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A Note on the $(\lambda, v)_h^\alpha-$Statistical Convergence of the Functions Defined on the Product of Time Scales

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Abstract. In this paper, we have introduced the concepts $(\lambda, v)_h^\alpha-$density of a subset of the product of time scales $\mathbb{T}^2$ and $(\lambda, v)_h^\alpha-$statistical convergence of order α $(0 < \alpha \leq 1)$ of $\Delta-$ measurable function f defined on the product time scale with the help of modulus function h and $\lambda = (\lambda_n)$, $v = (v_n)$ sequences. Later, we have discussed the connection between classical convergence, λ-statistical convergence and $(\lambda, v)_h^\alpha-$statistical convergence. In addition, we have seen that f is strongly $(\lambda, v)_h^\alpha-$summable on T then f is $(\lambda, v)_h^\alpha-$statistical convergent of order α .

Key Words and Phrases: time scale, statistical convergence, modulus function, λ sequence, order α.

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1. Introduction

The concept of statistical convergence which is a generalization of classical convergence was first given by Zygmund [1] and later were introduced independently by Steinhaus [2] and Fast [3]. This concept is discussed under different names in different spaces ([4],[5],[6],[7],[8],[9],[10], [11],[12]). Mursaleen [13] introduced the notion of λ-statistical convergence by using the sequence $\lambda = (\lambda_n)$ and then the λ-statistical convergence on the time scales was introduced by Yılmaz et al [14]. The order of statistical convergence of a sequence of positive linear operators was introduced by Gadjiev and Orhan [15]. Later, Çolak [16] introduced and investigated the statistical convergence of order α $(0 < \alpha \leq 1)$ and strong p-Cesaro summability of order α of number sequences.

The time scale calculus was first introduced by Hilger in his Ph.D. thesis in 1988 (see [17],[18],[19]). In later years, the integral theory on time scales was given by Guseinov [20] and further studies were developed by Cabada-Vivero [21] and Rzezuchowski [7]. Recently, Seyyidoğlu and Tan [8] defined the density of the subset of the time scale. By using this definition, they gave $\Delta-$convergence and $\Delta-$Cauchy concepts for a real valued function defined on the time scale. On the other side, the modulus function was first introduced by Nakano [22]. Aizpuru et al.[23] defined a new density concept with the help of a modulus function and obtained a new convergence concept between ordinary convergence and statistical convergence. Gürdal and Özgür [24] introduced ideal h-statistical convergence and ideal h-statistical Cauchy concepts in normed space using the modulus function h and ideals.

In this paper, we have aimed to define $(\lambda, v)_h^\alpha-$statistical convergence of $\Delta-$ measurable functions of order α $(0 < \alpha \leq 1)$ defined on the product time scale by using modulus function h, $\lambda = (\lambda_n)$ and $v = (v_r)$ sequences in light of works of Çınar et al [25], Seyyidoğlu and Tan [8] and [20].

2. Preliminaries

The concept of statistical convergence is based on the asymptotic (natural) density of a subset B in $\mathbb{N}$ (the set of positive integers) which is defined as

$$\delta(B) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in B\}|}{n}, \tag{1}$$

where $|B|$ denotes the number of elements in B (see [3],[5],[4]). It has been generalized to α-density of a subset $B \subset \mathbb{N}$ and given the definition of $\alpha-$statistically convergence ($\alpha \in (0, 1]$) by Çolak [16]. The notion of λ-statistical convergence was introduced by Mursaleen [13] using the sequence $\lambda = (\lambda_n)$ which is a non-decreasing sequence of positive numbers tending to ∞ as $n \to \infty$ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, and $I_n = [n - \lambda_n + 1, n]$. Lets denote by Λ the set of such $\lambda = (\lambda_n)$ sequences. The λ- density of $B \subset \mathbb{N}$ is defined by

$$\delta_\lambda(B) = \lim_{n\to\infty} \frac{|\{k \in I_n : k \in B\}|}{\lambda_n} \tag{2}$$

and $\delta_\lambda(B)$ reduces to the natural density $\delta(B)$ in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ (see [14]). A sequence $x = (x_n)$ is said to be λ- statistically convergent to L of order α ($\alpha \in (0, 1]$) if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in I_n : |x_k - L| \geq \epsilon\}|}{(\lambda_n)^\alpha} = 0. \tag{3}$$

In this case, we write $s_{\lambda^\alpha} - \lim_{n\to\infty} x_n = L$ (see [26],[27],[13],[28],[29],[30],[14]) and we denote by S_{λ^α} the set of λ^α- statistically convergent sequences of order α. If

$\lambda_n = n$, S_{λ^α} reduces to S^α the set of statistically convergent number sequences of order α. For applications of statistical convergence and λ-statistical convergence, see [31], [32].

On the other hand, we recall that $h : [0, \infty) \to [0, \infty)$ is called modulus function, or simply modulus, if it is satisfies:

(1) $h(s) = 0$ if and only if $s = 0$,

(2) $h\,(s + p) \leq h\,(s) + h\,(p)$ for every $s, p \in [0, \infty)$,

(3) h is increasing,

(4) h is continuous from the right at 0.

A modulus may be bounded or unbounded . For instance, $h(x) = x^p$, where $0 < p \leq 1$, is unbounded, but $h(x) = \frac{x}{1+x}$ is bounded (see [33],[34]).

Let h be an unbounded modulus function. The λ_h^α−density of order α $(0 < \alpha \leq 1)$ of a set $B \subseteq \mathbb{N}$ is defined by

$$\delta^{\lambda_h^\alpha}(B) = \lim_{n\to\infty} \frac{h(|\{n - \lambda_n + 1 \leq k \leq n : k \in B\}|)}{h((\lambda_n)^\alpha)} \tag{4}$$

whenever this limit exists.

In this study, we shall give a notion of $(\lambda, v)_h^\alpha$-statistical convergence on any time scales product and its properties using the sequences $\lambda, v \in \Lambda$, modulus function h and any real number α $(0 < \alpha \leq 1)$. Throughout this paper, we consider the time scales which are unbounded from above and have a minimum point. Lets remember some concepts.

A nonempty closed subset of $\mathbb{R}$ is called a time scale and is denoted by $\mathbb{T}$. We suppose that a time scale has the topology inherited from $\mathbb{R}$ with the standard topology. For $t \in \mathbb{T}$, we consider the forward (backward) jump operator $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma\,(t) := \inf\{s \in \mathbb{T} : s > t\}, \rho\,(t) := \sup\{s \in \mathbb{T} : s < t\}$. and graininess function : $\mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$. In this definition, we take $\inf \emptyset = sup\mathbb{T}$. For $t \in \mathbb{T}$ with $a \leq b$, it is defined the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Let $\mathbb{T}$ be a time scale. Denote by $\mathcal{F}$ the family of all left-closed and right-open intervals of $\mathbb{T}$ of the form $[a, b) = \{t \in \mathbb{T} : a \leq t < b\}$ with $a, b \in \mathbb{T}$ and $a \leq b$. It is clear that the interval $[a, a)$ is an empty set, $\mathcal{F}$ is semiring of subsets of $\mathbb{T}$. Let $m : \mathcal{F} \to [0, \infty)$ be the set function on $\mathcal{F}$ that assigns to each interval $[a, b)$ its length $b - a$, $m\,([a, b)) = b - a$. Then m is a countably additive measure on $\mathcal{F}$. We denote by μ_Δ the Caratheodory extension of the set function m associated with family $\mathcal{F}$ (for the Caratheodory extension see [8]) and is denoted by μ_Δ, the Lebesgue Δ-measure on $\mathbb{T}$, and that is a countably additive measure . In this case, it is known that if $a \in \mathbb{T} - \{max\mathbb{T}\}$, then the single point set $\{a\}$ is Δ-measurable and $\mu_\Delta(a) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$ then $\mu_\Delta(a, b)_\mathbb{T} = b - \sigma(a)$. If $a, b \in \mathbb{T} - \{max\mathbb{T}\}$, $a \leq b$; $\mu_\Delta(a, b]_\mathbb{T} = \sigma(b) - \sigma(a)$ and $\mu_\Delta[a, b]_\mathbb{T} = \sigma(b) - a$. It can be easily seen that the measure of a subset of $\mathbb{N}$ is equal to its cardinality (see [8],[6]).

Suppose that $\mathbb{T}_1$ and $\mathbb{T}_2$ are times scales and σ_j, ρ_j and μ_j are forward (backward) jump operators and graininess functions on $\mathbb{T}_j$ for $1 \leq j \leq 2$, respectively. Set $\mathbb{T}^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (t_1, t_2) : t_1 \in \mathbb{T}_1 \text{ and } t_2 \in \mathbb{T}_2\}$. $\mathbb{T}^2$ is called product (or 2-dimensional) time scale . $\mathbb{T}^2$ is complete metric space with the metric defined by

$$d(t, r) = (\sum_{i=1}^{2} |t_i - r_i|^2)^{\frac{1}{2}} \text{ for } t, r \in \mathbb{T}^2.$$

Recently, the λ-statistical convergence on time scale was introduced by Yılmaz et al [35] and then the notion of $(\lambda, v)-$statistical convergence of Δ-measurable real-valued function defined on product time scale was introduced by Çınar et al [25]. They also introduced the concept of the $(\lambda, v)-$density of Ω on $\mathbb{T}^2$ as follows.

Let $\lambda, v \in \Lambda$ be two sequences of positive real numbers. Throughout the paper we denote $A = \{[t - \lambda_t + t_0, t]_{\mathbb{T}_1} \times [r - v_r + r_0, r]_{\mathbb{T}_2}\}$, $B = \{[t_0, t]_{\mathbb{T}_1} \times [r_0, r]_{\mathbb{T}_2}\}$, where $t_0 = \min \mathbb{T}_1$, $r_0 = \min \mathbb{T}_2$. Suppose that Ω be a Δ-measurable subset of $\mathbb{T}^2 = \mathbb{T}_1 \times \mathbb{T}_2$. Then, the set $\Omega\ (t, r, \lambda, v)$ is defined by $\Omega\ (t, r, \lambda, v) =: \{(s, u) \in A : (s, u) \in \Omega\}$ for $(t, r) \in \mathbb{T}^2$. That is $\Omega\ (t, r, \lambda, v) = \Omega \cap A$. In this case, the density of Ω on $\mathbb{T}^2$ is defined as

$$\delta_{\mathbb{T}^2}^{(\lambda, v)}(\Omega) = \lim_{t \to \infty} \frac{\mu_\Delta(\Omega(t, r, \lambda, v))}{\mu_\Delta(A)} \tag{5}$$

provided that the limit exists. In case of $\mathbb{T}^2 = \mathbb{N}^2$, this reduces to the classical concept of the product asymptotic density.

Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a $\Delta-$ measurable function. It is said that f is (λ, v)-statistically convergent to a real number L on $\mathbb{T}^2$ if

$$\lim_{(t,r) \to \infty} \frac{\mu_\Delta(\{(s, u) \in A : |f(s, u) - L| \geq \epsilon\})}{\mu_\Delta(A)} = 0 \tag{6}$$

for every $\epsilon > 0$. In this case, we can write $s_{\mathbb{T}^2}^{(\lambda, v)} - \lim_{(t,r) \to \infty} f(t, r) = L$. The set of all $(\lambda, v)-$ statistically convergent functions on $\mathbb{T}^2$ will be denoted by $S_{\mathbb{T}^2}^{(\lambda, v)}$.

If one take $\lambda_t = t$ and $v_r = r$ in (6), we get the classical statistically convergent function to a real number L on $\mathbb{T}^2$, for the function f, which is defined as :

$$\lim_{(t,r)\to\infty} \frac{\mu_\Delta(\{(s,u)\in B: \ |f(s,u)-L|\geq \epsilon\})}{\mu_\Delta(B)} = 0$$

3. Main Results

Definition 1. *Let Ω be a Δ-measurable subset of $\mathbb{T}^2 = \mathbb{T}_1\times\mathbb{T}_2$, h be a modulus function, α be any real number $(0<\alpha\leq 1)$ and be the set $\Omega(t,r,\lambda,v) =: \{(s,u)\in A : (s,u)\in \Omega\}$ for $(t,r)\in\mathbb{T}^2 = \mathbb{T}_1\times\mathbb{T}_2$. In this case, the $(\lambda,v)_h^\alpha$-density of Ω on $\mathbb{T}^2$ of order α is defined by*

$$\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = \lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(\Omega(t,r,\lambda,v)))}{h((\mu_\Delta(A))^\alpha)}$$

provided that the limit exists.

When $\alpha = 1$, the $(\lambda,v)_h^\alpha$−density of Ω on $\mathbb{T}^2$ returns to the $(\lambda,v)_h$−density and the density will denoted by $\delta_{\mathbb{T}^2}^{(\lambda,v)_h}(\Omega)$. In case $h(x) = x$, $(\lambda,v)_h^\alpha$−density becomes $(\lambda,v)^\alpha$−density and is denoted by $\delta_{\mathbb{T}^2}^{(\lambda,v)^\alpha}(\Omega)$. If $\alpha = 1$ and $h(x) = x$, then $(\lambda,v)_h^\alpha$−density reduces to (λ,v)−density of Ω on $\mathbb{T}^2$which is denoted by $\delta_{\mathbb{T}^2}^{(\lambda,v)}(\Omega)$. We can easily get $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = \delta_{\mathbb{T}^2}^{\alpha_h}(\Omega)$ if $\lambda_t = t$ and $v_r = r$ and $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = \delta_{\mathbb{T}^2}^{(\lambda,v)^\alpha}(\Omega)$ if we take $h(x) = x$ on $\mathbb{T}^2$. If $\alpha = 1$, $h(x) = x$, $\lambda_t = t$ and $v_r = r$ then $(\lambda,v)_h^\alpha$−density reduces to Δ−density of Ω on $\mathbb{T}^2$

Definition 2. *Let $f : \mathbb{T}^2\to\mathbb{R}$ be a $\Delta-$ measurable function. Then, we call that f is $(\lambda,v)_h^\alpha$−statistically convergent function to a real number L of order α $(0<\alpha\leq 1)$ on $\mathbb{T}^2$ if*

$$\lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(\{(s,u)\in A: \ |f(s,u)-L|\geq\epsilon\}))}{h((\mu_\Delta(A))^\alpha)} = 0 \tag{7}$$

for every $\epsilon > 0$.

In this case, we write $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim_{(t,r)\to\infty} f(t,r) = L$. The set of all $(\lambda,v)_h^\alpha$−statistically convergent functions on $\mathbb{T}^2$ will be denoted by $S_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}$.

As will be noted that, when $\alpha = 1$, $(\lambda,v)_h^\alpha$−statistically convergent function on $\mathbb{T}^2$ of order α returns to $(\lambda,v)_h$−statistically convergent function. If $\alpha = 1$, $h(x) = x$, $\lambda_t = t$ and $v_r = r$ then $(\lambda,v)_h^\alpha$−statistically convergent function on $\mathbb{T}^2$ reduces to $\Delta-$convergent function on $\mathbb{T}^2$ and which is denoted by $\Delta - \lim_{(t,r)\to\infty} f(t,r) = L$.

The equality $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) + \delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}\left(\mathbb{T}^2 \diagdown \Omega\right) = 1$ does not hold for α $(0 < \alpha \leq 1)$ and an unbounded modulus h, in general. For instance, if we take $h(x) = x^p$, $0 < p \leq 1$, $0 < \alpha < 1$ and $\Omega = \{(2n, 2m) : n, m \in \mathbb{N}\}$, then $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = \delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}\left(\mathbb{T}^2 \diagdown \Omega\right) = \infty$. Also, finite sets have zero $(\lambda, v)_h^\alpha-$density for any unbounded modulus h and α $(0 < \alpha \leq 1)$ (see [27],[39]).

Lemma 1. *Let α be any real number $(0 < \alpha \leq 1)$, Ω be a Δ-measurable subset of $\mathbb{T}^2 = \mathbb{T}_1 \times \mathbb{T}_2$, h be an unbounded modulus function . If $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = 0$ then $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}\left(\mathbb{T}^2 \diagdown \Omega\right) \neq 0$.*

Proof. Let α $(0 < \alpha \leq 1)$ be any given real number and the equality $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = 0$ be valid for any unbounded modulus h. Suppose that $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\mathbb{T} \diagdown \Omega) = 0$. Let us say $\Omega(t, r, \lambda, v) = \Omega(t, r) \cap A$ for $(t, r) \in \mathbb{T}^2$ and $\mathbb{T}^2 \diagdown \Omega(t, r, \lambda, v) =: \{(s, u) \in A : (s, u) \in \mathbb{T}^2 \diagdown \Omega(t, r)\}$ for $(t, r) \in \mathbb{T}^2$. Since $\mu_\Delta(A) = \mu_\Delta(\Omega(t, r, \lambda, v)) + \mu_\Delta(\mathbb{T}^2 \diagdown \Omega(t, r, \lambda, v))$ for $(t, r) \in \mathbb{T}^2$ and h is subadditive, we have

$$h(\mu_\Delta(A)) \leq h(\ \mu_\Delta \Omega(t, r, \lambda, v)) + h(\ \mu_\Delta(\mathbb{T}^2 \diagdown \Omega(t, r, \lambda, v))) \tag{8}$$

Hence we may write

$$\lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(A)}{h((\mu_\Delta(A))^\alpha)} \tag{9}$$

$$\leq \lim_{(t,r)\to\infty} \frac{h(\ \mu_{\Delta_\lambda} \Omega(t, r, \lambda, v))}{h((\mu_\Delta(A))^\alpha)} + \lim_{(t,r)\to\infty} \frac{h(\ \mu_\Delta(\mathbb{T} \diagdown \Omega(t, r, \lambda, v)))}{h((\mu_\Delta(A))^\alpha)}. \tag{10}$$

Since $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = 0$ and $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}\left(\mathbb{T}^2 \diagdown \Omega\right) = 0$, the right side of the inequality is equal to zero and thus

$$\lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(A))}{h((\mu_\Delta(A)^\alpha)} = 0.$$

This is a contradiction. Because $\frac{h(\mu_\Delta(A))}{h((\mu_\Delta(A)^\alpha)} \geq 1$ for α $(0 < \alpha \leq 1)$ and therefore

$$\lim_{(t,r)\to\infty} \frac{h((\mu_\Delta(A))}{h((\mu_\Delta(A)^\alpha)} \geq 1. \tag{11}$$

For any unbounded modulus h and $0 < \alpha \leq 1$, if $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^{\alpha}}(\Omega) = 0$ then $\delta_{\mathbb{T}^2}^{(\lambda,v)^{\alpha}}(\Omega) = 0$, but the inverse of this does not need to be true ([36]). Namely, a set having zero α-density for some α $(0 < \alpha \leq 1)$ might have non-zero $(\lambda, v)_h^{\alpha}-$density for some unbounded modulus h, with the same α. Similarly a set having zero $(\lambda, v)-$ density might have non-zero $(\lambda, v)_h^{\alpha}-$density for some unbounded modulus h and $0 < \alpha \leq 1$. For example, let $h(x) = log(x + 1)$ and $\Omega = \{\{1, 4, 9, ...\}\times\{1, 4, 9, ...\}\}$. Then $\delta_{\mathbb{T}^2}(\Omega) = 0$ and $\delta_{\mathbb{T}^2}^{(\lambda,v)^{\alpha}}(\Omega) = 0$ for $1/2 < \alpha \leq 1$, but $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^{\alpha}}(\Omega) \geq \delta_{\mathbb{T}^2}^{(\lambda,v)_h}(\Omega) = 1/2$ and therefore $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^{\alpha}}(\Omega) \neq 0$.

If $\Phi \subseteq \mathbb{T}^2$ has zero $(\lambda, v)_h^{\alpha}-$density for some unbounded modulus h and for some α $(0 < \alpha \leq 1)$, then it has zero $(\lambda, v)^{\alpha}-$density and hence zero $(\lambda, v)-$density (see [35]).

Lemma 2. *Let h be unbounded modulus and $\Phi \subseteq \mathbb{T}^2$. If $0 < \alpha \leq \beta \leq 1$ then $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^{\beta}}(\Phi) \leq \delta_{\mathbb{T}^2}^{(\lambda,v)_h^{\alpha}}(\Phi)$.*

Thus, for any unbounded modulus h and $0 < \alpha \leq \beta \leq 1$, if Φ has zero $(\lambda, v)_h^{\alpha}-$density in that case, it has zero $(\lambda, v)_h^{\beta}-$density. Specially, a set having zero $(\lambda, v)_h^{\alpha}-$density for some α $(0 < \alpha \leq 1)$ has zero $(\lambda, v)_h-$density. But, the inverse is not correct. For instance, let $h(x) = x^p$ for $0 < p \leq 1$ and $\Phi = \{\{1, 4, 9, ...\}\times\{1, 4, 9, ...\}\}$. Then

$$\delta_{\mathbb{T}^2}^{(\lambda,v)_h}(\Phi) = \lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(\Phi(t, r, \lambda, v)_{\mathbb{T}^2}))}{h(\mu_\Delta(A))} \tag{12}$$

$$\leq \lim_{(t,r)\to\infty} \frac{h(\lceil\sqrt{\mu_\Delta(\Phi(t, r, \lambda, v)_{\mathbb{T}^2})}\rceil)}{h(\mu_\Delta(A))} \tag{13}$$

$$= \lim_{(t,r)\to\infty} \frac{(\lceil\sqrt{\mu_\Delta(\Phi(t, r, \lambda, v)_{\mathbb{T}^2})}\rceil)^p}{(\mu_\Delta(A)^p} = 0$$

but, if we get $0 < \alpha \leq 1/2$,

$$\delta_{\mathbb{T}^2}^{(\lambda,v)_h^{\alpha}}(\Phi) = \lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(\Phi(t, r, \lambda, v)_{\mathbb{T}^2})))}{h((\mu_\Delta(A)^{\alpha})} \tag{14}$$

$$= \lim_{(t,r)\to\infty} \frac{(\lceil\sqrt{\mu_\Delta(\Phi(t, r, \lambda, v)_{\mathbb{T}^2})}\rceil)^p}{((\mu_\Delta(A)^{\alpha})^p} = \infty$$

where $\lceil r\rceil$ denotes the integer part of number r.

Proposition 1. *Let* $f,g : \mathbb{T}^2 \to \mathbb{R}$ *be a* $\Delta-$ *measurable functions such that* $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = L_1$ *and* $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} g(t,r) = L_2$ *and* h *and* k *be modulus functions. Then the following statements hold:*

(i) $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} (f(t,r)+g(t,r)) = L_1 + L_2$,

(ii) $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} (cf(t,r)) = cL_1 \quad (c \in \mathbb{R})$

(iii) If $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = L_1$, then $s_{\mathbb{T}^2}^{(\lambda,v)^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = L_1$.

(iv) If $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = \ell$ and $s_{\mathbb{T}^2}^{(\lambda,v)_k^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = m$, then $\ell = m$.

(v) $\lim\limits_{(t,r)\to\infty} f(t,r) = \ell \;\Rightarrow\; s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = \ell \;\Rightarrow\; \Delta - \lim\limits_{(t,r)\to\infty} f(t,r) = \ell$.

Proof. It is easy to prove and we omit it.

Theorem 1. $s_{\mathbb{T}^2}^{\alpha_h} \subseteq s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}$ *if and only if*

$$\liminf_{(t,r)\to\infty} \frac{h((\mu_\Delta(A)^\alpha)}{h((\mu_\Delta((B)^\alpha)} > 0. \tag{15}$$

Proof. For given $\epsilon > 0$*, we have*

$$h(\mu_\Delta(\{(s,u) \in B : |f(s,u) - L| \geq \epsilon\})) \supset h(\mu_\Delta(\{(s,u) \in A : |f(s,u) - L| \geq \epsilon\})).$$

Then

$$\begin{aligned} & \frac{h(\mu_\Delta(\{(s,u) \in B : |f(s,u) - L| \geq \epsilon\}))}{h((\mu_\Delta(B)^\alpha)} \\ \geq \; & \frac{h(\mu_\Delta(\{(s,u) \in A : |f(s,u) - L| \geq \epsilon\}))}{h((\mu_\Delta(B)^\alpha)} \\ = \; & \frac{h(\mu_\Delta(A\ )^\alpha)}{h((\mu_\Delta(B)^\alpha)} \frac{1}{h(\mu_\Delta(A)^\alpha)} h(\mu_\Delta(\{(s,u) \in A : |f(s,u) - L| \geq \epsilon\})) \end{aligned}$$

Hence by using (15) and taking the limit as $(t,r) \to \infty$*, we get* $s_{\mathbb{T}^2}^{\alpha_h} - \lim\limits_{(t,r)\to\infty} f(t,r) \to L$ *implies* $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = L$.

The definition of $p-$strongly (W,λ,v) summable functions on $\mathbb{T}^2$ was given by Çınat et al [25] as follows.

Definition 3. *Let* $f : \mathbb{T}^2 \to \mathbb{R}$ *be a* $\Delta-$ *measurable function,* $\lambda, v \in \Lambda$ *and* $0 < p < \infty$. *We say that* f *is* $p-$*strongly* $(W, \lambda, v)-$*summable functions on* $\mathbb{T}^2$ *if there exists* $L \in \mathbb{R}$ *such that*

$$\lim_{(t,r)\to\infty} \frac{1}{(\mu_\Delta(A))} \iint\limits_A |f(s,u) - L|^p \Delta s \ \Delta u = 0. \tag{16}$$

The set of all $p-$strongly $(W, \lambda, v)-$summable functions on $\mathbb{T}^2$ is denoted by $[W, \lambda, v]_{\mathbb{T}^2}^p$.

We need to emphasize that measure theory on time scales was first constructed by Guseinov [20] and *Lebesque* $\Delta -$ *integral* on time scales has been introduced by Cabada and Vivero [38].

Definition 4. *Let* $f : \mathbb{T}^2 \to \mathbb{R}$ *be a* $\Delta-$ *measurable function,* $\lambda, v \in \Lambda$. *We say that f is strongly* $(W, (\lambda, v)_h^\alpha)$*-summable function on* $\mathbb{T}^2$ *if there exists some* $L \in \mathbb{R}$ *such that*

$$\lim_{(t,r)\to\infty} \frac{1}{(\mu_\Delta(A))^\alpha} \iint\limits_A h(|f(s,u) - L|) \ \Delta s \ \Delta u = 0. \tag{17}$$

In this case we write $(W, (\lambda, v)_h^\alpha)_{\mathbb{T}^2} - \lim\limits_{(t,r)\to\infty} f(t,r) = L$. The set of all strongly $(W, (\lambda, v)_h^\alpha)_{\mathbb{T}^2}-$summable functions on $\mathbb{T}^2$ will be denoted by $[W, (\lambda, v)_h^\alpha]_{\mathbb{T}^2}$. If we take $h(x) = x^p$ $(0 < p < \infty)$ and $\alpha = 1$ then we get $[W, (\lambda, v)_p]_{\mathbb{T}^2}$, the set of all $p-$strongly $(W, \lambda, v)-$summable functions on $\mathbb{T}^2$ (see [14]).

Lemma 3. *Let* $f : \mathbb{T}^2 \to \mathbb{R}$ *be a* $\Delta-$ *measurable function and* $\Omega(t, r, \lambda, v, h) = \{(s, u) \in A : h(\ |f(s, u) - L|) \geq \epsilon\}$ *for* $\epsilon > 0$. *In this case , we have*

$$\begin{aligned} h(\mu_\Delta(\Omega(t, r, \lambda, v, h))) &\leq \frac{1}{\epsilon} \iint\limits_{\Omega(t,r,\lambda,v,h)} h(|f(s,u) - L|) \ \Delta s \ \Delta u && (18) \\ &\leq \frac{1}{\epsilon} \iint\limits_A h(|f(s,u) - L|) \ \ \Delta s \ \Delta u && (19) \end{aligned}$$

Proof. It can be proved by using similar method with [39].

Theorem 2. *Let* $f : \mathbb{T}^2 \to \mathbb{R}$ *be a* $\Delta-$ *measurable function,* $\lambda, v \in \Lambda$, $L \in \mathbb{R}$. *Then we get*

(i) If f is strongly $(W,(\lambda,v)_h^\alpha)_{\mathbb{T}^2}$ −summable function to L, then $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}-\lim_{(t,r)\to\infty} f(t,r)=L$.

(ii) If $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}-\lim_{(t,r)\to\infty} f(t,r)=L$, f is a bounded function and h unbounded modulus such that $h(x)-x\geq 0$, then f is strongly $(W,(\lambda,v)_h^\alpha)$−summable function to L.

Proof. (i) Let f is strongly $(W,(\lambda,v)_h^\alpha)$−summable function to L. For given $\epsilon>0$, let $\Omega(t,r,\lambda,v,h)=\{\ (s,u)\in A: h(|f(s,u)-L|)\geq\epsilon\ \}$ on time scale $\mathbb{T}^2$. Then, it follows from lemma 3

$$\varepsilon\ h(\mu_\Delta(\Omega(t,r,\lambda,v,h)))\leq\iint_A h(|f(s,u)-L|)\ \Delta s\ \Delta u.$$

Dividing both sides of the last equality by $h(\mu_\Delta(A)^\alpha)$ and taking limit as $(t,r)\to\infty$, we obtain

$$\begin{aligned}\varepsilon\quad&\lim_{(t,r)\to\infty}\frac{h(\mu_\Delta(\Omega(t,r,\lambda,v,h)))}{h((\mu_\Delta(A)^\alpha)}\\ \leq\quad&\lim_{(t,r)\to\infty}\frac{1}{h((\mu_\Delta(A)^\alpha)}\iint_A h(|f(s,u)-L|)\ \Delta s\ \Delta u=0\end{aligned}\tag{20}$$

which yields that $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}-\lim_{(t,r)\to\infty} f(t,r)=L$.

(ii) Let f be bounded and $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}$−statistically convergent to L on $\mathbb{T}^2$. Then, there exists a positive number M such that $|f(t,r)-L|\leq M$ for all $(t,r)\in\mathbb{T}^2$, and also

$$\lim_{(t,r)\to\infty}\frac{h(\mu_\Delta(\Omega(t,r,\lambda,v,h)))}{h((\mu_\Delta(A)^\alpha)}=0$$

where $\Omega(t,r,\lambda,v,h)=\{\ (s,u)\in A: h(|f(s,u)-L|)\geq\epsilon\ \}$ as stated before. Since

$$\begin{aligned}&\iint_A h(|f(s,u)-L|)\ \Delta s\ \Delta u\\ =\quad&\iint_{\Omega(t,r,\lambda,v,h)} h(|f(s,u)-L|)\ \ \Delta s\ \Delta u\end{aligned}\tag{21}$$

$$+ \iint\limits_{\mathbb{T}^2/\Omega(t,r,\lambda,v,h)} h(|f(s,u)-L|) \;\; \Delta s \; \Delta u \tag{22}$$
$$< \;\; (h(M)) \iint\limits_{\Omega(t,r,\lambda,v,h)} \Delta s \; \Delta u + \varepsilon \iint\limits_{\mathbb{T}^2\setminus\Omega(t,r,\lambda,v,h} \Delta s \; \Delta u$$
$$\leq \;\; (h(M) \; (h(\mu_\Delta(\Omega(t,r,\lambda,v,h)))) + \varepsilon \;\; (h(\mu_\Delta(A)))$$

we obtain

$$\lim_{(t,r)\to\infty} \frac{1}{h((\mu_\Delta(A)^\alpha)} \iint\limits_A h(|f(s,u)-L|) \;\; \Delta s \; \Delta u \tag{23}$$
$$\leq \;\; [(h(M)] \lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(\Omega(t,r,\lambda,v,h)))}{h((\mu_\Delta(A)^\alpha)} + \varepsilon \lim_{(t,r)\to\infty} \frac{h(\mu_\Delta(A))}{h((\mu_\Delta(A)^\alpha)}$$

Since $\epsilon > 0$ is arbitrary, the proof follows from (20) and (23).

Theorem 3. *Let f be a $\Delta-$ measurable function. Then $s_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha} - \lim\limits_{(t,r)\to\infty} f(t,r) = L$ if and only if there exists a $\Delta-$ measurable $\Omega \subseteq \mathbb{T}^2$ such that $\delta_{\mathbb{T}^2}^{(\lambda,v)_h^\alpha}(\Omega) = 1$ and $\lim\limits_{(t,r)\to\infty} h(|f(t,r)-L|) = 0$, $((t,r) \in \Omega(t,r,\lambda,v,h))$.*

Proof. It can be easily proved by using similar way in Theorem 3.9 of Turan and Duman (see [39]).

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