, r(B_2, a_2)\right] \times$$

$$\times \left[r^{\frac{2\gamma}{n}}(B_0, 0)\, r(B_2, a_2)\, r(B_3, a_3)\right] \cdot \ldots \cdot \left[r^{\frac{2\gamma}{n}}(B_0, 0)\, r(B_n, a_n)\, r(B_1, a_1)\right].$$

Analogously to Theorem 1, for each multiplier

$$r^{\frac{2\gamma}{n}}(B_0, 0)\, r(B_k, a_k)\, r(B_{k+1}, a_{k+1}), \quad k = \overline{1, n}, \quad a_{n+1} := a_1,$$

we construct its conformal automorphism of the complex plane. The equality implies that

$$r^{\frac{2\gamma}{n}}(B_0, 0) r(B_k, a_k) r(B_{k+1}, a_{k+1}) =$$

$$= \widetilde{Y_2^{(k)}\left(\frac{2\gamma}{n}\right)}|a_k a_{k+1}|^{\frac{2\gamma}{n}}\left(\frac{1}{2}|a_k - a_{k+1}|\right)^{2-\frac{2\gamma}{n}}, \quad k=\overline{1,n}, \quad a_{n+1} := a_1.$$

With regard for the result in [7] for three arbitrary pairwise non-overlapping domains, we finally obtain the relation (3). Theorem 2 is proved.◀

Corollary 3. *Let $n \in \mathbb{N}$, $n \geqslant 3$, $\gamma \in (0, n]$. Then, for any fixed n-radial system of points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}\backslash\{0\}$ such that*

$$\left(\prod_{k=1}^n |a_k a_{k+1}|\right)^{\frac{\gamma}{n}} \leqslant 1, \quad \left(\prod_{k=1}^n \frac{1}{2}|a_k - a_{k+1}|\right)^{1-\frac{\gamma}{n}} \leqslant 1,$$

and for any collection of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the inequality

$$r^\gamma(B_0, 0)\prod_{k=1}^n r(B_k, a_k) \leqslant \left(Y_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}}$$

is proper.

Corollary 4. *Let $n \in \mathbb{N}$, $n \geqslant 3$, $\gamma \in (1, n]$. Then, for any system of different points $A_n = \{a_k\}_{k=1}^n$ of a unit circle and for any collection of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the inequality*

$$r^\gamma(B_0, 0)\prod_{k=1}^n r(B_k, a_k) \leqslant \left(\sin\frac{\pi}{n}\right)^{n-\gamma}\left(Y_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}}$$

holds.

3. Conclusions

In the paper the problem of finding a maximum of the product of inner radii of mutually non-overlapping domains with respect to n-radial systems of points on the complex plane on a certain positive degree of $\gamma \in (0,\, n]$ of the inner radius of the domain with respect to the origin is considered. We obtained a solution of this problem for $n = 2$ and $\gamma \in (0,\, 2]$. And for $n \geqslant 3$ an upper estimate for the maximum of the functional $Y_n(\gamma)$ on an arbitrary n-radial systems of points on the complex plane is established. As a consequence, the corresponding results are obtained for the case when the points are located on the unit circle.

Acknowledgements

The present work is financially held up by the budget program "Support of the development of priority trends of scientific researches" (KPKVK 6541230).

The author thanks the anonymous referees for the very careful analysis of this work and remarks.

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Iryna Denega
Institute of mathematics of the National Academy of Sciences of Ukraine,
Department of complex analysis and potential theory,
Tereschenkivska 3, 01024, Kyiv, Ukraine
E-mail: iradenega@gmail.com

Received 17 February 2020
Accepted 20 July 2020