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Fixed Point on Complete $\mathcal{M}-$Metric Spaces via $F(\psi, \varphi)$-Contraction Mappings and Application to Periodic Differential Equation

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Abstract. We demonstrate the presence of a single fixed point on $\mathcal{M}-$ metric spaces for generalized $F(\psi, \varphi)$-contractions using $\alpha-$admissibility and $\mathcal{C}-$class functions. Further, we provide an answer to an open problem regarding the existence and uniqueness of a fixed point of a classical Chatterjea contraction on an $\mathcal{M}-$metric space, which is still open. Our obtained results are the extensions, unifications and generalizations of some of the well known previous results. Also, we furnish some illustrative examples and an application to validate the findings.

Key Words and Phrases: $\mathcal{M}$-metric spaces, continuity, $\mathcal{C}-$ class function, α-admissibility.

2010 Mathematics Subject Classifications: 47H10, 54H25

1. Introduction and Preliminaries

Distance is one of the earliest perceptions appreciated by human. Euclid was the first to formulate the notion of distance. Maurice René Frechét [7] considered the general and more axiomatic form of distance and named it "L-space". Felix Hausdorff [10] reviewed it as a metric space which has been refined, discussed and generalized in numerous ways. Latterely, Asadi et al. [4] exhibited the idea of an $\mathcal{M}$-metric space as an improvement of a partial metric space introduced by Mathews [13]. Acknowledging the works of Ansari [2] and Samet et al. [16], we familiarize with generalized $F(\psi, \varphi)$-contraction using $\mathcal{C}$-class functions and α-admissibility in complete $\mathcal{M}$-metric spaces to set up adequate conditions for the presence of possibly unique fixed point. Further we prove Theorem 3.3 of Asadi et al. [4], which is still an open problem and whose partial answer

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is provided in Kumrod and Sintunavarat [12] and Monfared et al. [14]. Kumrod and Sintunavarat [12] assumed additional conditions: $\sigma_m(\mathcal{T}^n\mathfrak{u}_0, \mathcal{T}^n\mathfrak{u}_0) \leq \sigma_m(\mathcal{T}^{n-1}\mathfrak{u}_0, \mathcal{T}^{n-1}\mathfrak{u}_0)$ and $\sigma_m(\mathcal{T}^{n-1}\mathfrak{u}_0, \mathcal{T}^{n-1}\mathfrak{u}_0) \leq \sigma_m(\mathcal{T}^n\mathfrak{u}_0, \mathcal{T}^n\mathfrak{u}_0)$ to give two partial answers and left two challenging open questions for further work related to using additional conditions used in providing partial answers. Monfared et al. [14] proved the result when $k \in [0, \frac{\sqrt{3}-1}{2}]$, instead of $k \in [0, \frac{1}{2})$ to prove a fixed point. However, we do not assume any additional conditions on Chatterjea contraction mappings in $\mathcal{M}$-metric spaces for proving a single fixed point of a discontinuous mapping. In the sequel, we also answered the two challenging open problems posed by Kumrod and Sintunavarat [12]. The validity of results are substantiated by suitable examples and an application in solving periodic differential equation. Motivation behind this application is significance of periodic differential equations in celestial mechanics, the theory of nonlinear oscillators or in population dynamics with seasonal effects.

Definition 1. *[4]. A distance function $\sigma_m : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ on a nonempty set $\mathcal{U}$ is an $\mathcal{M}$-metric if*

($\sigma_m 1$) $\mathfrak{u} = \mathfrak{v} \Leftrightarrow \sigma_m(\mathfrak{u},\mathfrak{u}) = \sigma_m(\mathfrak{u},\mathfrak{v}) = \sigma_m(\mathfrak{v},\mathfrak{v})$,

($\sigma_m 3$) $\sigma_m(\mathfrak{u},\mathfrak{v}) = \sigma_m(\mathfrak{v},\mathfrak{u})$,

($\sigma_m 4$) $\sigma_m(\mathfrak{u},\mathfrak{v}) - \sigma_{m_{\mathfrak{uv}}} \leq \sigma_m(\mathfrak{u},\mathfrak{w}) - \sigma_{m_{\mathfrak{uw}}} + \sigma_m(\mathfrak{w},\mathfrak{v}) - \sigma_{m_{\mathfrak{wv}}}$,
where $\sigma_{m_{\mathfrak{uv}}} = \min\{\sigma_m(\mathfrak{u},\mathfrak{u}), \sigma_m(\mathfrak{v},\mathfrak{v})\}$.

A pair $(\mathcal{U}, \sigma_m)$ is an $\mathcal{M}-$ metric space.

We denote $\sigma_{M_{\mathfrak{uv}}} = \max\{\sigma_m(\mathfrak{u},\mathfrak{u}), \sigma_m(\mathfrak{v},\mathfrak{v})\}$. It is interesting to see here that $\sigma_{m_{\mathfrak{uu}}} = \sigma_{M_{\mathfrak{uu}}} = \sigma_m(\mathfrak{u},\mathfrak{u}), \mathfrak{u},\mathfrak{v} \in \mathcal{U}$. Further, one may notice that ($\sigma_m 1$) may be obtained from ($\sigma_m 4$) and metric $(d) \Longrightarrow$ partial metric $(\rho) \Longrightarrow \mathcal{M}-$metric (σ_m). However reverse implication is not applicable.

Remark 1. *[4] 1.* $0 \leq \sigma_{M_{\mathfrak{uv}}} + \sigma_{m_{\mathfrak{uv}}} = \sigma_m(\mathfrak{u},\mathfrak{u}) + \sigma_m(\mathfrak{v},\mathfrak{u})$, *2.* $0 \leq \sigma_{M_{\mathfrak{uv}}} - \sigma_{m_{\mathfrak{uv}}} = |\sigma_m(\mathfrak{u},\mathfrak{u}) - \sigma_m(\mathfrak{v},\mathfrak{v})|$, *3.* $\sigma_{M_{\mathfrak{uv}}} - \sigma_{m_{\mathfrak{uv}}} \leq \sigma_{M_{\mathfrak{uw}}} - \sigma_{m_{\mathfrak{uw}}} + \sigma_{M_{\mathfrak{wv}}} - \sigma_{m_{\mathfrak{wv}}}, \mathfrak{u},\mathfrak{v} \in \mathcal{U}$.

Example 1. *Let $\mathcal{U} = [0,\infty)$. Then $\sigma_m(\mathfrak{u},\mathfrak{v}) = \frac{\mathfrak{u}+\mathfrak{v}}{2}$ on $\mathcal{U}$ is an $\mathcal{M}$-metric.*

Observe that it is not a partial metric as $\sigma_m(5,5) = 5 \not\leq \sigma_m(6,2) = 4$.

Example 2. *Let $\mathcal{U} = \{1,3,5\}$. Define $\sigma_m(1,1) = 1,\ \sigma_m(3,3) = 10,\ \sigma_m(5,5) = 12, \sigma_m(1,3) = \sigma_m(3,1) = 8,\ \sigma_m(1,5) = \sigma_m(5,1) = 6,\ \sigma_m(3,5) = \sigma_m(5,3) = 12$. So σ_m is an $\mathcal{M}$-metric, however it is not a partial metric as $\sigma_m(5,5) = 12 \not\leq \sigma_m(1,5) = 6$.*

Remark 2. *[4]. Each $\mathcal{M}$-metric σ_m on $\mathcal{U}$ give rise to a $\mathcal{T}_0$ topology τ_m on $\mathcal{U}$. The set $\{B_{\sigma_m}(\mathfrak{u},\epsilon) : \mathfrak{u} \in \mathcal{U}, \epsilon > 0\}$, where $B_{\sigma_m}(\mathfrak{u},\epsilon) = \{\mathfrak{v} \in \mathcal{U} : \sigma_m(\mathfrak{u},\mathfrak{v}) < \sigma_{m_{\mathfrak{u},\mathfrak{v}}} + \epsilon\}$, forms the base of τ_{σ_m}.*

Definition 2. *[4]. In an $\mathcal{M}$-metric space $(\mathcal{U}, \sigma_m)$, for $m, n \in \mathbb{N}$.*

1. *A sequence converges to $\mathfrak{u} \in \mathcal{U}$ iff $\lim_{n\to\infty}\left(\sigma_m(\mathfrak{u}_n,\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}}}\right) = 0$.*
2. *A sequence $\{\mathfrak{u}_n\}$ is σ_m-Cauchy if $\lim_{n,m\to\infty}\left(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_m) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}_m}}\right)$ and $\lim_{n,m\to\infty}\left(\sigma_{M(\mathfrak{u}_n,\mathfrak{u}_m)} - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}_m}}\right)$ exist (finitely).*
3. *$(\mathcal{U}, \sigma_m)$ is complete if each σ_m-Cauchy sequence $\{\mathfrak{u}_n\}$ in $\mathcal{U}$ converges with respect to τ_{σ_m} to $\mathfrak{u} \in \mathcal{U}$, i.e., $\lim_{n,m\to\infty}\left(\sigma_m(\mathfrak{u}_n,\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}}}\right) = 0$ and $\lim_{n,m\to\infty}\left(\sigma_{M(\mathfrak{u}_n,\mathfrak{u})} - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}}}\right) = 0$.*

Lemma 1. *[4]. Let $(\mathcal{U}, \mathcal{M})$ be an $\mathcal{M}$-metric space. Assume that $\mathfrak{u}_n \to \mathfrak{u}$ and $\mathfrak{v}_n \to \mathfrak{v}$ as $n \to \infty, \mathfrak{u}, \mathfrak{v} \in \mathcal{U}$ and $n \in \mathbb{N}$. Then we have*

1. *$\lim_{n\to\infty}\left(\sigma_m(\mathfrak{u}_n,\mathfrak{v}_n) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{v}_n}}\right) = \sigma_m(\mathfrak{u},\mathfrak{v}) - \sigma_{m_{\mathfrak{u}\mathfrak{v}}}$.*
2. *$\lim_{n\to\infty}\left(\sigma_m(\mathfrak{u}_n,\mathfrak{v}) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{v}}}\right) = \sigma_m(\mathfrak{u},\mathfrak{v}) - \sigma_{m_{\mathfrak{u}\mathfrak{v}}}$.*
3. *$\exists\, r \in [0, 1)$, $\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n) \le r\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})$, then (i) $\lim_{n\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1}) = 0$. (ii) $\lim_{n\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n) = 0$. (iii) $\lim_{m,n\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) = 0$. (iv) $\{\mathfrak{u}_n\}$ is an σ_m-Cauchy sequence.*
4. *$\sigma_m(\mathfrak{u},\mathfrak{v}) = \sigma_{m_{\mathfrak{u}\mathfrak{v}}}$. Further if $\sigma_m(\mathfrak{u},\mathfrak{u}) = \sigma_m(\mathfrak{v},\mathfrak{v})$, then $\mathfrak{u} = \mathfrak{v}$.*

Definition 3. *[16]. A function $f : \mathcal{U} \to \mathcal{U}$ is called an α-admissible if there exists a function $\alpha : \mathcal{U} \times \mathcal{U} \to (0,\infty)$ so that $\alpha(\mathfrak{u},\mathfrak{v}) \ge 1$ implies $\alpha(f\mathfrak{u}, f\mathfrak{v}) \ge 1$, $\mathfrak{u}, \mathfrak{v} \in \mathcal{U}$.*

Example 3. *Let $\mathcal{U} = \{A, B, C, D, E\} \subseteq \mathbb{R}^2$, where $A = (0, 0)$, $B = (1, 0)$, $C = (1, 2)$, $D = (1, 3)$, $E = (1, 4)$. Let $f : \mathcal{U} \to \mathcal{U}$ be defined as $f\mathfrak{u} = \begin{cases} C, & \text{if } \mathfrak{u} \in \mathcal{U}\backslash\{E\} \\ D, & \text{if } \mathfrak{u} = E \end{cases}$. Let $\alpha : \mathcal{U} \times \mathcal{U} \to (0,\infty)$ be given by $\alpha(\mathfrak{u},\mathfrak{v}) = \begin{cases} 1, & \text{if } \mathfrak{u},\ \mathfrak{v} \in \mathcal{U}\backslash\{E\} \\ \frac{3}{2}, & \text{if } \mathfrak{u} = E \end{cases}$. If $\mathfrak{u} \in \mathcal{U}\backslash\{E\}$, then $\alpha(f\mathfrak{u}, f\mathfrak{v}) = \alpha(C, C) = 1$ and if $\mathfrak{u} = E$ then $\alpha(f\mathfrak{u}, f\mathfrak{v}) = \alpha(D, D) = 1$, $\mathfrak{u}$, $\mathfrak{v} \in \mathcal{U}$. Using Definition 3, f is an α-admissible.*

Example 4. *Let $\mathcal{U} = [0, \frac{\pi}{2}]$ be the subset of $\mathbb{R}$. Define $f : \mathcal{U} \to \mathcal{U}$ so that $f\mathfrak{u} = sin\mathfrak{u}$. If an $\alpha : \mathcal{U} \times \mathcal{U} \to (0,\infty)$ is given by $\alpha(\mathfrak{u},\mathfrak{v}) = e^{\mathfrak{u}+\mathfrak{v}}$ and $\alpha(\mathfrak{u},\mathfrak{v}) = e^{\mathfrak{u}+\mathfrak{v}} \ge 1, \mathfrak{u},\ \mathfrak{v} \in \mathcal{U}$. Then we have $\alpha(f\mathfrak{u}, f\mathfrak{v}) = e^{f\mathfrak{u}+f\mathfrak{v}} = e^{sin\mathfrak{u}+sin\mathfrak{v}} \ge 1$. Therefore all the conditions of Definition 3 are fulfilled. So, $f : \mathcal{U} \to \mathcal{U}$ is an α-admissible.*

Definition 4. *[11] Let Ψ be the set of functions $\psi : [0,\infty) \to [0,\infty)$ so that (i) ψ is lower semicontinuous and strictly increasing. (ii) $\psi(t) = 0$ iff $t = 0$.*

Definition 5. *[2]. An ultra altering distance function is a nondecreasing and continuous mapping $\varphi : [0,\infty) \to [0,\infty)$ so that $\varphi(t) > 0$, for all $t > 0$. For instance, $\varphi(t) = kt, k \in (0,1)$. Φ is the class of the ultra altering distance functions.*

Definition 6. *[2]. A continuous mapping $\mathcal{F} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a $\mathcal{C}$-class function if*
(i) $\mathcal{F}(\mathfrak{u},\mathfrak{v}) \leq \omega$, (ii) $\mathcal{F}(\mathfrak{u},\mathfrak{v}) = \mathfrak{u}$ implies that either $\mathfrak{u} = 0$ or $\mathfrak{v} = 0$, $\mathfrak{u},\mathfrak{v} \in [0,\infty)$.

Example 5. *[2]. Suppose that $\mathcal{F} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a C-class function. Then following are elements of $\mathcal{C}$: (i) $\mathcal{F}(\mathfrak{u},\mathfrak{v}) = \mathfrak{u} - \mathfrak{v}$, (ii) $\mathcal{F}(\mathfrak{u},\mathfrak{v}) = \mu\mathfrak{u}$, $0 < \mu < 1$, (iii) $\mathcal{F}(\mathfrak{u},\mathfrak{v}) = \frac{\mathfrak{u}}{(1+\mathfrak{v})^{\mathfrak{s}}}$, $\mathfrak{s} \in (0,\infty)$, (iv) $F(\mathfrak{u},\mathfrak{v}) = \mathfrak{u}\gamma(\mathfrak{u})$, $\gamma : [0,\infty) \to (0,1)$ is a continuous function,*
(v) $\mathcal{F}(\mathfrak{u},\mathfrak{v}) = \mathfrak{u} - \left(\frac{2+\mathfrak{u}}{1+\mathfrak{v}}\right)\mathfrak{v}$, (vi) $F(\mathfrak{u},\mathfrak{v}) = \sqrt[n]{\ln(1+\mathfrak{u}^n)}$.

2. Main Results

First we familiarize the class of generalized $F(\psi,\varphi)-$contractions using $\alpha-$admissibility and $\mathcal{C}-$class functions to investigate the fixed points on $\mathcal{M}-$metric spaces.

Theorem 1. *Let $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ be a continuous mapping of a complete $\mathcal{M}$-metric space $(\mathcal{U},\sigma_m)$ satisfying*

$$\begin{aligned}\big(\psi\big(\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v})\big)+l\big)^{\alpha(\mathfrak{u},\mathcal{T}\mathfrak{u})\alpha(\mathfrak{v},\mathcal{T}\mathfrak{v})} &\leq \Big[a_1F(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{v}))) + a_2F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})))\\ &\quad + a_3F(\psi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v}))) + a_4F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})))\\ &\quad + a_5F(\psi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})))\Big]^{\beta(\mathfrak{u},\mathcal{T}\mathfrak{u})\beta(\mathfrak{v},\mathcal{T}\mathfrak{v})} + l, \qquad (1)\end{aligned}$$

$\mathfrak{u}$, $\mathfrak{v} \in \mathcal{U}$, $1 \leq l \in \mathbb{R}$, a_i ($i = 1$ to 5) are nonnegative constants with $\sum_{i=1}^{5} a_i < 1$ and $\beta : [0,\infty) \times [0,\infty) \to [0,k]$ a function such that $k \in (0,1)$. We also assume that
***(i)** $\mathcal{T}$ is α-admissible, **(ii)** there exists a $\mathfrak{u}_0 \in \mathcal{U}$ so that $\alpha(\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0) \geq 1$.*
Then $\mathcal{T}$ has a unique fixed point in $\mathcal{U}$.

Proof. Consider $\mathfrak{u}_0 \in \mathcal{U}$ so that $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$. Construct a sequence $\{\mathfrak{u}_n\}$ so that $\mathfrak{u}_n = \mathcal{T}\mathfrak{u}_{n-1}$, $n \in \mathbb{N}$. Since $\alpha(\mathfrak{u}_0, \mathfrak{u}_1) = \alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$ and $\alpha(\mathfrak{u}_1, \mathfrak{u}_2) = \alpha(\mathcal{T}\mathfrak{u}_0, \mathcal{T}\mathcal{T}\mathfrak{u}_0) \geq 1$. Continuing these steps, we obtain $\alpha(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \geq 1$, $n \in \mathbb{N} \cup \{0\}$. If for some n, $\sigma_b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0$, then $\mathfrak{u}_n$ is a fixed point of $\mathcal{T}$. Let $\mathfrak{u}_n \neq \mathfrak{u}_{n+1}$, $n \in \mathbb{N} \cup \{0\}$. Applying (1) for $\mathfrak{u} = \mathfrak{u}_{n-1}$ and $\mathfrak{v} = \mathfrak{u}_n$,

$$
\begin{aligned}
&\big(\psi(\sigma_m(\mathcal{T}\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_n) + l)\big) \leq \big(\psi(\sigma_m(\mathcal{T}\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_n)) + l\big)^{\alpha(\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_{n-1})\alpha(\mathfrak{u}_n, \mathcal{T}\mathfrak{u}_n)} \\
&\leq \Bigg[a_1 F(\psi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n)), \varphi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n))) \\
&\qquad + a_2 F(\psi(\sigma_m(\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_{n-1})), \varphi(m(\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_{n-1}))) \\
&\qquad + a_3 F(\psi(\sigma_m(\mathfrak{u}_n, \mathcal{T}\mathfrak{u}_n)), \varphi(m(\mathfrak{u}_n, \mathcal{T}\mathfrak{u}_n))) \\
&\qquad + a_4 F(\psi(\sigma_m(\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_n)), \varphi(\sigma_m(\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_n))) \\
&\qquad + a_5 F(\psi(\sigma_m(\mathfrak{u}_n, \mathcal{T}\mathfrak{u}_{n-1})), \varphi(\sigma_m(\mathfrak{u}_n, \mathcal{T}\mathfrak{u}_{n-1})))\Bigg]^{\beta(\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_{n-1})\beta(\mathfrak{u}_n, \mathcal{T}\mathfrak{u}_n)} + l \\
&= \Bigg[a_1 F(\psi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n)), \varphi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n))) \\
&\qquad + a_2 F(\psi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n)), \varphi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n))) \\
&\qquad + a_3 F(\psi(\sigma_m(\mathfrak{u}_n, \mathfrak{u}_{n+1})), \varphi(\sigma_m(\mathfrak{u}_n, \mathfrak{u}_{n+1}))) \\
&\qquad + a_4 F(\psi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_{n+1})), \varphi(\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_{n+1}))) \\
&\qquad + a_5 F(\psi(\sigma_m(\mathfrak{u}_n, \mathfrak{u}_n)), \varphi(\sigma_m(\mathfrak{u}_n, \mathfrak{u}_n)))\Bigg]^{\beta(\mathfrak{u}_{n-1}, \mathfrak{u}_n)\beta(\mathfrak{u}_n, \mathfrak{u}_{n-1})} + l. \qquad (2)
\end{aligned}
$$

Let us assume that $\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_{n+1}) = A_{\mathfrak{u}\mathfrak{v}}$. Therefore, we have

$$
\begin{aligned}
A_{\mathfrak{u}\mathfrak{v}} &= \sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_{n+1}) = \sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_{n+1}) - \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}} + \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}} \\
&\leq \sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n) - \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_n}} + \sigma_m(\mathfrak{u}_n, \mathfrak{u}_{n+1}) - \sigma_{m_{\mathfrak{u}_n\mathfrak{u}_{n+1}}} + \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}}. \qquad (3)
\end{aligned}
$$

Since, $\sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}} \leq \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_n}} + \sigma_{m_{\mathfrak{u}_n\mathfrak{u}_{n+1}}} - \sigma_{m_{\mathfrak{u}_n\mathfrak{u}_n}}$. Then (3) implies, $\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_{n+1}) \leq \sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n, \mathfrak{u}_{n+1}) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}_n}}$. Using Definition 1 ($\sigma_m 2$) in above inequality, we have

$$\sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_{n+1}) \leq \sigma_m(\mathfrak{u}_{n-1}, \mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n, \mathfrak{u}_{n+1}) - \sigma_m(\mathfrak{u}_n, \mathfrak{u}_n). \qquad (4)$$

Using (2) and (4),

$$\big(\psi(\sigma_m(\mathfrak{u}_n, \mathfrak{u}_{n+1})) + l\big) = \big(\psi(\sigma_m(\mathcal{T}\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_n)) + l\big)$$

$$\leq \big(\psi(\sigma_m(\mathcal{T}\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_n)) + l\big)^{\alpha(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_{n-1})\alpha(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_n)}$$

$$\leq \Bigg[a_1F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)))$$

$$+a_2F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)))$$

$$+a_3F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}))) + a_4F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)$$

$$+\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) - \sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})$$

$$-\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))) + a_5F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)))\Bigg]^{\beta(\mathfrak{u}_{n-1},\mathfrak{u}_n)\beta(\mathfrak{u}_n,\mathfrak{u}_{n-1})},$$

i.e.

$$\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) \leq \Bigg[a_1\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)) + a_2\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n))$$

$$+a_3\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) + a_4\big(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) - \sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))\big)$$

$$+a_5\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))\Bigg]^{\beta(\mathfrak{u}_{n-1},\mathfrak{u}_n)\beta(\mathfrak{u}_n,\mathfrak{u}_{n-1})} = \Bigg[(a_1+a_2+a_4)\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n))+$$

$$(a_3+a_4)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) + (a_5-a_4)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))\Bigg]^{\beta(\mathfrak{u}_{n-1},\mathfrak{u}_n)\beta(\mathfrak{u}_n,\mathfrak{u}_{n-1})}. \tag{5}$$

Since, $\beta : [0,\infty) \to [0,k]$ is a function, where $k \in (0,1)$. Then (5) becomes

$$\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) = \Bigg[(a_1+a_2+a_4)\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n))$$

$$+(a_3+a_4)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) + (a_5-a_4)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))\Bigg]. \tag{6}$$

Similarly, applying (1) with $\mathfrak{u} = \mathfrak{u}_{n+1}$ and $\mathfrak{v} = \mathfrak{u}_n$, we obtain

$$\big(\psi(\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n)) + l\big) = \big(\psi(\sigma_m(\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_{n-1})) + l\big)$$
$$\leq \big(\psi(\sigma_m(\mathcal{T}v_n,\mathcal{T}\mathfrak{u}_{n-1})) + l\big)^{\alpha(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_{n-1})\alpha(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_n)}$$

$$\leq \Bigg[a_1F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})))$$
$$+ a_2F(\psi(\sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_n)))$$
$$+ a_3F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_{n-1})),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_{n-1})))$$
$$+ a_4F(\psi(\sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_{n-1})),\varphi(\sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_{n-1})))$$
$$+ a_5F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_n)))\Bigg]^{\beta(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_n)\beta(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_{n-1})} + l$$
$$= \Bigg[a_1F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})))$$
$$+ a_2F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})))$$
$$+ a_3F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)))$$
$$+ a_4F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)))$$
$$+ a_5F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1})),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1})))\Bigg]^{\beta(\mathfrak{u}_n,\mathfrak{u}_{n-1})\beta(\mathfrak{u}_{n-1},\mathfrak{u}_n)} + l \tag{7}$$

Let $\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1}) = B_{\mathfrak{u}\mathfrak{v}}$, we have

$$B_{\mathfrak{u}\mathfrak{v}} = \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1}) = \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1}) - \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}} + \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}}$$
$$\leq \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) - \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_n}} + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) - \sigma_{m_{\mathfrak{u}_n\mathfrak{u}_{n+1}}} + \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}}. \tag{8}$$

Since, $\sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n+1}}} \leq \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_n}} + \sigma_{m_{\mathfrak{u}_n\mathfrak{u}_{n+1}}} - \sigma_{m_{\mathfrak{u}_n\mathfrak{u}_n}}$. Therefore (8) becomes

$$\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1}) \leq \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}_n}}.$$

Using Definition 1 (σ_m2) in above inequality, we obtain

$$\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1}) \leq \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) - \sigma_m(\mathfrak{u}_n,\mathfrak{u}_n). \tag{9}$$

Using (7) and (9),

$$\psi(\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n)) \leq \Bigg[a_1F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})))$$
$$+ a_2F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})))$$
$$+ a_3F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)))$$

$$+ a_5F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) - \sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)), \tag{10}$$

$$\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) - \sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)))$$

$$\left. + a_4F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)))\right]^{\beta(\mathfrak{u}_n,\mathfrak{u}_{n-1})\beta(\mathfrak{u}_{n-1},\mathfrak{u}_n)}, \tag{11}$$

i.e.,

$$\psi(\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n)) \le \Big[a_1\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1}))$$

$$+ a_2\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) + a_3\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)) + a_4\big(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)$$

$$\left. + \sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n) - \sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))\big) + a_5\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n))\right]^{\beta(\mathfrak{u}_n,\mathfrak{u}_{n-1})\beta(\mathfrak{u}_{n-1},\mathfrak{u}_n)}$$

$$= \Big[(a_1 + a_3 + a_5)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})) + (a_2 + a_5)\psi(\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n))$$

$$\left. + (a_4 - a_5)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))\right]^{\beta(\mathfrak{u}_n,\mathfrak{u}_{n-1})\beta(\mathfrak{u}_{n-1},\mathfrak{u}_n)}. \tag{12}$$

Since $\beta : [0,\infty) \to [0,k]$ is a function, where $k \in (0,1)$

$$\psi(\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n)) = \Big[(a_1 + a_3 + a_5)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1}))$$

$$+(a_2 + a_5)\psi(\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_n)) + (a_4 - a_5)\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n))\Big]. \tag{13}$$

On adding (6) and (11), we obtain

$$\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) \le \lambda F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)) \le \lambda\psi(m(\mathfrak{u}_{n-1},\mathfrak{u}_n)), \tag{14}$$

with $0 \le \lambda = \frac{2a_1+a_2+a_3+a_4+a_5}{2-(a_2-a_3+a_4+a_5)} < 1$. Now, we assert that $\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) \to 0$ as $n \to \infty$. If for some $n_0 \in \mathbb{N}$, $\mathfrak{u}_{n_0} = \mathfrak{u}_{n_0+1}$, then by (12)

$$0 \le \sigma_m(\mathfrak{u}_{n_0+1},\mathfrak{u}_{n_0+2})$$

$$\le F\big(\psi(\sigma_m(\mathfrak{u}_{n_0},\mathfrak{u}_{n_0+1})),\varphi(\sigma_m(\mathfrak{u}_{n_0},\mathfrak{u}_{n_0+1}))\big) \le \psi\big(\sigma_m(\mathfrak{u}_{n_0},\mathfrak{u}_{n_0+1})\big),$$

i.e., $\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) = 0$, $n \ge n_0$. Thus $\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) \to 0$ as $n \to \infty$. Now let $\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) > 0$, $n \in \mathbb{N}$. Inequality (12) gives that $\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})) \le$

$\lambda\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n))$. It follows that sequence $\{\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})\}$ is decreasing. So, there exists $m\in\mathbb{R}$ and $\lim_{n\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})=m$. Now we will assert that $m=0$. If possible, let $m>0$. Using (12),

$$\begin{aligned}\lim_{n,m\to\infty}\sup\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}))&\le\lambda\lim_{n,m\to\infty}\sup F(\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)),\varphi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n))\\&\le\lambda\lim_{n,m\to\infty}\sup\psi(\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n)).\end{aligned}$$

Hence we get, $\psi(m)\le\lambda F\big(\psi(m),\varphi(m)\big)\le\lambda\psi(m)$. Since $\lambda\in(0,1)$. So $\psi(m)=0$ or $\varphi(m)=0$, i.e., $m=0$, a contradiction. So

$$\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})\to0\ \text{ as }\ n\to\infty.\tag{15}$$

Next we submit that $\{\mathfrak{u}_n\}$ is a σ_m-Cauchy sequence. We have, $\lim_{n\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})=0$, $0\le\sigma_{m_{\mathfrak{u}_n\mathfrak{u}_{n+1}}}\le\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1})\Rightarrow\lim_{n\to\infty}\sigma_{m_{\mathfrak{u}_n\mathfrak{u}_{n+1}}}=0$ and $\sigma_{m_{\mathfrak{u}_n\mathfrak{u}_{n+1}}}=\min\{\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n),\sigma_m(\mathfrak{u}_{n+1},\mathfrak{u}_{n+1})\}\Rightarrow\lim_{n\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n)=0$. Also $\sigma_{m_{\mathfrak{u}_n\mathfrak{u}_m}}=\min\{\sigma_m(\mathfrak{u}_n,\mathfrak{u}_n),\sigma_m(\mathfrak{u}_m,\mathfrak{u}_m)\}\Rightarrow\lim_{n,m\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}_m)=0$. Hence, $\lim_{n,m\to\infty}(\sigma_{m_{\mathfrak{u}_n,\mathfrak{u}_m}}-\sigma_{m_{\mathfrak{u}_n,\mathfrak{u}_m}})=0$. We show that $\lim_{n,m\to\infty}(\sigma_m(\mathfrak{u}_n,\mathfrak{u}_m)-\sigma_{m_{\mathfrak{u}_n,\mathfrak{u}_m}})=0$. Suppose $\sigma_{M^*}(\mathfrak{u},\mathfrak{v})=\sigma_m(\mathfrak{u},\mathfrak{v})-\sigma_{m_{\mathfrak{u}\mathfrak{v}}},\ \forall\,\mathfrak{u},\,\mathfrak{v}\in\mathcal{U}$. If $\lim_{n,m\to\infty}\sigma_{M^*}(\mathfrak{u}_n,\mathfrak{u}_m)\ne0$, there exists $\epsilon>0$ and $\{l_k\}\subset\mathbb{N}$, $\sigma_{M^*}(\mathfrak{u}_{l_k-1},\mathfrak{u}_{n_k})<\epsilon$. Now from Definition 1 (σ_m4), we get, $\epsilon\le\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{n_k})\le\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{l_k-1})+\sigma_{M^*}(\mathfrak{u}_{l_k-1},\mathfrak{u}_{n_k})+\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{l_k-1})+\epsilon$. Therefore $\lim_{k\to\infty}\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{n_k})=\epsilon$, which means that $\lim_{k\to\infty}\big(\sigma_m(\mathfrak{u}_{l_k},\mathfrak{u}_{n_k})-\sigma_{m_{\mathfrak{u}_{l_k},\mathfrak{u}_{n_k}}}\big)=\epsilon$. On the other hand $\lim_{k\to\infty}\sigma_{m_{\mathfrak{u}_{l_k},\mathfrak{u}_{n_k}}}=0$, so we have

$$\lim_{k\to\infty}\sigma_m(\mathfrak{u}_{l_k},\mathfrak{u}_{n_k})=\epsilon.\tag{16}$$

Again by Definition 1 (σ_m4), we get

$$\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{n_k})\le\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{l_k+1})+\sigma_{M^*}(\mathfrak{u}_{l_k+1},\mathfrak{u}_{n_k+1})+\sigma_{M^*}(\mathfrak{u}_{n_k+1},\mathfrak{u}_{n_k})$$

and

$$\sigma_{M^*}(\mathfrak{u}_{l_k+1},\mathfrak{u}_{n_k+1})\le\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{l_k+1})+\sigma_{M^*}(\mathfrak{u}_{l_k},\mathfrak{u}_{n_k})+\sigma_{M^*}(\mathfrak{u}_{n_k+1},\mathfrak{u}_{n_k}),\quad\text{as }k\to\infty,$$

together with (13) and (14), we have

$$\lim_{k\to\infty}\sigma_m(\mathfrak{u}_{l_k+1},\mathfrak{u}_{n_k+1})=\epsilon.\tag{17}$$

Using (13), (14) and (15), we get

$$\begin{aligned}&\big(\psi\big(\sigma_m(\mathfrak{u}_{m_k+1},\mathfrak{u}_{n_k+1})\big)+l\big)\le\big(\psi\big(\sigma_m(\mathcal{T}\mathfrak{u}_{m_k},\mathcal{T}\mathfrak{u}_{n_k})\big)+l\big)^{\alpha(\mathfrak{u}_{m_k},\mathcal{T}\mathfrak{u}_{m_k})\alpha(\mathfrak{u}_{n_k},\mathcal{T}\mathfrak{u}_{n_k})}\\&\quad\le\lambda F(\psi(\sigma_m(\mathfrak{u}_{m_k},\mathfrak{u}_{n_k})),\varphi(\sigma_m(\mathfrak{u}_{m_k},\mathfrak{u}_{n_k}))+l\le\lambda\psi(\sigma_m(\mathfrak{u}_{m_k},\mathfrak{u}_{n_k}))+l.\end{aligned}$$

Therefore, $\psi\big(\sigma_m(\mathfrak{u}_{m_k+1},\mathfrak{u}_{n_k+1})\big) \le \lambda F(\psi(\sigma_m(\mathfrak{u}_{m_k},\mathfrak{u}_{n_k})),\varphi(\sigma_m(\mathfrak{u}_{m_k},\mathfrak{u}_{n_k})) \le \lambda\psi(\sigma_m(\mathfrak{u}_{m_k},\mathfrak{u}_{n_k}))$, where $\lambda \in (0,1)$. As $k \to \infty$, $\psi(\epsilon) \le F\big(\psi(\epsilon),\varphi(\epsilon)\big) \le \psi(\epsilon)$. Using the property of F, φ and φ, $\psi(\epsilon) = 0$ or $\varphi(\epsilon) = 0$, i.e., $\epsilon = 0$, a contradiction. So, $\{\mathfrak{u}_n\}$ is a σ_m-Cauchy sequence.
Using completeness of $\mathcal{U}$, $\mathfrak{u}_n \to \mathfrak{u}$, $\mathfrak{u} \in \mathcal{U}$ in the τ_m topology, i.e.,

$$\lim_{n\to\infty}\big(\sigma_m(\mathfrak{u}_n,\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_n\mathfrak{u}}}\big) = 0 \text{ and } \lim_{n\to\infty}\big(\sigma_{M_{\mathfrak{u}_n,\mathfrak{u}}} - \sigma_{m_{\mathfrak{u}_n\mathfrak{u}}}\big) = 0.$$

However, $\lim_{n\to\infty}\sigma_{m_{\mathfrak{u}_n\mathfrak{u}}} = 0$. Hence $\lim_{n\to\infty}\sigma_m(\mathfrak{u}_n,\mathfrak{u}) = 0$ and by Remark 1, $\sigma_m(\mathfrak{u},\mathfrak{u}) = 0$.
(a) By using continuity of $\mathcal{T}$, $\lim_{n\to\infty}\big(\sigma_m(\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u}) - m_{\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u}}\big) = 0$,
i.e., $\lim_{n\to\infty}\big(\sigma_m(\mathfrak{u}_{n+1},\mathcal{T}\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_{n+1},\mathcal{T}\mathfrak{u}}}\big) = 0$. Also, $\lim_{n\to\infty}\sigma_{m_{\mathfrak{u}_{n+1}\mathcal{T}\mathfrak{u}}} = 0$. Hence, $\lim_{n\to\infty}\sigma_m(\mathfrak{u}_{n+1},\mathcal{T}\mathfrak{u}) = 0$ and by Remark 1, $\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}) = 0$. Also $\mathfrak{u}_n \to \mathfrak{u}$ as $n \to \infty$. Therefore by Lemma 1, we obtain

$$\big(\sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_n,\mathcal{T}\mathfrak{u}}}\big) \to \big(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) - \sigma_{m_{\mathfrak{u},\mathfrak{u}_n}}\big) = \sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}), \quad n \in \mathbb{N}.$$

But $\big(\sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_n,\mathcal{T}\mathfrak{u}}}\big) \to 0$. Thus $\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) = 0$.
Therefore $\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) = \sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}) = \sigma_m(\mathfrak{u},\mathfrak{u}) = 0$ and by Definition 1 ($\sigma_m 1$), we get $\mathcal{T}\mathfrak{u} = \mathfrak{u}$.

(b) By using conditions (i), (ii) and (12)

$$\begin{aligned}\big(\psi\big(\sigma_m(\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u})\big) + l\big) &\le \big(\psi\big(\sigma_m(\mathcal{T}\mathfrak{u}_n,T\mathfrak{u})\big) + l\big)^{\alpha(\mathfrak{u}_n,\mathcal{T}\mathfrak{u}_n)\alpha(\mathfrak{u},\mathcal{T}\mathfrak{u})}\\ &\le \lambda F(\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u})) + l \le \lambda\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u})) + l,\end{aligned}$$

where $\lambda \in (0,1)$, i.e., $\psi\big(\sigma_m(\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u})\big) \le \lambda F(\psi(\sigma_m(u_n,\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u}_n,\mathfrak{u})) \le \lambda\psi(\sigma_m(\mathfrak{u}_n,\mathfrak{u}))$, so we get $\sigma_m(\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u}) \to 0$ as $n \to \infty$. Also $0 \le \sigma_{m_{\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u}}} \le \sigma_m(\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u}) \to 0$ as $n \to \infty$. Thus $\mathcal{T}\mathfrak{u}_n \to \mathcal{T}\mathfrak{u}$ in the τ_m topology. Hence $\mathcal{T}\mathfrak{u} = \mathfrak{u}$ follows as in (a).

Next we assert that the fixed point $\mathfrak{u}$ is unique. Let $\mathfrak{v}$ be also a fixed point of $\mathcal{T}$. Using (1), we get

$$\begin{aligned}&\big(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})) + l\big) \le \big(\psi(\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v})) + l\big) \le \big(\psi(\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v})) + l\big)^{\alpha(\mathfrak{u},\mathcal{T}\mathfrak{u})\alpha(\mathfrak{v},\mathcal{T}\mathfrak{v})}\\ &\le \Big[a_1F(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{v}))) + a_2F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})))\\ &+a_3F(\psi(m(\mathfrak{v},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v}))) + a_4F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})))\\ &+a_5F(\psi(m(\mathfrak{v},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})))\Big]^{\beta(\mathfrak{u},\mathcal{T}\mathfrak{u})\beta(\mathfrak{v},\mathcal{T}\mathfrak{v})} + l\end{aligned}$$

$$= \Big[a_1 F(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{v}))) + a_2 F(\psi(\sigma_m(\mathfrak{u},\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{u}))) + a_3 F(\psi(m(\mathfrak{v},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{v},\mathfrak{v}))) + a_4 F(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{v}))) + a_5 F(\psi(m(\mathfrak{v},\mathfrak{u})),\varphi(\sigma_m(\mathfrak{v},\mathfrak{u})))\Big]^{\beta(\mathfrak{u},\mathfrak{u})\beta(\mathfrak{v},\mathfrak{v})} + l,$$

i.e., $\psi\big(\sigma_m(\mathfrak{u},\mathfrak{v})\big) \leq \lambda\psi\big(\sigma_m(\mathfrak{u},\mathfrak{v})\big)$, where $\lambda \in (0,1)$, a contradiction. Hence $\mathfrak{u} = \mathfrak{v}$. ◀

Following example is given to appreciate the effectiveness of the $\mathcal{M}-$metric space and to validate the result proved herein. Here, it is fascinating to point out that this example can not be used in the context a partial metric space and consequently, Theorem 1 is a genuine extension and improvement using $\mathcal{C}-$class function that envelops a huge class of contractions and $\alpha-$admissible mappings in an $\mathcal{M}-$metric space.

Example 6. *Let $\mathcal{U} = \{1,2,3\}$ and usual $\mathcal{M}$-metric space be defined as $\sigma_m(1,1) = 6, \sigma_m(2,2) = 1, \sigma_m(3,3) = 1, \sigma_m(1,2) = \sigma_m(2,1) = 3, \sigma_m(1,3) = \sigma_m(3,1) = 5, \sigma_m(2,3) = \sigma_m(3,2) = 1$. Let $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ be defined as $\mathcal{T}\mathfrak{u} = \mathfrak{u}$, $\mathfrak{u} \in \mathcal{U}$. Next, we define a function $\alpha : \mathcal{U} \times \mathcal{U} \to (0,\infty)$ as $\alpha(\mathfrak{u},\mathfrak{v}) = \begin{cases} 1, & \text{if } \mathfrak{u},\ \mathfrak{v} \in \mathcal{U} \\ 0, & \text{otherwise,} \end{cases}$ $\varphi,\ \psi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t$, $\psi(t) = \frac{t}{2}$, $l = 1$ and $\beta : [0,\infty)\times[0,\infty) \to [0,k]$ be a function such that for $k \in (0,1)$, $\beta(\mathfrak{u},\mathfrak{v}) = \frac{1}{2}$. Taking $\mathfrak{u} = 2$, $\mathfrak{v} = 3$, $a_1 = a_2 = a_3 = a_4 = \frac{1}{5}$ and $a_5 = \frac{1}{10}$ in (1), we get*

$$\big[\psi(\sigma_m(2,3)+1)\big]^{\alpha(2,2)\alpha(3,3)}$$

$$\leq \Big[a_1 F(\psi(\sigma_m(2,3)),\varphi(\sigma_m(2,3))) + a_2 F(\psi(\sigma_m(2,2)),\varphi(\sigma_m(2,2))) + a_3 F(\psi(\sigma_m(3,3)),\varphi(\sigma_m(3,3))) + a_4 F(\psi(\sigma_m(2,3)),\varphi(\sigma_m(2,3))) + a_5 F(\psi(\sigma_m(3,2)),\varphi(\sigma_m(3,2)))\Big]^{\beta(2,2)\beta(3,3)} + 1,$$

i.e.

$$\psi(1) \leq \left[\frac{1}{5}\psi(1) + \frac{1}{5}\psi(1) + \frac{1}{5}\psi(1) + \frac{1}{5}\psi(1) + \frac{1}{10}\psi(1)\right]^{\frac{1}{4}} + 1.$$

Also $\alpha(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v}) = \alpha(\mathfrak{u},\mathfrak{v}) \geq 1$, for all $\mathfrak{u} \in \mathcal{U}$, i.e., $\mathcal{T}$ is α-admissible. Each one of the postulates of Theorem 1 is fulfilled and $\mathcal{T}$ has a unique fixed point at $\mathfrak{u} = 1$ in $\mathcal{U}$.

Theorem 2. *Theorem* 1 *is true if inequality* (1) *is replaced by* (16).

$$\alpha(\mathfrak{u},\mathcal{T}\mathfrak{u})\alpha(\mathfrak{v},\mathcal{T}\mathfrak{v})\big[\psi(\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v})+l)\big]$$

$$\leq \Bigg[a_1F(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{v})))+a_2F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})))$$

$$+a_3F(\psi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{v},T\mathfrak{v})))+a_4F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})))$$

$$+a_5F(\psi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})))\Bigg]^{\beta(\mathfrak{u},\mathcal{T}\mathfrak{u})\beta(\mathfrak{u},\mathcal{T}\mathfrak{u})}+l. \tag{18}$$

Proof. Proof follows the pattern of Theorem 1. ◄

Now we furnish an illustrative example which cannot be used in the context partial metric space and consequently, Theorem 2 is an extended, improved, sharpened and generalized result using $\alpha-$admissibility and $\mathcal{C}-$class functions in an $\mathcal{M}$-metric space.

Example 7. *Let* $\mathcal{U}=\{1,2,3\}$ *and usual* $\mathcal{M}$*-metric space be defined as* $\sigma_m(1,1)=1$, $\sigma_m(2,2)=9$, $\sigma_m(3,3)=5$, $\sigma_m(1,2)=\sigma_m(2,1)=1$, $\sigma_m(1,3)=\sigma_m(3,1)=7$, $\sigma_m(2,3)=\sigma_m(3,2)=7$. *Let a continuous self-mapping* $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ *be defined as* $\mathcal{T}\mathfrak{u}=\mathfrak{u}$, $\mathfrak{u}\in\mathcal{U}$. *Let function* $\alpha:\mathcal{U}\times\mathcal{U}\to(0,\infty)$ *be given by* $\alpha(\mathfrak{u},\mathfrak{v})=\begin{cases}1, & \text{if } \mathfrak{u},\ \mathfrak{v}\in\mathcal{U}\\ 0, & \text{otherwise}\end{cases}$, $\varphi,\ \psi:[0,\infty)\to[0,\infty)$ *by* $\varphi(t)=t$, $\psi(t)=\frac{t}{2}$, $l=1$ *and* $\beta:[0,\infty)\times[0,\infty)\to[0,k]$ *be a function such that for* $k\in(0,1)$, $\beta(\mathfrak{u},\mathfrak{v})=\frac{1}{2}$. *Taking* $\mathfrak{u}=1$, $\mathfrak{v}=2$ $a_1=a_2=a_3=a_4=\frac{1}{5}$ *and* $a_5=\frac{1}{10}$ *in (16), we get*

$$\alpha(1,1)\alpha(2,2)\big[\psi(\sigma_m(1,2)+1)\big]\leq\Bigg[a_1F(\psi(\sigma_m(1,2)),\varphi(\sigma_m(1,2)))$$

$$+a_2F(\psi(\sigma_m(1,1)),\varphi(\sigma_m(1,1)))+a_3F(\psi(\sigma_m(2,2)),\varphi(\sigma_m(2,2)))$$

$$+a_4F(\psi(\sigma_m(1,2)),\varphi(\sigma_m(1,2)))+a_5F(\psi(\sigma_m(2,1)),\varphi(\sigma_m(2,1)))\Bigg]^{\beta(1,1)\beta(2,2)}+1,$$

i.e.

$$\psi(1)\leq\left[\frac{1}{5}\psi(1)+\frac{1}{5}\psi(1)+\frac{1}{5}\psi(9)+\frac{1}{5}\psi(1)+\frac{1}{10}\psi(1)\right]^{\frac{1}{4}}+1.$$

Also $\alpha(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v})=\alpha(\mathfrak{u},\mathfrak{v})\geq 1$, *for all* $\mathfrak{u}\in\mathcal{U}$, *i.e.,* $\mathcal{T}$ *is* α*-admissible. Each one of the postulates of Theorem 2 is fulfilled and* $\mathcal{T}$ *has a fixed point at* $\mathfrak{u}=1$ *in* $\mathcal{U}$.

Theorem 3. *Theorem* 1 *remains true if inequality* (1) *is replaced by following inequality*

$$\alpha(\mathfrak{u},\mathcal{T}\mathfrak{u})\alpha(\mathfrak{v},\mathcal{T}\mathfrak{v})\big[\psi(\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v})+l)\big]$$
$$\leq \beta(\mathfrak{u},\mathcal{T}\mathfrak{u})\beta(\mathfrak{v},\mathcal{T}\mathfrak{v})\Bigg[a_1F(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{v})))$$
$$+a_2F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})))+a_3F(\psi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v})))$$
$$+a_4F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})))+a_5F(\psi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})))\Bigg]+l.$$

Theorem 4. *Theorem* 1 *remains true if inequality* (1) *is replaced by following inequality*

$$\big[\alpha(\mathfrak{u},\mathcal{T}\mathfrak{u})\alpha(\mathfrak{v},\mathcal{T}\mathfrak{v})-1+l)\big]^{\psi(\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v})}\leq l^{\beta(\mathfrak{u},\mathcal{T}\mathfrak{u})\beta(\mathfrak{v},\mathcal{T}\mathfrak{v})A_k\mathfrak{u}\mathfrak{v}},$$

where,

$$A_k\mathfrak{u}\mathfrak{v}=a_1F(\psi(\sigma_m(\mathfrak{u},\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathfrak{v})))+a_2F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})))$$
$$+a_3F(\psi(\sigma_m(\mathfrak{v},T\mathfrak{v})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v})))+a_4F(\psi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})),\varphi(\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})))$$
$$+a_5F(\psi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})),\varphi(\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u}))).$$

Now we solve Theorem 3.3, left as an open problem in Asadi et al. [4]

Theorem 5. *Let $\mathcal{T}:\mathcal{U}\longrightarrow\mathcal{U}$ be a self mapping of a complete $\mathcal{M}-$metric space $(\mathcal{U},\sigma_m)$ satisfying*

$$\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u})\leq k\big[\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v})+\sigma_m(\mathcal{T}\mathfrak{u},\mathfrak{v})\big]\ \ ,\exists\ k\in[0,\frac{1}{2}),\ \ \mathfrak{u},\mathfrak{v}\in\mathcal{U}. \tag{19}$$

Then $\mathcal{T}$ has a unique fixed point.

Proof. Consider $\mathfrak{u}_0\in\mathcal{U}$. We construct a sequence $\{\mathfrak{u}_n\}$ so that $\mathfrak{u}_n=\mathcal{T}\mathfrak{u}_{n-1}$, $n\in\mathbb{N}$. If for some n, $\mathfrak{u}_n=\mathfrak{u}_{n+1}$, then $\mathfrak{u}_n$ is a fixed point of $\mathcal{T}$. Let $\mathfrak{u}_n\neq\mathfrak{u}_{n+1}$, $n\in\mathbb{N}\cup\{0\}$. Using (17) for $\mathfrak{u}=\mathfrak{u}_{n-1}$ and $\mathfrak{v}=\mathfrak{u}_n$,

$$\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1})=\sigma_m(\mathcal{T}\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_{n-2})\leq k[\sigma_m(\mathfrak{u}_{n-1},\mathcal{T}\mathfrak{u}_{n-2})+\sigma_m(\mathfrak{u}_{n-2},\mathcal{T}\mathfrak{u}_{n-1})]$$
$$=k[\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n-1})+\sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_n)].$$

Since,

$$\sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_n)=\sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_n)-\sigma_{m_{\mathfrak{u}_{n-2}\mathfrak{u}_n}}+\sigma_{m_{\mathfrak{u}_{n-2}\mathfrak{u}_n}}$$

$$\leq \sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_{n-1}) - \sigma_{m_{\mathfrak{u}_{n-2}\mathfrak{u}_{n-1}}} + \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) - \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) + \sigma_{m_{\mathfrak{u}_{n-2}\mathfrak{u}_n}}. \tag{20}$$

From Definition 1 ($\sigma_m 3$), we have, $\sigma_{m_{\mathfrak{u}_{n-2},\mathfrak{u}_n}} \leq \sigma_{m_{\mathfrak{u}_{n-2}\mathfrak{u}_{n-1}}} + \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_n}} - \sigma_{m_{\mathfrak{u}_{n-1}\mathfrak{u}_{n-1}}}$.
Inequality (19) becomes $\sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_n) \leq \sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_{n-1}) + \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_n) - \sigma_{m_{\mathfrak{u}_{n-1},\mathfrak{u}_{n-1}}}$.
Using Definition 1 ($\sigma_m 2$) in above inequality,

$$\sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_n) \leq \sigma_m(\mathfrak{u}_{n-2},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1}) - \sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n-1}). \tag{21}$$

By using (20) in (18), we get $\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n-1}) \leq \lambda\sigma_m(\mathfrak{u}_{n-1},\mathfrak{u}_{n-2})$, where, $0 \leq \lambda = \frac{k}{1-k} < 1$.

By Lemma 1, $\{\mathfrak{u}_n\}$ is a Cauchy sequence and using completeness of $\mathcal{U}$, $\mathfrak{u}_n \to \mathfrak{u}$, for some $\mathfrak{u} \in \mathcal{U}$. Therefore $\sigma_m(\mathfrak{u}_n,\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}}} \to 0$, $\sigma_M(\mathfrak{u}_n,\mathfrak{u}) - \sigma_{m_{\mathfrak{u}_n,\mathfrak{u}}} \to 0$, since $\sigma_{m_{\mathfrak{u}_n,\mathfrak{u}}} \to 0$, $\sigma_m(\mathfrak{u}_n,\mathfrak{u}) \to 0$ and $\sigma_{M_{\mathfrak{u}_n,\mathfrak{u}}} \to 0$. Using Remark 1, $\sigma_m(\mathfrak{u},\mathfrak{u}) = 0 = \sigma_{m_{\mathfrak{u},\mathcal{T}\mathfrak{u}}}$; $\sigma_m(\mathfrak{u}_{n+1},\mathcal{T}\mathfrak{u}) = \sigma_m(\mathcal{T}\mathfrak{u}_n,\mathcal{T}\mathfrak{u}) \leq k[\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) + \sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})]$, therefore $\sigma_m(\mathfrak{u}_n,\mathfrak{u}_{n+1}) \to 0$. Since, $\lim_{n\to\infty}\sup\sigma_m(\mathfrak{u}_{n+1},\mathcal{T}\mathfrak{u}) = \lim_{n\to\infty}\sup\sigma_m(\mathcal{T}\mathfrak{u}_n, T\mathfrak{u}) \leq k\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})$. Also $\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) - \sigma_{m_{\mathfrak{u},\mathcal{T}\mathfrak{u}}} \leq \sigma_m(\mathfrak{u},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u})$, i.e., $\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) \leq \lim_{n\to\infty}\sup(\sigma_m(\mathfrak{u},\mathfrak{u}_n) + \sigma_m(\mathfrak{u}_n,\mathcal{T}\mathfrak{u})) \leq k\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u})$, since $\sigma_{m_{\mathfrak{u},\mathcal{T}\mathfrak{u}}} = 0$ and $\sigma_m(\mathfrak{u}_n,\mathfrak{u}) \to 0$. So $\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) = 0$. By using inequality (17), $\sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}) \leq 2k\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) = 0$, therefore $\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) = 0 = \sigma_m(\mathfrak{u},\mathfrak{u}) = \sigma_m(\mathcal{T}\mathfrak{u}, T\mathfrak{u})$, i.e., $\mathfrak{u} = \mathcal{T}\mathfrak{u}$ by Definition 1 ($\sigma_m 1$).
Now we assert that the fixed point of $\mathcal{T}$ is unique. Let $\mathcal{T}\mathfrak{u} = \mathfrak{u}$ and $\mathcal{T}\mathfrak{v} = \mathfrak{v}$, however, $\mathfrak{u} \neq \mathfrak{v}$. From (17), we get

$$\sigma_m(\mathfrak{u},\mathfrak{v}) = \sigma_m(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{v}) \leq k[\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v}) + \sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{u})] = 2k\sigma_m(\mathfrak{u},\mathfrak{v}),$$

which is impossible. Hence $\mathfrak{u} = \mathfrak{v}$. ◄

Following examples support the open problem:

Example 8. *Let* $\mathcal{U} = [0,\frac{1}{2})$ *and* $\sigma_m(\mathfrak{u},\mathfrak{v}) = \frac{\mathfrak{u}+\mathfrak{v}}{2}$ *be an* $\mathcal{M}$*-metric. Consider* $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ *as:* $\mathcal{T}\mathfrak{u} = \mathfrak{u}^2$, $\mathfrak{u}\in\mathcal{U}$. *For* $\mathfrak{u} = \frac{1}{4}$, $\mathfrak{v} = \frac{1}{4}$, *using* (19), *we have*

$$\sigma_m(\frac{1}{16},\frac{1}{16}) \leq \frac{2}{5}\big[\sigma_m(\frac{1}{4},\frac{1}{16}) + \sigma_m(\frac{1}{16},\frac{1}{4})\big], \quad k = \frac{2}{5} \in [0,\frac{1}{2}).$$

Each one of the postulates of Theorem 5 *is fulfilled and* $\mathcal{T}$ *has a unique fixed point at* $\mathfrak{u} = 0$ *in* $\mathcal{U}$.

Example 9. *Let* $\mathcal{U} = \{1,2,4\}$ *and an* $\mathcal{M}$*-metric be defined as* $\sigma_m(1,1) = 1$, $\sigma_m(2,2) = 9$, $\sigma_m(4,4) = 5$, $\sigma_m(1,2) = \sigma_m(2,1) = 7$, $\sigma_m(1,4) = \sigma_m(4,1) = 2$, $\sigma_m(2,4) = \sigma_m(4,2) = 7$. *Consider* $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ *as:* $\mathcal{T}\mathfrak{u} = \mathfrak{u}$, $\mathfrak{u}\in\mathcal{U}$. *For* $\mathfrak{u} = 1$, $\mathfrak{v} = 2$ *,using* (19), *we have*
$\sigma_m(1,4) \leq k[\sigma_m(1,4) + \sigma_m(1,2)], k = \frac{1}{4} \in [0,\frac{1}{2})$. *Each one of the postulates of Theorem* 5 *is fulfilled and* $\mathcal{T}$ *has a unique fixed point at* $\mathfrak{u} = 1$ *in* $\mathcal{U}$.

Example 10. *Let* $\mathcal{U} = [0, \frac{1}{2})$ *and* $\sigma_m(\mathfrak{u}, \mathfrak{v}) = \frac{\mathfrak{u}+\mathfrak{v}}{2}$ *be an* $\mathcal{M}$*-metric. Consider* $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ *as:* $\mathcal{T}\mathfrak{u} = \begin{cases} \frac{1}{16}, & if \ \ \mathfrak{u} \in [0, \frac{1}{3}) \\ \frac{1}{2}, & if \ \ \mathfrak{u} \in [\frac{1}{3}, \frac{1}{2}) \end{cases}$.

Case (i) *When* $\mathfrak{u}, \mathfrak{v} \in [0, \frac{1}{3})$, *for* $\mathfrak{u} = \frac{1}{4}$, $\mathfrak{v} = \frac{1}{4}$, *using (17), we have*

$$\sigma_m(\frac{1}{16}, \frac{1}{16}) \leq \frac{2}{5}\big[\sigma_m(\frac{1}{4}, \frac{1}{16}) + \sigma_m(\frac{1}{16}, \frac{1}{4})\big].$$

Case (ii) *When* $\mathfrak{u}, \mathfrak{v} \in [\frac{1}{3}, \frac{1}{2})$, *for* $\mathfrak{u} = \frac{1}{3}$, $\mathfrak{v} = \frac{5}{12}$, *using (17), we have*

$$\sigma_m(\frac{1}{2}, \frac{1}{2}) \leq \frac{2}{5}\big[\sigma_m(\frac{1}{3}, \frac{1}{2}) + \sigma_m(\frac{5}{12}, \frac{1}{3})\big].$$

Case (iii) *When either* $\mathfrak{u} \in [0, \frac{1}{3})$ *or* $\mathfrak{v} \in [\frac{1}{3}, \frac{1}{2})$, *for* $\mathfrak{u} = \frac{1}{4}$, $\mathfrak{v} = \frac{5}{12}$, *using (17), we have*

$$\sigma_m(\frac{1}{16}, \frac{1}{2}) \leq \frac{2}{5}\big[\sigma_m(\frac{1}{4}, \frac{1}{2}) + \sigma_m(\frac{5}{12}, \frac{1}{16})\big],$$

$k = \frac{2}{5} \in [0, \frac{1}{2}]$, $\forall \mathfrak{u}, \mathfrak{v} \in \mathcal{U}$. *Each one of the postulates of Theorem* 5 *is fulfilled and* $\mathcal{T}$ *has a unique fixed point at* $\mathfrak{u} = \frac{1}{16}$ *in* $\mathcal{U}$.

Remark 3.

1. *Theorem* 5 *is an extension and generalisation of classical Chatterjea result [6] to an* $\mathcal{M}$*-metric space which is an improvement of a partial metric space. One may observe that the self mapping need not be continuous for the existence of a fixed point of Chatterjea contraction in* $\mathcal{M}$*-metric spaces. In the sequel, we have provided a solution to open problems posed by Asadi et. al [4] and Kumrod and Sintunavarat [12] without assuming additional condition on a mapping.*

2. *It is worth mentioning here that on changing the elements of* C *[2], different type of contractions, which are already present in the literature, may be found. Theorems* 1, 2, 3, 4 *and* 5 *still holds if we replace inequality under consideration by using any one element of* C *in Example 5* (*for details one may refer to [2], [17]- [19] and so on*).

3. *Noticeably, the fixed point results in, Altun et al. [1], Asadi et al. [4], Banach [5], Chatterjea [6], Geraghty [8], Hammache et al. [9], Kumrod and Sintunavarat [12] and Monfared et al. [14] and references therein, are instant outcomes of obtained results.*

2.1. Application

Now we solve periodic differential equation with given boundary conditions to demonstrate the applicability of Theorem 1. Let $\mathcal{U} = \mathbf{C}[I, \mathbb{R}]$ be the set of continuous functions on $[0,1]$. Define $\sigma_m : \mathcal{U} \times \mathcal{U} \longrightarrow [0, \infty)$ as $\sigma_m(\mathfrak{u}, \mathfrak{v}) = \sup_{t\in[0,1]} \frac{|\mathfrak{u}(t)+\mathfrak{v}(t)|}{2}$. Here $(\mathcal{U}, \sigma_m)$ is an $\mathcal{M}-$metric space.

Theorem 6. *Let us examine the periodic differential equation*

$$\frac{d\mathfrak{u}}{dt} = f(t, \mathfrak{u}(t)), \ \textit{with}\ \mathfrak{u}(0) = \mathfrak{u}(1),\ t \in [0,1], \tag{22}$$

and continuous function $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$. *If there exists* $\eta \geqslant 1$ *so that*

$$|f(t, \mathfrak{u}(t)) + f(t, \mathfrak{v}(t))| \leq \eta\, |\mathfrak{u}(t) + \mathfrak{v}(t)| \tag{23}$$

and

$$|f(t, \mathfrak{u}(t))| \leq \eta \sup_{t\in[0,1]} |\mathfrak{u}(t)|\,. \tag{24}$$

Then boundary value problem (22) has only one solution.

Proof. The equation in problem (22) may be rewritten as

$$\tfrac{d\mathfrak{u}}{dt} + \eta\mathfrak{u}(t) = f(t, \mathfrak{u}(t)) + \eta\mathfrak{u}(t), t \in [0,1], \text{ with } \mathfrak{u}(0) = \mathfrak{u}(1).$$

Equivalently,

$$\mathfrak{u}(t) = \int_0^1 G(t,\xi)[f(\xi, \mathfrak{u}(\xi)) + \eta\mathfrak{u}(\xi)]d\xi,\ for\ t \in [0,1], \tag{25}$$

where Green function $G(t,\xi) = \begin{cases} \frac{e^{\eta(\xi-t+1)}}{e^{\eta}-1}, & 0 \leqslant \xi \leqslant t \leqslant 1 \\ \frac{e^{\eta(\xi-t)}}{e^{\eta}-1}, & 0 \leqslant t \leqslant \xi \leqslant 1 \end{cases}$. Now $\mathfrak{u} \in \mathcal{U}$ is a solution of (25) iff it is the solution of boundary value problem (22). Define a map $\mathcal{T} : \mathcal{U} \longrightarrow \mathcal{U}$ by
$\mathcal{T}\mathfrak{u}(t) = \int_0^1 G(t,\xi)[f(\xi, x(\xi)) + \eta\mathfrak{u}(\xi)]d\xi$. Now,

$$\sigma_m(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) = \sup_{t\in[0,1]} \frac{|\mathcal{T}\mathfrak{u}(t) + \mathcal{T}\mathfrak{v}(t)|}{2}$$

$$= \frac{1}{2} \sup_{t\in[0,1]} \left| \int_0^1 G(t,\xi)[f(\xi, \mathfrak{u}(\xi)) + \eta\mathfrak{u}(\xi)]d\xi + \int_0^1 G(t,\xi)[f(\xi, \mathfrak{v}(\xi)) + \eta\mathfrak{v}(\xi)]d\xi \right|$$

$$\leq \frac{1}{2} \big[\eta \sup_{t\in[0,1]} |\mathfrak{u}(t) + \mathfrak{v}(t)| \int_0^1 G(t,\xi)d\xi + \sup_{t\in[0,1]} |(f(\xi, \mathfrak{u}(\xi))) + f(\xi, \mathfrak{v}(\xi))| \int_0^1 G(t,\xi)d\xi\big]$$

$$\leq \frac{1}{2}\big[\eta \sup_{t\in[0,1]} |\mathfrak{u}(t)+\mathfrak{v}(t)| \int_0^1 G(t,\xi)d\xi + \eta \sup_{t\in[0,1]} |\mathfrak{u}(t)+\mathfrak{v}(t)| \int_0^1 G(t,\xi)d\xi\big]$$

$$= \frac{\eta}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)+\mathfrak{v}(t)| \int_0^1 G(t,\xi)d\xi$$

$$= \frac{\eta}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)+\mathfrak{v}(t)| \big[\int_0^t \frac{e^{\eta(\xi-t+1)}}{e^{\eta}-1}d\xi + \int_t^1 \frac{e^{\eta(\xi-t)}}{e^{\eta}-1}\big] = \frac{\eta}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)+\mathfrak{v}(t)| \frac{1}{\eta},$$

$$\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{u}) = \sup_{t\in[0,1]} \frac{|\mathfrak{u}(t)+\mathcal{T}\mathfrak{u}(t)|}{2}$$

$$= \frac{1}{2} \sup_{t\in[0,1]} \left|\mathfrak{u}(t) + \int_0^1 G(t,\xi)[f(\xi,\mathfrak{u}(\xi)) + \eta\mathfrak{u}(\xi)]d\xi\right|$$

$$\leq \frac{1}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| + \frac{1}{2} \sup_{t\in[0,1]} \left|\int_0^1 G(t,\xi)f(\xi,\mathfrak{u}(\xi))d\xi\right| + \frac{\eta}{2} \sup_{t\in[0,1]} \left|\int_0^1 G(t,\xi)\mathfrak{u}(\xi)d\xi\right|$$

$$\leq \frac{1}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| + \frac{\eta}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| \int_0^1 G(t,\xi)d\xi + \frac{\eta}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| \int_0^1 G(t,\xi)d\xi$$

$$= \frac{1}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| + \eta \sup_{t\in[0,1]} |\mathfrak{u}(t)| \frac{1}{\eta} = \frac{3}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)|.$$

Similarly, $\sigma_m(\mathfrak{v},\mathcal{T}\mathfrak{v}) \leq \frac{3}{2}\sup_{t\in[0,1]} |y(t)|,$

$$\sigma_m(\mathfrak{u},\mathcal{T}\mathfrak{v}) = \sup_{t\in[0,1]} \frac{|\mathfrak{u}(t)+\mathcal{T}\mathfrak{v}(t)|}{2}$$

$$= \frac{1}{2} \sup_{t\in[0,1]} \left|\mathfrak{u}(t) + \int_0^1 G(t,\xi)[f(\xi,\mathfrak{v}(\xi)) + \eta\mathfrak{v}(\xi)]d\xi\right|$$

$$\leq \frac{1}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| + \frac{1}{2} \sup_{t\in[0,1]} \left|\int_0^1 G(t,\xi)f(\xi,y(\xi))\right| + \frac{\eta}{2} \sup_{t\in[0,1]} \left|\int_0^1 G(t,\xi)\mathfrak{v}(\xi)d\xi\right|$$

$$\leq \frac{1}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| + \frac{\eta}{2} \sup_{t\in[0,1]} |\mathfrak{v}(t)| \int_0^1 G(t,\xi)d\xi + \frac{\eta}{2} \sup_{t\in[0,1]} |\mathfrak{v}(t)| \int_0^1 G(t,\xi)d\xi$$

$$= \frac{1}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| + \eta \sup_{t\in[0,1]} |\mathfrak{v}(t)| \frac{1}{\eta} = \frac{1}{2} \sup_{t\in[0,1]} |\mathfrak{u}(t)| + \sup |\mathfrak{v}(t)|$$

$$\leq \sup_{t\in[0,1]} |\mathfrak{u}(t)+\mathfrak{v}(t)|.$$

Similarly, $\sigma_m(\mathfrak{v}, \mathcal{T}\mathfrak{u}) \leq \sup_{t\in[0,1]} |\mathfrak{u}(t) + \mathfrak{v}(t)|$. Now if we define an $\alpha : \mathcal{U} \times \mathcal{U} \longrightarrow (0, \infty)$ by $\alpha(\mathfrak{u}, \mathfrak{v}) = \begin{cases} 1, & \mathfrak{u}, \mathfrak{v} \in \mathcal{U} \\ 0, & otherwise \end{cases}$, φ, $\psi : [0, \infty) \longrightarrow [0, \infty)$ by $\varphi(t) = t$, $\psi(t) = \frac{t}{2}, t \in [0, 1]$ and $\beta : [0, \infty) \longrightarrow [0, \infty)$ by $\beta(\mathfrak{u}, \mathfrak{v}) = \frac{1}{\sqrt{2}}$. Hence $\mathcal{T}$ is $\alpha-$admissible and by taking $a_1 = \frac{1}{5}, a_2 = \frac{1}{10}, a_3 = \frac{1}{10}, a_4 = \frac{1}{6}, a_5 = \frac{1}{7}$ and $l = 1$, all the postulates of Theorem 1 are verified and consequently, unique solution of Problem 22 exists in $\mathcal{U}$. ◄

3. Conclusion

We established unique fixed point for the generalized $F(\psi, \varphi)-$contraction using $\mathcal{C}-$class functions and $\alpha-$admissibility in a complete $\mathcal{M}$-metric space, which is fascinating, generalised and distinct than a usual metric space due to the fact it admits non-zero self-distance at a point and is more general than the partial metric space. Our theorems are sharpened versions of the well-known results. Further, we provided an answer to the open problem of Asadi et al. [4] on the question of existence and uniqueness of fixed point for Chatterjea contraction mapping without assuming any additional condition on mappings. In the sequel, we answered two challenging open problems posed by Kumrod and Sintunavarat [12]. Examples and an application to find the solution of a periodic differential equation with given boundary conditions substantiate the utility of these extensions.

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