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# Estimation of Fixed Points of Hardy and Rogers Generalized Non-Expansive Mappings

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**Abstract.** In the present paper, we study a three-step iterative scheme to approximate the fixed points of Hardy and Rogers generalized non-expansive mappings. Some weak and strong convergence results are proved for such mappings in uniformly convex Banach spaces. Further, it is showed numerically that the considered iterative scheme has a better speed of convergence than some known and leading schemes for generalized non-expansive mappings. The results of this paper are the refinement and generalization of several relevant results in the literature.

**Key Words and Phrases**: generalized non-expansive mappings, fixed point, weak and strong convergence results, uniformly convex Banach space.

**2010 Mathematics Subject Classifications**: 47H09, 47H10

## 1. Introduction

Throughout the paper, $\mathbb{Z}_+$ denotes the set of all nonnegative integers. We consider that $C$ is a nonempty subset of a Banach space $X$ and $F(T)$, the set of all fixed points of the mapping $T$ on $C$. A mapping $T : C \to C$ is said to be non-expansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. It is called quasi non-expansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$.

The study of fixed points of non-expansive mappings and its generalized forms is more complicated as compared to the study of fixed points of contractive mappings. The fixed point theory of contractiveness rotates around the Banach contraction principle wherein the fixed point of a contraction can be approximated by Picard iterative scheme in complete metric space. But in general, fixed points of non-expansive mappings can't be approximated by the same scheme. The approximating fixed points of non-expansive and its generalized mappings has been

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studied in a variety of ways and its applications in the theory of optimization, variational inequality problems, convex feasibility problems are fairly well known.

In 1973, Hardy and Rogers [1] introduced the concept of generalized non-expansive mapping which is defined as follows:
A self map $T$ on a nonempty subset $C$ of a Banach space $X$ is called generalized non-expansive if for all $x, y \in C$,

$$\|Tx-Ty\| \leq a_1\|x-y\|+a_2\|x-Tx\|+a_3\|y-Ty\|+a_4\|x-Ty\|+a_5\|y-Tx\|, \quad (1)$$

where $a_1, ..., a_5$ are nonnegative real numbers with $a_1 + a_2 + a_3 + a_4 + a_5 \leq 1$.
The condition (1) appears to be quite natural from a geometric point of view. It is obvious that condition (1) is equivalent to the following condition (cf. [2]).

$$\|Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|), \quad (2)$$

for all $x, y \in C$, where $a, b, c$ are nonnegative constants with $a + 2b + 2c \leq 1$ and $a = a_1$, $b = \frac{a_2+a_3}{2}$, $c = \frac{a_4+a_5}{2}$.

It is well known that if $T$ has a fixed point then $T$ is quasi non-expansive mapping. Thus, the class of generalized non-expansive mappings is bigger than the class of non-expansive mappings and smaller than the class of quasi non-expansive mappings. The existence and convergence theorems for non-expansive and generalized non-expansive mappings have been studied by notable authors, e.g. see [3, 4, 5].

Quite recently, Ali et al. [6] proved the following eminent lemma for the generalized non-expansive mappings due to Hardy and Rogers.

**Lemma 1.** *[6] Let $C$ be a nonempty subset of a Banach space $X$ and $T : C \to C$ a generalized non-expansive mapping satisfying* (2)*. Then*

$$\|x - Ty\| \leq \|x - y\| + \frac{1 + b + c}{1 - b - c}\Big\|x - Tx\Big\|, \ \forall x, y \in C.$$

We observe the following fact.

**Remark 1.** *Every generalized non-expansive mapping due to Hardy and Rogers satisfies condition $(E)$ due to García-Falset et al. [7] for $\mu = \frac{1+b+c}{1-b-c} \geq 1$.*

In 2008, Suzuki [8] introduced generalized non-expansive mapping, also called condition $(C)$ on mapping which is defined as follows:
A self map $T$ on a nonempty subset $C$ of a Banach space $X$ is said to satisfy condition $(C)$ if,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

Suzuki proved existence of fixed point and convergence theorems for such mappings. Suzuki also showed that every non-expansive mapping satisfies condition $(C)$, but the converse is not true in general. Moreover, if $T$ satisfies condition $(C)$ and has at least one fixed point, then it is quasi non-expansive mapping. Thus the notion of mappings satisfying condition $(C)$ is weaker than non-expansiveness and stronger than quasi non-expansiveness.

From the beginning of the twentieth century, a large number of eminent researchers studied and proved the existence of a fixed point of various classes of linear and nonlinear mappings in various classes of space. So, it is natural to consider the question that: "when the existence of a fixed point of an operator is accomplished, then how to find the fixed point"? The fixed point of a linear or nonlinear mapping can find out by the two types of methods: (i) the direct methods and (ii) iterative methods. Sometimes, direct method fails to find the fixed point of a mapping due to various reasons, so iterative methods take the place to find the fixed point. Thus the iterative approximations of fixed points become a major tool in the fixed point theory and applied mathematics.
Since Picard iterative scheme need not converge to the fixed point of non-expansive mapping even map has a fixed point. So in 1953, Mann [9] introduced an iterative scheme which has been extensively used to approximate fixed point of non-expansive mappings. In this scheme the sequence $\{x_n\}$ is generated by an arbitrary point $x_0 \in C$ and defined in the following manner:

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n,\ n \in \mathbb{Z}_+, \tag{3}$$

where $\{a_n\}$ is a sequence in $(0, 1)$.

In 1974, Ishikawa [10] introduced a two-step iterative scheme to approximate fixed point of pseudo contractive mappings. In this scheme the sequence $\{x_n\}$ is generated by an arbitrary point $x_0 \in C$ and defined as follows:

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T x_n\ , n \in \mathbb{Z}_+, \end{cases} \tag{4}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$.

In 2000, Noor [11] introduced the following three-step iterative scheme for the solution of general variational inequalities. In this scheme the sequence $\{x_n\}$ is generated by an arbitrary point $x_0 \in C$ and defined as follows:

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T z_n, \\ z_n = (1 - c_n)x_n + c_n T x_n,\ n \in \mathbb{Z}_+, \end{cases} \tag{5}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0, 1)$.

In 2007, Agrawal et al. [12] introduced the following two-step iterative scheme, called S iterative scheme and approximated the fixed points of nearly asymptotically non-expansive mappings in uniformly convex Banach spaces. In this scheme the sequence $\{x_n\}$ is generated by an arbitrary guess $x_0 \in C$ and defined in the following manner:

$$\begin{cases} x_{n+1} = (1-a_n)Tx_n + a_nTy_n, \\ y_n = (1-b_n)x_n + b_nTx_n, \ n \in \mathbb{Z}_+, \end{cases} \tag{6}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1)$. They also claimed that this scheme converges to a fixed point of a contraction at the rate same as Picard iterative scheme and faster than the Mann scheme.

In 2014, Abbas and Nazir [13] introduced the following three-step iterative scheme to approximate fixed points of non-expansive mappings in uniformly convex Banach space. In this scheme the sequence $\{x_n\}$ is generated by an initial guess $x_0 \in C$ and defined as follows:

$$\begin{cases} x_{n+1} = (1-a_n)Ty_n + a_nTz_n, \\ y_n = (1-b_n)Tx_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \ n \in \mathbb{Z}_+, \end{cases} \tag{7}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. Also, they claimed that this scheme converges to a fixed point of a contraction faster than S iterative scheme (6).

Recently, Sahu et al. [14] and Thakur et al. [15] introduced the following same iterative scheme, independently to approximate the fixed points of non-expansive mappings in uniformly convex Banach spaces. In this scheme the sequence $\{x_n\}$ is generated by an initial point $x_0 \in C$ and defined in the following manner:

$$\begin{cases} x_{n+1} = (1-a_n)Tz_n + a_nTy_n, \\ y_n = (1-b_n)z_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \ n \in \mathbb{Z}_+, \end{cases} \tag{8}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. They proved that this scheme converges to a fixed point of contraction faster than Picard, Mann, Ishikawa, Noor, S and Abbas iterative schemes. They also presented an example to support their claim.

In 1981, Bose and Mukherjee [16] proved that the Mann iterative scheme converges strongly to the fixed point of generalized non-expansive mappings due to Hardy and Rogers in uniformly convex Banach spaces. In process, Maiti and Ghosh [17] proved strong convergence theorem for the same mappings via Ishikawa iterative scheme in uniformly convex Banach spaces. While Park [18] proved weak convergence result for the same mappings via Mann iterative scheme in uniformly convex Banach spaces.

Very recently, Ali et al. [19] approximated the fixed points of Suzuki's generalized non-expansive mappings via iterative scheme (8) in uniformly convex Banach spaces. Since the mappings due to Hardy and Rogers and, Suzuki are generalized non-expansive. So, most recently, Ali et al. [6] compared the generalized non-expansive mappings due to Suzuki and, Hardy and Rogers and showed that both the mappings do not imply each other. Also, the class of generalized non-expansive mappings due to Hardy and Rogers is very natural than the class of mappings satisfying Suzuki's condition $(C)$.

So, motivated by the above, we prove some weak and strong convergence results for generalized non-expansive mappings due to Hardy and Rogers via iterative scheme (8) in uniformly convex Banach spaces. Further, we show numerically that the scheme (8) converges to fixed points of generalized non-expansive mapping faster than Mann, Ishikawa, Noor, S and Abbas iterative schemes. The results of the present paper are different from the results of Ali et al. [19] and generalize several relevant results in the literature and particularly those which are contained in Sahu et al. [14] and Thakur et al. [15].

## 2. Preliminaries

We now recall some definitions and lemmas to be used in the main results.

**Definition 1.** *(cf. [20]) A Banach space $X$ is called uniformly convex if for each $\epsilon \in (0,2]$ and for all $x, y \in X$, with $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x-y\| \geq \epsilon$ there is a $\delta = \delta(\epsilon) > 0$ such that $\|\frac{x+y}{2}\| \leq 1-\delta$.*

**Definition 2.** *[21] Let $C$ be a nonempty, closed and convex subset of a Banach space $X$. A mapping $T : C \to X$ is called demiclosed with respect to $y \in X$, if for each sequence $\{x_n\}$ in $C$ and each $x \in C$, $\{x_n\}$ converges weakly at $x$ and $\{Tx_n\}$ converges strongly at $y$ imply that $Tx = y$.*

**Definition 3.** *A Banach space $X$ is said to satisfy Opial's condition [22] if for each weakly convergent sequence $\{x_n\}$ to $x \in X$,*

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$$

*holds, for all $y \in X$, with $y \neq x$.*

**Definition 4.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and let $\{x_n\}$ be a bounded sequence in $X$. For $x \in X$, we set*

$$r(x, \{x_n\}) = \lim_{n\to\infty} \sup \|x_n - x\|.$$

*The asymptotic radius of $\{x_n\}$ relative to $C$ is defined by*

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

*The asymptotic center of $\{x_n\}$ relative to $C$ is the set*

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

*It is known that in a uniformly convex Banach space, $A(C, \{x_n\})$ consists exactly one point.*

**Lemma 2.** *[23] Suppose $X$ is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of $X$ such that $\lim_{n\to\infty} \sup \|x_n\| \leq r$, $\lim_{n\to\infty} \sup \|y_n\| \leq r$ and $\lim_{n\to\infty} \sup \|t_n x_n + (1-t_n)y_n\| = r$ hold, for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

**Lemma 3.** *[6] Let $T : C \to C$ be a generalized non-expansive mapping satisfying* (2)*, where $C$ is a nonempty closed and convex subset of a uniformly convex Banach space $X$. If the sequence $\{x_n\}$ converges weakly to $x \in C$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tx = x$. That is, $I - T$ is demiclosed at zero.*

**Lemma 4.** *[24] Let $T$ be a generalized non-expansive mapping on a weakly compact convex subset $Y$ of a uniformly convex Banach space $X$ satisfying* (2)*. Then $T$ has a fixed point.*

## 3. Main Results

In this section, we prove some weak and strong convergence results for generalized non-expansive mappings satisfying (2) via iterative scheme (8) in uniformly convex Banach spaces. First, we establish the following useful lemmas for the next results.

**Lemma 5.** *Let $C$ be a nonempty closed and convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a generalized non-expansive mapping. Let $\{x_n\}$ be a sequence defined by iterative scheme* (8)*, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

*Proof.* Since $T$ is a generalized non-expansive mapping, so we can easily obtain that

$$\|Tx_n - p\| \quad \leq \quad \|x_n - p\|, \ \forall x_n \in C, \ p \in F(T).$$

Now, by iterative scheme (8), we get

$$\begin{aligned}\|z_n - p\| &= \|(1-c_n)x_n + c_n T x_n - p\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|Tx_n - p\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{9}$$

Using (9), we obtain that

$$\begin{aligned}\|y_n - p\| &= \|(1-b_n)z_n + b_n T z_n - p\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|Tz_n - p\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|z_n - p\| \\ &= \|z_n - p\| \leq \|x_n - p\|. \end{aligned} \tag{10}$$

Again using (9) and (10), we get

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-a_n)Tz_n + a_n T y_n - p\| \\ &\leq (1-a_n)\|Tz_n - p\| + a_n\|Ty_n - p\| \\ &\leq (1-a_n)\|z_n - p\| + a_n\|y_n - p\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{11}$$

This means that, the sequence $\{\|x_n - p\|\}$ is non-increasing and bounded below for all $p \in F(T)$. Hence, $\lim_{n\to\infty} \|x_n - p\|$ exists. ◄

**Lemma 6.** *Let $C$ be a nonempty closed and convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a generalized non-expansive mapping. Let $\{x_n\}$ be a sequence defined by iterative scheme (8). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

*Proof.* Suppose $F(T) \neq \emptyset$ and $p \in F(T)$. Then by Lemma 5, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Assume that

$$\lim_{n\to\infty} \|x_n - p\| = \alpha. \tag{12}$$

From (9), (10) and (12), we have

$$\lim_{n\to\infty} \sup \|z_n - p\| \leq \lim_{n\to\infty} \sup \|x_n - p\| \leq \alpha. \tag{13}$$

$$\lim_{n\to\infty} \sup \|y_n - p\| \leq \lim_{n\to\infty} \sup \|x_n - p\| \leq \alpha. \tag{14}$$

Since $T$ is a generalized non-expansive mapping, we have

$$
\begin{aligned}
\|Tx_n - p\| &= \|Tx_n - Tp\| \leq \|x_n - p\| \\
\Longrightarrow \qquad \lim_{n\to\infty} \sup \|Tx_n - p\| &\leq \lim_{n\to\infty} \sup \|x_n - p\| \leq \alpha. \qquad (15)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\lim_{n\to\infty} \sup \|Ty_n - p\| \leq \lim_{n\to\infty} \sup \|y_n - p\| \leq \alpha. \qquad (16) \\
\lim_{n\to\infty} \sup \|Tz_n - p\| \leq \lim_{n\to\infty} \sup \|z_n - p\| \leq \alpha. \qquad (17)
\end{aligned}
$$

Again,

$$
\begin{aligned}
\alpha &= \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1-a_n)Tz_n + a_n Ty_n - p\| \\
&= \lim_{n\to\infty} \|(1-a_n)(Tz_n - p) + a_n(Ty_n - p)\|. \qquad (18)
\end{aligned}
$$

From (16), (17), (18) and using Lemma 2, we have

$$
\lim_{n\to\infty} \|Tz_n - Ty_n\| = 0. \qquad (19)
$$

Now,

$$
\|x_{n+1} - p\| = \|(1-a_n)Tz_n + a_n Ty_n - p\| \leq \|Tz_n - p\| + a_n\|Ty_n - Tz_n\|.
$$

Taking the $\liminf$ on both sides, we get

$$
\begin{aligned}
\alpha &= \lim_{n\to\infty} \inf \|x_{n+1} - p\| \leq \lim_{n\to\infty} \inf \|Tz_n - p\| \\
&\leq \lim_{n\to\infty} \inf \|z_n - p\|. \qquad (20)
\end{aligned}
$$

So that, (13) and (20) give,

$$
\lim_{n\to\infty} \|z_n - p\| = \alpha.
$$

Thus,

$$
\begin{aligned}
\alpha &= \lim_{n\to\infty} \|z_n - p\| = \lim_{n\to\infty} \|(1-c_n)x_n + c_n Tx_n - p\| \\
&= \lim_{n\to\infty} \|(1-c_n)(x_n - p) + c_n(Tx_n - p)\|. \qquad (21)
\end{aligned}
$$

From (12), (15), (21) and using Lemma 2, we have

$$
\lim_{n\to\infty} \|x_n - Tx_n\| = 0.
$$

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $p \in A(C, \{x_n\})$, then using Lemma 1, we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \lim_{n\to\infty} \sup \|x_n - Tp\| \\ &\leq \lim_{n\to\infty} \sup(\|x_n - p\| + \frac{1+b+c}{1-b-c}\|Tx_n - x_n\|) \\ &= \lim_{n\to\infty} \sup \|x_n - p\| \\ &= r(p, \{x_n\}) = r(C, \{x_n\}). \end{aligned}$$

This means that, $Tp \in A(C, \{x_n\})$. Since $X$ is uniformly convex, $A(C, \{x_n\})$ is singleton, hence $Tp = p$. This completes the proof. ◄

Now, we prove weak and strong convergence results for generalized non-expansive mappings satisfying (2) via iterative scheme (8).

**Theorem 1.** *Let $C$ be a nonempty closed and convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a generalized non-expansive mapping. Assume that $X$ satisfies the Opial's condition, then the sequence $\{x_n\}$ defined by iterative scheme (8) converges weakly to a point of $F(T)$.*

*Proof.* In view of Lemma 5, the $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Now, we prove that $\{x_n\}$ has a unique weak sub-sequential limit in $F(T)$. Let $x$ and $y$ be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of the sequence $\{x_n\}$, respectively. By Lemma 6, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed at zero by Lemma 3. This implies that, $(I - T)x = 0$, that is $x = Tx$, similarly $Ty = y$. Next, we show uniqueness. If $x \neq y$, then by Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - x\| &= \lim_{n_i\to\infty} \|x_{n_i} - x\| < \lim_{n_i\to\infty} \|x_{n_i} - y\| \\ &= \lim_{n\to\infty} \|x_n - y\| = \lim_{n_j\to\infty} \|x_{n_j} - y\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - x\| \\ &= \lim_{n\to\infty} \|x_n - x\|. \end{aligned}$$

This is a contradiction, so $x = y$. Consequently, $\{x_n\}$ converges weakly to a point of $F(T)$. This completes the proof. ◄

**Theorem 2.** *Let $C$ be a nonempty closed and convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a generalized non-expansive mapping. Then the sequence $\{x_n\}$ defined by iterative scheme* (8) *converges strongly to a point of $F(T)$ if and only if $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf\{\|x_n - p\| : p \in F(T)\}$.*

*Proof.* Necessity is obvious.
Conversely, assume that $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$ and $p \in F(T)$. By Lemma 5, $\lim_{n\to\infty} \|x_n - p\|$ exists, for all $p \in F(T)$ and $\lim_{n\to\infty} d(x_n, F(T)) = 0$ by assumption. Now, we will show that $\{x_n\}$ is a Cauchy sequence in $C$. As $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for a given $\epsilon > 0$, there exists $m_0 \in \mathbb{Z}_+$ such that for all $n \geq m_0$

$$\begin{aligned} d(x_n, F(T)) &< \frac{\epsilon}{2} \\ \text{or } \inf\{\|x_n - p\| : p \in F(T)\} &< \frac{\epsilon}{2}. \end{aligned}$$

In particular, $\inf\{\|x_{m_0} - p\| : p \in F(T)\} < \frac{\epsilon}{2}$. Therefore there exists $p \in F(T)$ such that

$$\|x_{m_0} - p\| < \frac{\epsilon}{2}.$$

Now for $m, n \geq m_0$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq \|x_{m_0} - p\| + \|x_{m_0} - p\| \\ &= 2\|x_{m_0} - p\| < \epsilon. \end{aligned}$$

That is, $\{x_n\}$ is a Cauchy sequence in $C$. As $C$ is a closed subset of a Banach space $X$, so there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$. By assumption, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ which gives $d(q, F(T)) = 0 \implies q \in F(T)$. ◀

**Theorem 3.** *Let $C$ be a nonempty compact and convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a generalized non-expansive mapping. Then the sequence $\{x_n\}$ defined by iterative scheme* (8) *converges strongly to a fixed point of $T$.*

*Proof.* In view of Lemma 6, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since $C$ is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ strongly for some $p \in C$. By Lemma 1, we have

$$\|x_{n_j} - Tp\| \leq \|x_{n_j} - p\| + \frac{1+b+c}{1-b-c}\|Tx_{n_j} - x_{n_j}\|, \ \forall j \geq 1.$$

Letting $j \to \infty$, we get that $x_{n_j} \to Tp$. This implies that $Tp = p$, *i.e.*, $p \in F(T)$. Also, $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 5. Thus, $p$ is the strong limit of the sequence $\{x_n\}$. ◀

In 1974, Senter and Dotson [25] introduced the notion of condition (I) on the mappings which is defined as follows:

**Definition 5.** *A mapping $T : C \to C$ is said to satisfy condition $(I)$, if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$.*

We now prove a strong convergence result using condition $(I)$.

**Theorem 4.** *Let $C$ be a nonempty closed and convex subset of a uniformly convex Banach space $X$ and $T : C \to C$ a generalized non-expansive mapping which satisfies condition $(I)$. Then the sequence $\{x_n\}$ defined by iterative scheme (8) converges strongly to a fixed point of $T$.*

*Proof.* We proved in Lemma 6 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{22}$$

By using condition $(I)$ and (22), we get

$$\begin{aligned} 0 \leq \lim_{n\to\infty} f(d(x_n, F(T))) &\leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \\ \implies \lim_{n\to\infty} f(d(x_n, F(T))) &= 0. \end{aligned}$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0, \forall r > 0$, hence we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Now, all the conditions of Theorem 2 are satisfied therefore by its conclusion $\{x_n\}$ converges strongly to a fixed point of $T$. ◄

Since every non-expansive mapping is a generalized non-expansive due to Hardy and Rogers, so we have the following remark.

**Remark 2.** *All the results in this paper generalize the corresponding results of Sahu et al. [14], Thakur et al. [15] and many others because mappings here are generalized non-expansive and iterative scheme is more general than the others.*

## 4. An Illustrative Numerical Example

In this section, we furnish the following example to approximate the fixed point of generalized non-expansive mapping satisfying (2) via iterative scheme (8) and also compare rate of convergence of considered iterative scheme with some leading schemes for the same mapping.

**Example 1.** *Let $X = \mathbb{R}$ be a Banach space with the usual norm and $C = [0, \infty)$ a subset of $X$. Define a mapping $T : C \to C$ by*

$$Tx = \begin{cases} \frac{1}{3} sinx, & if\ x \in [0, \frac{\pi}{2}] \\ \\ \frac{1}{6} sinx, & if\ x \in (\frac{\pi}{2}, \infty). \end{cases}$$

*Here $T$ is a generalized non-expansive mapping due to Hardy and Rogers but not a non-expansive mapping and $T$ has a unique fixed point 0.*

**Verification**. It is obvious that the mapping $T$ is discontinuous at $x = \frac{\pi}{2}$ and hence not a non-expansive mapping because non-expansive mappings are continuous. Now, we have the following cases:

**Case-I** If $x, y \in [0, \frac{\pi}{2}]$, then

$$\|Tx - Ty\| = \left\| \frac{1}{3} sinx - \frac{1}{3} siny \right\| \leq \frac{1}{3} \|x - y\| + \frac{3}{5} \left[ \left\| x - \frac{1}{3} sinx \right\| + \left\| y - \frac{1}{3} siny \right\| \right].$$

**Case-II** If $x, y \in (\frac{\pi}{2}, \infty)$, then

$$\|Tx - Ty\| = \left\| \frac{1}{6} sinx - \frac{1}{6} siny \right\| \leq \frac{1}{3} \|x - y\| + \frac{3}{5} \left[ \left\| x - \frac{1}{6} sinx \right\| + \left\| y - \frac{1}{6} siny \right\| \right].$$

**Case-III** If $x \in [0, \frac{\pi}{2}]$ and $y \in (\frac{\pi}{2}, \infty)$, then

$$\|Tx - Ty\| = \left\| \frac{1}{3} sinx - \frac{1}{6} siny \right\| \leq \frac{1}{3} \|x - y\| + \frac{3}{5} \left[ \left\| x - \frac{1}{3} sinx \right\| + \left\| y - \frac{1}{6} siny \right\| \right].$$

**Case-IV** If $x \in (\frac{\pi}{2}, \infty)$ and $y \in [0, \frac{\pi}{2}]$, then

$$\|Tx - Ty\| = \left\| \frac{1}{6} sinx - \frac{1}{3} siny \right\| \leq \frac{1}{3} \|x - y\| + \frac{3}{5} \left[ \left\| x - \frac{1}{6} sinx \right\| + \left\| y - \frac{1}{3} siny \right\| \right].$$

Hence, for $a = \frac{1}{3}$, $b = \frac{3}{5}$ and $c = 0$ $(a + 2b + 2c = \frac{14}{15} < 1)$ $T$ is a generalized non-expansive mapping.

With the help of Matlab program Software 2015a, we compute that the sequence $\{x_n\}$ defined by iterative iterative scheme (8) converges faster than some known and leading iterative schemes to a fixed point 0 of $T$ which is shown by Table 1 and Figure 1. For this, we choose control sequences in $(0, 1)$ as follows: $a_n = \frac{28n+1}{30n+2}$, $b_n = \frac{3n+1}{20n+2}$ and $c_n = \frac{9n+1}{19n+2}$ and initial guess $x_0 = 1.2$.

| Iter. | Sahu-Thakur | Mann | Ishikawa | Noor | S | Abbas |
|---|---|---|---|---|---|---|
| 1 | 1.2000000 | 1.2000000 | 1.2000000 | 1.2000000 | 1.2000000 | 1.2000000 |
| 2 | 0.2113311 | 0.3940535 | 0.3727584 | 0.3705836 | 0.2893847 | 0.2208345 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 12 | 0.0000000 | 0.0000252 | 0.0000099 | 0.0000085 | 0.0000017 | 0.0000001 |
| 13 | 0.0000000 | 0.0000096 | 0.0000034 | 0.0000029 | 0.0000005 | 0.0000000 |
| 14 | 0.0000000 | 0.0000036 | 0.0000012 | 0.0000010 | 0.0000002 | 0.0000000 |
| 15 | 0.0000000 | 0.0000014 | 0.0000004 | 0.0000003 | 0.0000000 | 0.0000000 |
| 16 | 0.0000000 | 0.0000005 | 0.0000001 | 0.0000001 | 0.0000000 | 0.0000000 |
| 17 | 0.0000000 | 0.0000002 | 0.0000001 | 0.0000000 | 0.0000000 | 0.0000000 |
| 18 | 0.0000000 | 0.0000001 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 19 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

Table 1: Comparison of speed of the convergence of different iterative schemes.

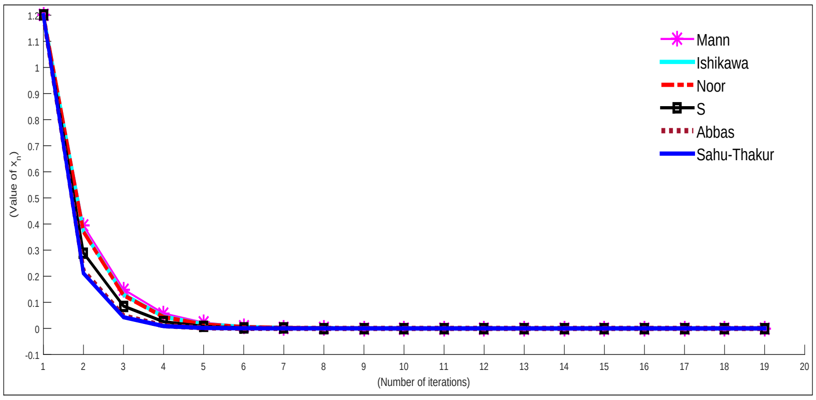


Figure 1: Convergence behavior of the sequences defined by different iterative schemes.

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