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Solution of Fourth Order Parabolic Partial Differential Equation Using Haar Wavelet and Finite-Difference Method

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Abstract. In this paper, we proposed a method based on Haar wavelet embedded with finite-difference approach for solving fourth order parabolic partial differential equation. We approximate space derivative by Haar wavelet and time derivative by finite-difference to find the solution. Further, to validate the accuracy and efficiency of the proposed method, we provide examples and compared our maximum absolute errors with existing methods such as parametric septic splines [7], sextic spline [8], parametric quintic spline [9] and Quintic B-Spline [10].

Key Words and Phrases: Haar wavelet, finite-difference, parabolic PDE.

2010 Mathematics Subject Classifications: 65D07, 65M12, 65M99, 65N35, 65N55, 65L10, 65L12.

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1. Introduction

We consider the following fourth order parabolic partial differential equation (PDE)

$$\frac{\partial^2 u}{\partial \tau^2} + \mu \frac{\partial^4 u}{\partial x^4} = f(x,\tau), \quad x \in (0,1), \tag{1}$$

with initial conditions

$$u(x,0) = g_0(x), \quad u_\tau(x,0) = g_1(x), \tag{2}$$

and boundary conditions,

$$u(0,\tau) = f_0(\tau), \quad u_{xx}(0,\tau) = p_0(\tau), \tau > 0. \tag{3}$$

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$$u(1, \tau) = f_1(\tau), \quad u_{xx}(1, \tau) = p_1(\tau), \tau > 0. \tag{4}$$

The equation of the form (1) occurs in the mathematical modelling of undamped transverse vibration of a flexible straight beam, μ is the ratio of flexural rigidity of the beam to its mass per unit length, u is the transverse displacement of the beam, x and τ are space and time variables, respectively. $g_0(x)$, $g_1(x)$ are the continuous functions of x and f_0, f_1, p_0, p_1 are the continuous function of τ.

Solution of fourth order parabolic PDE have been proposed by many researchers such as Fairweather and Gourlay [1] applied explicit and implicit finite difference method. High accuracy method was given by Douglas [2], stable implicit finite difference method was proposed by Evans and Yousif [3]. Adomain decomposition method was given by Wazwaz [4], Alternating group explicit method which achieved better accuracy level was given by Adomain and Rach [5], Jain et al. [6] proposed direct and splitting approach finite difference method and Khan and Sultana [7] presented the method based on parametric septic spline. Three level methods based on sextic spline and parametric quintic spline have been given by Khan et al. [8] and Aziz et al. [9]. Quintic B-Spline method was given by Siddiqui and Arshed [10]. In this paper, we proposed Haar wavelet for approximation of space derivative and finite difference method for time derivative to solve the fourth order parabolic PDE.

2. Haar Wavelet

Nowadays, the Haar wavelet is becoming very popular tool for solving differential equations. Many researcher proposed various techniques based on Haar wavelet to solve the ordinary and partial differential equations, fractional order differential equations as well as integral equations. Islam et al [11] solved second order parabolic PDE using Haar and Legendre wavelet, Lepik [12] solved PDE's using two dimensional Haar wavelet, Raza and Khan ([13], [14] and [15]) solved neutral DDE and higher order BVP's respectively using Haar wavelet, Kumar and Pandit [16] proposed composite numerical scheme for the coupled Burgers equation, Jiwari ([17], [18]) proposed quasi-linearization approach and hybrid numerical scheme respectively to solve the Burgers equation, Kumar and Pandit [19] proposed an efficient algorithm based on Haar wavelet to solve Fokker-Planck equations. More literature on wavelet is given by Ahmad and Shah [20] and Lepik and Hein [21]. The Haar wavelet family for $x \in [0,1]$ is defined as follows:

$$\mathcal{H}_i(x) = \begin{cases} 1, & \nu_1(i) \le x < \nu_2(i), \\ -1, & \nu_2(i) \le x < \nu_3(i), \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

The integration $\mathcal{P}_i(x)$ of Haar wavelet can be obtained as follows:

$$\mathcal{P}_i(x) = \begin{cases} x - \nu_1(i), & \nu_1(i) \le x < \nu_2(i), \\ \nu_3(i) - x, & \nu_2(i) \le x < \nu_3(i), \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

Further, the double integration $\mathcal{Q}_i(x)$ of Haar wavelet can be obtained as follows:

$$\mathcal{Q}_i(x) = \begin{cases} \frac{1}{2}(x - \nu_1(i))^2, & \nu_1(i) \le x < \nu_2(i), \\ \frac{1}{4m^2} - \frac{1}{2}(\nu_3(i) - x)^2, & \nu_2(i) \le x < \nu_3(i), \\ \frac{1}{4m^2}, & \nu_3(i) \le x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

The triple integration of Haar wavelet can be obtained as follows:

$$\mathcal{R}_i(x) = \begin{cases} 0, & x < \nu_1(i), \\ \frac{1}{3!}[x - \nu_1(i)]^3, & \nu_1(i) \le x < \nu_2(i), \\ \frac{1}{3!}[(x - \nu_1(i))(x - \nu_2(i))^3], & \nu_2(i) \le x < \nu_3(i), \\ \frac{1}{3!}[(x - \nu_1(i))(x - \nu_2(i))^3 + (x - \nu_3(i))^3], & \nu_3(i) \le x. \end{cases} \tag{8}$$

The fourth order integration of Haar wavelet is given by

$$\mathcal{S}_i(x) = \begin{cases} 0, & x < \nu_1(i), \\ \frac{1}{4!}[x - \nu_1(i)]^4, & \nu_1(i) \le x < \nu_2(i), \\ \frac{1}{4!}[(x - \nu_1(i))(x - \nu_2(i))^4], & \nu_2(i) \le x < \nu_3(i), \\ \frac{1}{4!}[(x - \nu_1(i))(x - \nu_2(i))^4 + (x - \nu_3(i))^4], & \nu_3(i) \le x. \end{cases} \tag{9}$$

The collocation grid is given as

$$X(i) = \frac{2i-1}{m}, \quad i = 1, 2, \ldots, m$$

and the time disctetization is given by

$$\tau(i) = \frac{i}{N}, \quad i = 0, 1, 2, \ldots, N.$$

3. Method for Solution

To solve the fourth order parabolic PDE, we apply finite difference method to discretize time derivative and Haar wavelet to approximate the space derivative. Let us consider the equation (1) with initial and boundary conditions (2)-(4). To solve the problem, we assume that

$$u_{xxxx}(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{H}_i(x). \tag{10}$$

Now integrating equation (10) four times from 0 to x, we get

$$u_{xxx}(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{P}_i(x) + u_{xxx}(0,\tau), \tag{11}$$

$$u_{xx}(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(x) + xu_{xxx}(0,\tau) + u_{xx}(0,\tau), \tag{12}$$

$$u_x(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{R}_i(x) + \frac{x^2}{2}u_{xxx}(0,\tau) + xu_{xx}(0,\tau) + u_x(0,\tau), \tag{13}$$

and

$$u(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{S}_i(x) + \frac{x^3}{6}u_{xxx}(0,\tau) + \frac{x^2}{2}u_{xx}(0,\tau) + xu_x(0,\tau) + u(0,\tau). \tag{14}$$

Now, to find the $u_{xxx}(0,\tau)$ integrating equation (11) from 0 to 1 and substituting the values from equations (3) and (4), we get

$$u_{xxx}(0,\tau) = p_1(\tau) - p_0(\tau) - \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(1) \tag{15}$$

Putting the value of $u_{xxx}(0,\tau)$ in equation (13), we get

$$\begin{aligned} u_x(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{R}_i(x) + \frac{x^2}{2}(p_1(\tau) - p_0(\tau) - \\ - \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(1)) + xu_{xx}(0,\tau) + u_x(0,\tau). \end{aligned} \tag{16}$$

To find the $u_x(0,\tau)$ integrating equation (16) from 0 to 1 and substituting the values from equations (3) and (4), we get

$$u_x(0,\tau) = f_1(\tau) - f_0(\tau) - \frac{1}{2}p_0(\tau) - \frac{1}{6}(p_1(\tau) - p_0(\tau) - \\ - \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(1)) - \sum_{i=1}^{N} a_i(\tau)\mathcal{S}_i(1). \tag{17}$$

Putting the values of $u_x(0,\tau)$ and $u_{xxx}(0,\tau)$ from equation (17) and (15) in equation (14), we get

$$u(x,\tau) = \sum_{i=1}^{N} a_i(\tau)\mathcal{S}_i(x) + \frac{x^3}{6}(p_1(\tau) - p_0(\tau) - \\ - \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(1)) + \frac{x^2}{2}u_{xx}(0,\tau) + x(f_1(\tau) - f_0(\tau) \\ - \frac{1}{2}p_0(\tau) - \frac{1}{6}(p_1(\tau) - p_0(\tau) - \sum_{i=1}^{N} a_i(\tau)\mathcal{Q}_i(1)) - \\ - \sum_{i=1}^{N} a_i(\tau)\mathcal{S}_i(1)) + u(0,\tau). \tag{18}$$

On simplifying (18) and using the boundary conditions (3) and (4), we get

$$u(x,\tau) = (1-x)f_0(\tau) + xf_1(\tau) + p_0(\tau)(-\frac{x}{3} + \frac{x^2}{2} - \frac{x^3}{6}) + p_1(\tau)(-\frac{x}{6} + \frac{x^3}{6}) \\ + \sum_{i=1}^{N} a_i(\tau)(\mathcal{S}_i(x) - x\mathcal{S}_i(1) + \mathcal{Q}_i(1)(\frac{x}{6} - \frac{x^3}{6})). \tag{19}$$

Equation (19) is Haar wavelet approximation of the fourth order parabolic PDE (1) with initial condition (2) and boundary conditions (3), (4). Now, to find the unknown Haar wavelet coefficients $a_i(\tau)$, we discretize time derivative by finite-difference method as follows:

$$u_{\tau\tau}(x_k,\tau_{j+1}) = \frac{u(x_k,\tau_{j+1}) - 2u(x_k,\tau_j) + u(x_k,\tau_{j-1})}{\triangle\tau^2}. \tag{20}$$

Now using (10) and (20) in (1), we get,

$$\frac{u(x_k,\tau_{j+1}) - 2u(x_k,\tau_j) + u(x_k,\tau_{j-1})}{\triangle\tau^2} + \mu\sum_{i=1}^{N} a_i(\tau_{j+1})\mathcal{H}_i(x_k) = f(x_k,\tau_j). \tag{21}$$

Using the values of $u(x_k,\tau_{j+1})$, $u(x_k,\tau_j)$ and $u(x_k,\tau_{j-1})$ from equation (19) in equation (21), which are solution of (1) at different time levels $j+1$, j and $j-1$ respectively, then we get

$$(1-x)f_0(\tau_{j+1}) + xf_1(\tau_{j+1}) + p_0(\tau_{j+1})(-\frac{x}{3}+\frac{x^2}{2}-\frac{x^3}{6})+$$
$$+p_1(\tau_{j+1})(-\frac{x}{6}+\frac{x^3}{6}) + \sum_{i=1}^{N} a_i(\tau_{j+1})(\mathcal{S}_i(x) - x\mathcal{S}_i(1)+$$
$$+\mathcal{Q}_i(1)(\frac{x}{6}-\frac{x^3}{6})) - 2((1-x)f_0(\tau) + xf_1(\tau) + p_0(\tau)(-\frac{x}{3}+\frac{x^2}{2}-\frac{x^3}{6})+$$
$$+p_1(\tau)(-\frac{x}{6}+\frac{x^3}{6}) + \sum_{i=1}^{N} a_i(\tau)(\mathcal{S}_i(x) - x\mathcal{S}_i(1) + \mathcal{Q}_i(1)(\frac{x}{6}-\frac{x^3}{6})))+$$
$$+(1-x)f_0(\tau_{j-1}) + xf_1(\tau_{j-1}) + p_0(\tau_{j-1})(-\frac{x}{3}+\frac{x^2}{2}-\frac{x^3}{6})+$$
$$+p_1(\tau_{j-1})(-\frac{x}{6}+\frac{x^3}{6}) + \sum_{i=1}^{N} a_i(\tau_{j-1})(\mathcal{S}_i(x) - x\mathcal{S}_i(1) + \mathcal{Q}_i(1)(\frac{x}{6}-\frac{x^3}{6}))+$$
$$+\triangle\tau^2\mu\sum_{i=1}^{N} a_i(\tau_{j+1})\mathcal{H}_i(x_k) = \triangle\tau^2 f(x_k,\tau_j) \tag{22}$$

On simplification, we get the following system of linear equation,

$$\sum_{i=1}^{N} a_i(\tau_{j+1})(\mathcal{W}_i(x_k) + \triangle\tau^2\mu\mathcal{H}_i(x_k)) - 2\sum_{i=1}^{N} a_i(\tau_j)\mathcal{W}_i(x_k) + \sum_{i=1}^{N} a_i(\tau_{j-1})\mathcal{W}_i(x_k)$$
$$= \triangle\tau^2 f(x_k,\tau_j) - A(x_k,\tau_{j+1}) + 2A(x_k,\tau_j) - A(x_k,\tau_{j-1}). \tag{23}$$

where
$\mathcal{W} = \mathcal{S}_i(x) - x\mathcal{S}_i(1) + \mathcal{Q}_i(1)(\frac{x}{6}-\frac{x^3}{6})$, and
$\mathcal{A}(x_k,\tau_j) = (1-x_k)f_0(\tau_j) + x_kf_1(\tau_j) + p_0(\tau_j)(-\frac{x_k}{3}+\frac{x_k^2}{2}-\frac{x_k^3}{6}) + p_1(\tau_j)(-\frac{x_k}{6}+\frac{x_k^3}{6})$

On expanding the system (23), we get the following matrix formulation

$$\begin{pmatrix} -2\mathcal{W} & \mathcal{W}+\mu\triangle\tau^2\mathcal{H} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \mathcal{W} & -2\mathcal{W} & \mathcal{W}+\mu\triangle\tau^2\mathcal{H} & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \mathcal{W} & -2\mathcal{W} & \mathcal{W}+\mu\triangle\tau^2\mathcal{H} & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \mathcal{W} & -2\mathcal{W} & \mathcal{W}+\mu\triangle\tau^2\mathcal{H} & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \mathcal{W} & -2\mathcal{W} & \mathcal{W}+\mu\triangle\tau^2\mathcal{H} \\ 0 & 0 & 0 & . & . & . & 0 & \mathcal{W} & -2\mathcal{W} \end{pmatrix} \times$$

$$\times \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ . \\ . \\ . \\ . \\ \mathcal{V}_{N-1} \\ \mathcal{V}_N \end{pmatrix} \tag{24}$$

where a_1, a_2 ... a_N and $\mathcal{V}_1$,....$\mathcal{V}_N$ are vectors of size $N \times 1$. We solve system (24) with the help of equation (2) and obtained unknown finite difference Haar wavelet coefficients $a_i(\tau_j)$ at each time level and then we put them in equation (19) to find the approximate finite-difference Haar wavelet solution of equation (1) with initial condition (2) and boundary conditions (3) and (4).

3.1. Convergence and Error Analysis

Lemma 1. *Let $u \in L^2([0,1])$ with bounded derivative and $u(x) = \sum_{i=1}^{N} a_i \mathcal{H}_i(x)$is the Haar wavelet series then, the Haar wavelet coefficient a_i satisfies the following inequality*

$$\|a_i\|^2 \leq M2^{-(3j-2)/2}, \quad where \quad |u'(x)| \leq M \tag{25}$$

Proof: See [16]

Lemma 2. *If $u(x)$ is the exact and $u_J(x)$ is the approximate solution of the equation (1) then error norm satisfies the following inequality*

$$\|E_J\|^2 = \|u(x) - u_J(x)\| \leq M\sqrt{C}\frac{2^{-3(2^j)-1}}{1-2^{-3/2}}, \quad where \quad |u'(x)| \leq M \tag{26}$$

Proof: See [16]

4. Numerical Examples

In this section, we demonstrate two examples of fourth order parabolic PDE to show the applicability, accuracy and efficiency of the Haar wavelet finite difference method. We have computed maximum absolute error (MAE) and compared with parametric septic splines [7], sextic spline [8], parametric quintic spline method [9] and Quintic B-Spline method [10].

Problem 1. Consider the equation (1) with initial and boundary conditions (2)-(4), where $\mu = 1$, $f(x, \tau) = (\pi^4 - 1)sin(\pi x)cos(\tau)$, $g_0(x) = sin(\pi x)$, $g_1(x) = 0$, $f_0(\tau) = f_1(\tau) = p_0(\tau) = p_1(\tau) = 0$, $\tau \geq 0$.
The exact solution is

$$u(x, \tau) = sin(\pi x)cos(\tau).$$

MAE obtained by finite-difference Haar wavelet method with different resolutions level have been given in the Table 1.1. Also, we have plotted the graph of exact and finite-difference Haar wavelet solution which is given in Figures 1-5.

Table 1.1 MAE obtained by finite-difference Haar wavelet method for different values of space resolution level J and time step level N for problem 1.

J/N	10	20	50	100	500
3	1.4401e-09	3.4167e-12	9.6952e-16	1.1342e-15	5.5816e-15
4	7.7223e-10	1.8318e-12	5.2234e-16	1.1269e-15	5.0187e-15
5	3.9921e-10	9.4687e-13	4.1200e-16	7.3238e-16	3.2813e-15
6	2.0288e-10	4.8119e-13	1.1604e-15	2.1908e-15	4.4784e-15
7	1.0226e-10	2.4254e-13	1.4687e-15	1.1858e-15	6.2811e-15
8	5.1336e-11	1.2175e-13	4.5647e-16	6.0378e-16	4.8669e-15

Table 1.2 MAE obtained by parametric septic splines [7], sextic spline [8], parametric quintic spline method [9] and Quintic B-Spline method [10] for time level 200 and different space level J for problem 1.

J	[10]	[7]	[8]	[9]
10	—	2.0500e-6	2.1300e-05	4.4000e-04
16	—	4.0400e-7	9.0700e-06	7.2000e-05
25	1.8155e-03	—	—	—
65	5.6442e-03	—	—	—
100	3.7583e-03	—	—	—

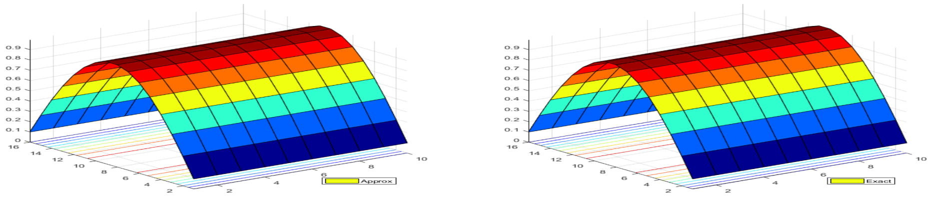

Figure 1. Surface plot of exact and finite-difference Haar wavelet solution of problem 1 with $J = 3$ and time step $N = 10$.

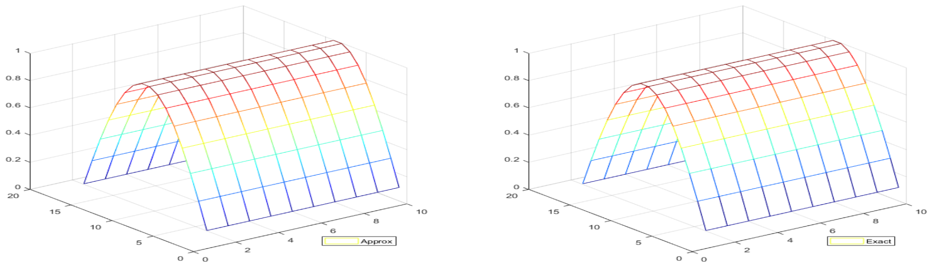

Figure 2. Mesh plot of exact and finite-difference Haar wavelet solution of problem 1 with $J = 3$ and time step $N = 10$.

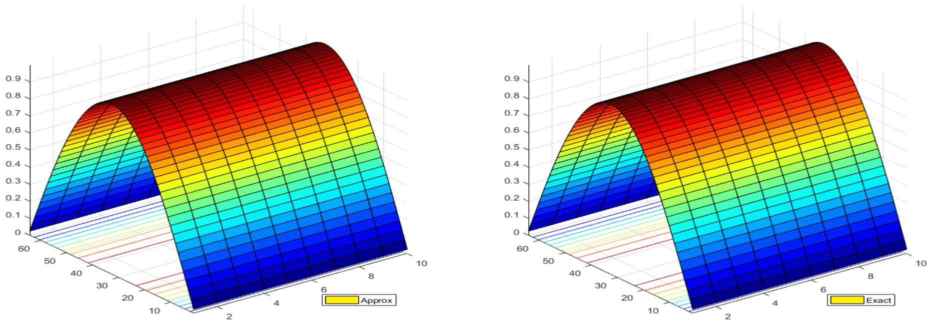

Figure 3. Surface plot of exact and finite-difference Haar wavelet solution of problem 1 with $J = 4$ and time step $N = 10$.

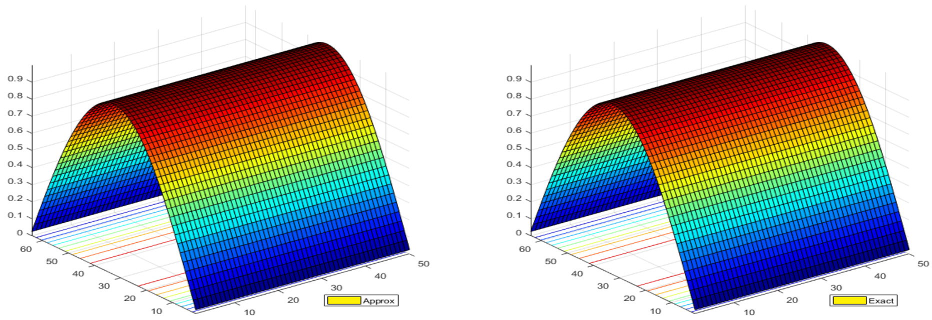

Figure 4. Surface plot of finite-difference Haar wavelet solution of problem 1 with $J = 5$ and time step $N = 50$.

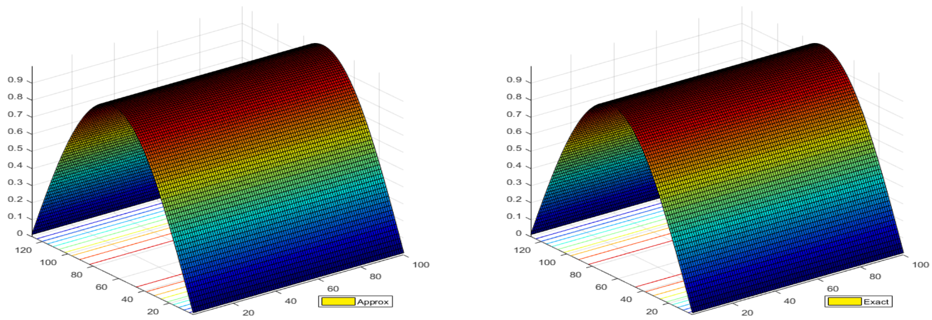

Figure 5. Surface plot of exact and finite-difference Haar wavelet solution of problem 1 with $J = 5$ and time step $N = 100$.

Problem 2. Consider the equation (1) with initial and boundary conditions (2)- (4), where $\mu = 1$, $f(x,\tau) = (\pi^4 + 1)e^{\tau} sin(\pi x)$, $g_0(x) = sin(\pi x)$, $g_1(x) = sin(\pi x)$, $f_0(\tau) = f_1(\tau) = p_0(\tau) = p_1(\tau) = 0$, $\tau \geq 0$.
The exact solution is

$$u(x,\tau) = e^{\tau} sin(\pi x).$$

MAE obtained by finite-difference Haar wavelet method with different resolutions level have been given in the Table 2.1. Further, the surface and mesh plot of exact and finite-difference Haar wavelet solution is given in Figures 6-10.

Table 2.1 MAE obtained by finite-difference Haar wavelet method for different values of space resolution level J and time step level N for problem 2.

J/N	10	20	50	100	500
3	6.2213e-08	9.7187e-10	3.9807e-12	6.2942e-14	4.6066e-15
4	3.1220e-08	4.8770e-10	1.9981e-12	3.1730e-14	5.4745e-15
5	1.5624e-08	2.4407e-10	1.0001e-12	1.5796e-14	3.8624e-15
6	7.8137e-09	1.2206e-10	5.0075e-13	9.4851e-15	4.5439e-15
7	3.9071e-09	6.1034e-11	2.5001e-13	5.1016e-15	5.4639e-15
8	1.9536e-09	3.0518e-11	1.2520e-13	2.3277e-15	5.7485e-15

Table 2.2 MAE obtained by Quintic B-Spline method [10] for time step level 20 and different values of space resolution level J for problem 2.

J	[10]
10	9.9400e-04
50	7.7250e-04
100	9.7375e-05

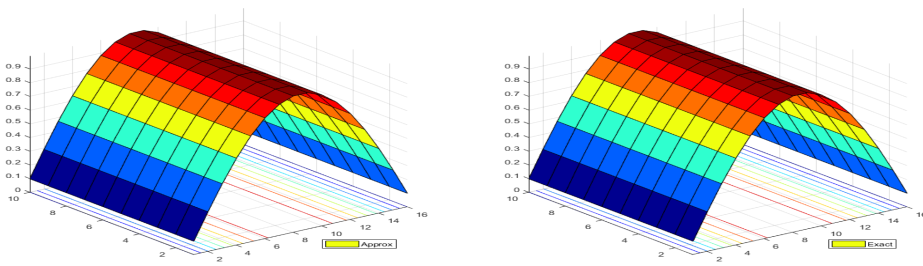

Figure 6. Surface plot of exact and finite-difference Haar wavelet solution of problem 2 with $J = 3$ and time step $N = 10$.

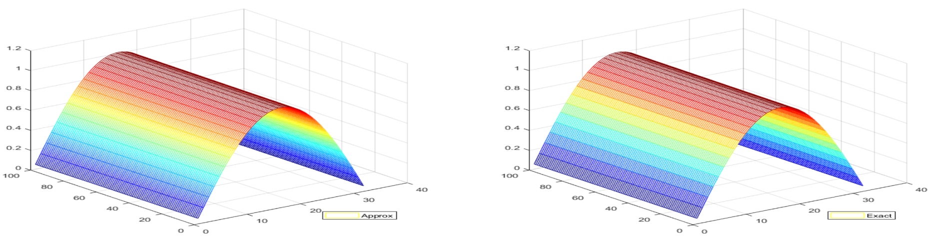

Figure 7. Mesh plot of exact and finite-difference Haar wavelet solution of problem 2 with $J = 4$ and time step $N = 100$.

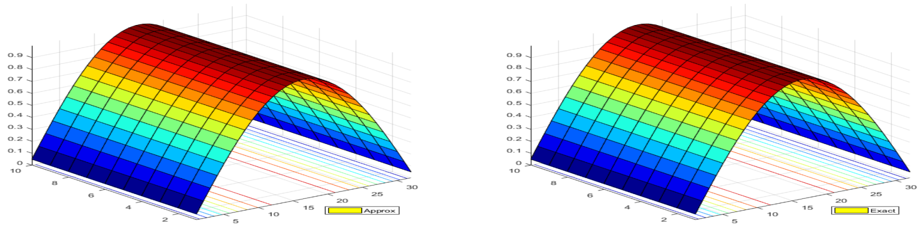

Figure 8. Surface plot of exact and finite-difference Haar wavelet solution of problem 2 with $J = 4$ and time step $N = 10$.

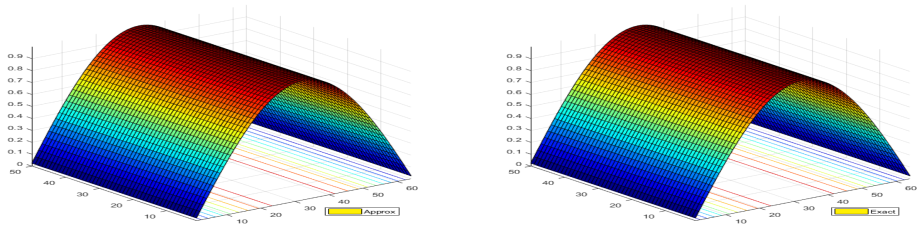

Figure 9. Surface plot of finite-difference Haar wavelet solution of problem 2 with $J = 5$ and time step $N = 50$.

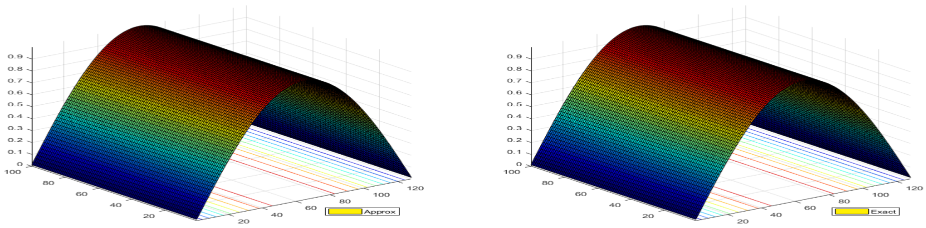

Figure 10. Surface plot of exact and finite-difference Haar wavelet solution of problem 2 with $J = 6$ and time step $N = 100$.

Conclusion

We have solved fourth order parabolic PDE using finite-difference Haar wavelet method and obtain the approximate solution. We compared our results with the existing methods such as parametric septic splines [7], sextic spline [8], parametric quintic spline method [9] and Quintic B-Spline method [10]. The tables $1.1 - 2.2$ clearly indicate that Haar wavelet produces better results.

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