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Numerical Algorithm for Coupled Viscous Burger's Equation Using Quasi-variable Meshes Compact Operators

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Abstract. We describe a quasi-variable meshes implicit compact finite-difference discretization having an accuracy of order four in the spatial direction and second-order in the temporal direction for obtaining numerical solution values of coupled viscous Burger's equations. The new difference scheme is derived on a quasi-variable meshes network to the extent that the magnitude of local truncation error of the high accuracy compact formulation remains unchanged even in the case of a uniform meshes network. Practically, quasi-variable meshes schemes yield a more precise solution compared with uniform meshes schemes of the same magnitude. The computational results with coupled Burger's equations are obtained using quasi-variable meshes high-order compact scheme and compared with a numerical solution using uniform meshes high-order schemes to demonstrate capability and accuracy.

Key Words and Phrases: compact scheme, Quasi-variable mesh, Coupled viscous Burger's equations, Maximum-absolute-errors.

2010 Mathematics Subject Classifications: 65M60, 65M12, 65M06

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1. Introduction

The parabolic partial differential equations assume a significant role in engineering and physical sciences such as convection effect, diffusion transport interaction, option pricing, fluid flow, and image processing. In the convection-diffusion phenomenon, when convection is significant comparing with diffusion, for as much second-order discretization of the convection term gives rise to oscillatory solution values. Such an oscillatory behavior can be resolved by unrealistic small mesh step-size or by a high-order accurate method. Many high-order schemes

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have been developed on uniformly spaced mesh-points, and nearly satisfactory results have been obtained earlier. In the finite differencing approaches, there is a direct connection between the truncation error and mesh spacing. The supra-convergence of discretization errors indicates that satisfactory numerical solution values can be computed when truncation error exhibits a lower order truncation error [1, 2, 3, 4]. There are two major limitations to finite-difference discretization on uniformly spaced mesh-points. The type of mesh network involves information about truncation error, and solution values in the finite-difference discretization relies on the mesh step-size as well as derivatives of a function. As a result, uniform distribution of local truncation error terms is not possible on uniform meshes. To gain uniform distribution of local truncation error, we need better networks in highly deviant derivative and coarse meshes for highly analytic function [5, 6]. Such an arrangement of meshes disperses the error uniformly all over the region and yields a more precise resolution. Therefore, high accuracy finite-difference discretization developed on a non-uniformly spaced mesh-point provides stable and accurate solution values.

The present work's contribution is to describe a high-order accurate, compact scheme and analyze the effect of variable mesh spacing on the truncation error (local). It is shown that the new scheme is fourth-order accurate in the spatial direction and is second-order exact in the time axis for both uniform mesh spacing as well as quasi-variable mesh spacing. The discretization takes one central mesh-point and two adjacent meshes at any time level, producing a compact scheme that is straightforward to implement. In the next section, we discuss quasi-variable meshes and algorithms to determine them. A two-level three-point implicit compact scheme of high-order has been presented in section 3. Numerical illustration of quasi-variable meshes compact scheme has been presented in section 4 and finalized with remarks.

2. Quasi-variable mesh network and compact operators

For the discretization purpose, the domain $\{(x,t); 0 \le x \le 1, t > 0\}$ is partitioned as $\{(x_n, t_j) : j = 0(1)J, n = 0(1)N+1\}$, where $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$, $t_j = jk, j = 0(1)J$, with J, N are positive integers and $k = 1/J$ is a fixed time-step. Let $h_n = x_n - x_{n-1}, n = 1(1)N+1$ be the unequal step-lengths along spatial direction. The subsequent step-length is obtained by $h_{n+1} = (1 + \mu h_n)h_n, n = 1(1)N$. Since, the length of diffusion space is one, thus $\sum_{n=1}^{N+1} h_n = 1$ and for a given value of mesh parameter μ, it is quite easy to find initial and subsequent mesh step-size. For particular value $\mu = 0$, the meshes are evenly spaced and $h = h_n$ for all n. The procedure to generate unequal step-length in spatial direction is presented in the following algorithm:

$r_1 = 1; r_2 = 1 + \mu; for(n = 3;\;\; n \leq 1 + N; n++)\;\; r_n = r_{n-1}(\mu r_{n-1} + 1);$
$sum = 0; for(n = 3;\;\; n \leq 1 + N; n++) sum+ = r_n;$
$h_1 = (x_{N+1} - x_0)/sum; for(n = 1;\;\; n < N; n++)\;\; h_{n+1} = (1 + \mu h_n/h_1)h_n;$
$x_{N+1} = 1; x_0 = 0; for(n = 1;\;\; n \leq N; n++) x_n = x_{n-1} + h_n; .$

Now, let us take a uniform segmentation of the domain $\mathcal{S} = [0, 1] = \{s_n = nh : n = 0(1)N + 1\}$, with the equal step-size $h = 1/(1 + N)$. Since $h_n > 0\ \forall n$, therefore, $h_{n+1} = h_n(1 + \mu h_n/h_1) > h_n$ for $\mu > 0$. Therefore, the finite sequence $\{h_n\}_{n=1}^{N+1}$ of mesh-step size is monotonic for $\mu > 0$. The monotonicity of mesh step sequence permits us to construct a one-one onto mapping

$$\mathcal{F} : \mathcal{S} \longrightarrow \Omega \text{ such that } \mathcal{F}(s_n) = x_n, n = 0(1)N + 1, \tag{1}$$

and the Jacobian $J(s) = d\mathcal{F}(s)/ds$ is a bounded function ($0 < r \leq J(s) \leq R < \infty,\ \forall s \in \mathcal{S}$). Thus,

$$J(s) > 0 \Rightarrow \frac{d\mathcal{F}(s)}{ds} > 0 \Rightarrow \frac{\mathcal{F}(s_n) - \mathcal{F}(s_{n+1})}{s_n - s_{n+1}} > 0 \Rightarrow \frac{x_n - x_{n+1}}{nh - (n+1)h} > 0. \tag{2}$$

This implies, $h_{n+1} > 0$ for all integer values of n. Also, $J(s) \leq R \Rightarrow \dfrac{d\mathcal{F}(s)}{ds} \leq R \Rightarrow \dfrac{x_{n+1} - x_n}{(n+1)h - nh} \leq R$. That is, $0 < h_{n+1} \leq Rh\ \forall\ n$. Consequently, we find that $\max_n |h_{n+1}| \leq Rh$, and $\| \boldsymbol{h} \|_\infty \leq Rh = \dfrac{R}{N+1} \leq \dfrac{R}{N}$, where $\boldsymbol{h} = [h_1, h_2, \cdots, h_n]$ is step-length vector. Therefore,

$$\| \boldsymbol{h} \|_\infty = O(h) = O\left(\frac{1}{N}\right), \text{ and if } N \to \infty,\ \ \| \boldsymbol{h} \|_\infty \to 0. \tag{3}$$

Such a quasi-variable mesh spacing permits three-points second-order accurate discretization to the second-order partial derivatives $U^{xx} = \partial^2 U/\partial x^2$. Private communication between Samarskii and Saul'yev on variable meshes can be found in the literature [7, 8]. Discussion of this type of mesh network in the circumstance of electrochemical simulation, ocean circulation wind-driven model, and convection-dominated phenomenon can be noticed in [9, 10, 11].

To derive the high-order compact scheme, we need to construct operators that approximate first and second-order derivatives along the spatial direction in such a manner that it uses the minimum number of stencils. At the mesh-point (x_n, t_j), let $U_{n,j} = U(x_n, t_j), V_{n,j} = V(x_n, t_j)$ and $u_{n,j}, v_{n,j}$ represents the exact and approximate solution values of the coupled viscous Burger's equations. We shall denote

$$U^x_{n,j} = \left(\frac{\partial U}{\partial x}\right)_{(x_n,t_j)}, U^{xx}_{n,j} = \left(\frac{\partial^2 U}{\partial x^2}\right)_{(x_n,t_j)}, U^t_{n,j} = \left(\frac{\partial U}{\partial t}\right)_{(x_n,t_j)},$$
$$V^x_{n,j} = \left(\frac{\partial V}{\partial x}\right)_{(x_n,t_j)}, V^{xx}_{n,j} = \left(\frac{\partial^2 V}{\partial x^2}\right)_{(x_n,t_j)}, V^t_{n,j} = \left(\frac{\partial V}{\partial t}\right)_{(x_n,t_j)}.$$

By the assistance of linear combinations of solution values $U_{n,j}$ and $U_{n\pm1,j}$ evaluated at the central mesh-point (x_n, t_j) and their neighbouring mesh-points $(x_{n\pm1}, t_j)$ respectively, one obtains

$$\mathcal{F}_x U_{n,j} = \frac{1}{(1+\mu h_n)(2+\mu h_n)} U_{n+1,j} + \frac{\mu h_n}{\mu h_n + 1} U_{n,j} - \frac{1+\mu h_n}{\mu h_n + 2} U_{n-1,j}, \quad (4)$$
$$\mathcal{S}_x U_{n,j} = \frac{2}{(1+\mu h_n)(2+\mu h_n)} U_{n+1,j} - \frac{2}{\mu h_n + 1} U_{n,j} + \frac{2}{\mu h_n + 2} U_{n-1,j}, \quad (5)$$
$$\mathcal{F}_t U_{n,j} = U_{n,j+1} - U_{n,j}. \quad (6)$$

Then, the application of series expansion yields

$$\mathcal{F}_x U_{n,j} = h_n U^x_{n,j} + O(h_n^3),\ \mathcal{S}_x U_{n,j} = h_n^2 U^{xx}_{n,j} + O(h_n^4),\ \mathcal{F}_t U_{n,j} = k U^t_{n,j} + O(k^2), \quad (7)$$

and in particular when $\mu = 0$, the operator $\mathcal{S}_x$ reduced to the of well known second central-difference operator $\delta^2 U_{n,j} = -2U_{n,j} + U_{n-1,j} + U_{n+1,j}$ commonly applied to discretize the diffusion term when a uniform mesh step-size is taken into the consideration. Similarly, $\mathcal{F}_x$ represents the twice multiple of the composite of averaging and central-difference operators. The application of such operators gives rise to *supra*-convergent scheme and is useful for solving fully nonlinear parabolic equations in one-dimension, with a lower order of accuracy.

3. Difference scheme for coupled viscous Burger's equations

Burger's equation describes a mathematical model of sedimentation or emergence of volume concentration to different sorts of particles in liquid suspensions, or colloids by the impact of gravitational attraction [12]. A differential quadrature formula was applied to solve coupled Burger's equation by Mittal and Jiwari [13]. Recently, He and Tang [14] described a lattice Boltzmann method, and high accuracy compact numerical scheme for the Burgers' equations was obtained in [15]. The viscous Burgers' model is presented as

$$-\epsilon U^{xx} + 2UU^x + \alpha(UV^x + VU^x) + U^t = 0, \quad (8)$$

$$-\epsilon V^{xx} + 2VV^x + \beta(UV^x + VU^x) + V^t = 0, \quad t \geq 0, \quad x \in [0,1], \tag{9}$$

along with the initial data

$$U(x,0) = \phi_1(x), \quad V(x,0) = \phi_2(x), \tag{10}$$

and the end-point values

$$U(0,t) = F_1(t), \quad V(0,t) = G_1(t), \quad U(1,t) = F_2(t), \quad V(1,t) = G_2(t), \tag{11}$$

where $0 < \epsilon \ll 1$, defines viscosity and $R_e = 1/\epsilon$ is the cell Reynolds number.

Let $U_{n,j}$ and $V_{n,j}$ be the specific arrangement estimations of $U(x,t)$ and $V(x,t)$ at the mesh-point (x_n, t_j). Then, the high-order discretization is defined in the following manner:

$$\tilde{t}_j = \zeta t_{j+1} + (1-\zeta) t_j, \tag{12}$$

$$\begin{bmatrix} \widetilde{U}_{n+\tau,\ j} \\ \widetilde{V}_{n+\tau,\ j} \end{bmatrix} = \zeta \begin{bmatrix} U_{n+\tau,\ j+1} \\ V_{n+\tau,\ j+1} \end{bmatrix} + (1-\zeta) \begin{bmatrix} U_{n+\tau,\ j} \\ V_{n+\tau,\ j} \end{bmatrix}, \tau = 0, \pm 1, \tag{13}$$

$$\begin{bmatrix} \widetilde{U}^t_{n+\tau,\ j+1} \\ \widetilde{V}^t_{n+\tau,\ j+1} \end{bmatrix} = \frac{1}{k} \begin{bmatrix} U_{n+\tau,\ j+1} - U_{n+\tau,\ j} \\ V_{n+\tau,\ j+1} - V_{n+\tau,\ j} \end{bmatrix}, \tau = 0, \pm 1, \tag{14}$$

$$\begin{bmatrix} \widetilde{U}^x_{n,\ j} \\ \widetilde{U}^x_{n+1,\ j} \\ \widetilde{U}^x_{n-1,\ j} \end{bmatrix} = \boldsymbol{\mathcal{M}} \begin{bmatrix} \widetilde{U}_{n,\ j} \\ \widetilde{U}_{n+1,\ j} \\ \widetilde{U}_{n-1,\ j} \end{bmatrix}, \begin{bmatrix} \widetilde{V}^x_{n,\ j} \\ \widetilde{V}^x_{n+1,\ j} \\ \widetilde{V}^x_{n-1,\ j} \end{bmatrix} = \boldsymbol{\mathcal{M}} \begin{bmatrix} \widetilde{V}_{n,\ j} \\ \widetilde{V}_{n+1,\ j} \\ \widetilde{V}_{n-1,\ j} \end{bmatrix}, \tag{15}$$

$$\begin{bmatrix} \tilde{\psi}_{n\pm1,j} \\ \tilde{\varphi}_{n\pm1,j} \end{bmatrix} = \begin{bmatrix} \widetilde{U}^t_{n\pm1,\ j} + 2\widetilde{U}_{n\pm1,\ j}\widetilde{U}^x_{n\pm1,\ j} + \alpha(\widetilde{U}_{n\pm1,\ j}\widetilde{V}^x_{n\pm1,\ j} + \widetilde{V}_{n\pm1,\ j}\widetilde{U}^x_{n\pm1,\ j}) \\ \widetilde{V}^t_{n\pm1,\ j} + 2\widetilde{V}_{n\pm1,\ j}\widetilde{V}^x_{n\pm1,\ j} + \beta(\widetilde{U}_{n\pm1,\ j}\widetilde{V}^x_{n\pm1,\ j} + \widetilde{V}_{n\pm1,\ j}\widetilde{U}^x_{n\pm1,\ j}) \end{bmatrix}, \tag{16}$$

$$\begin{bmatrix} \widetilde{\widetilde{U}}^x_{n,\ j} \\ \widetilde{\widetilde{V}}^x_{n,\ j} \end{bmatrix} = \begin{bmatrix} \widetilde{U}^x_{n,\ j} - \vartheta h_n(\tilde{\psi}_{n+1,\ j} - \tilde{\psi}_{n-1,\ j}) \\ \widetilde{V}^x_{n,j} - \vartheta h_n(\tilde{\vartheta}_{n+1,\ j} - \tilde{\vartheta}_{n-1,\ j}) \end{bmatrix}, \tag{17}$$

$$\begin{bmatrix} \widetilde{\widetilde{\psi}}^x_{n,j} \\ \widetilde{\widetilde{\varphi}}^x_{n,j} \end{bmatrix} = \begin{bmatrix} \widetilde{U}^t_{n,\ j} + 2\widetilde{U}_{n,\ j}\widetilde{\widetilde{U}}^x_{n,\ j} + \alpha(\widetilde{U}_{n,\ j}\widetilde{\widetilde{V}}^x_{n,\ j} + \widetilde{V}_{n,\ j}\widetilde{\widetilde{U}}^x_{n,\ j}) \\ \widetilde{V}^t_{n,\ j} + 2\widetilde{V}_{n,\ j}\widetilde{\widetilde{V}}^x_{n,\ j} + \beta(\widetilde{U}_{n,\ j}\widetilde{\widetilde{V}}^x_{n,\ j} + \widetilde{V}_{n,\ j}\widetilde{\widetilde{U}}^x_{n,\ j}) \end{bmatrix}, \tag{18}$$

$$\begin{bmatrix} \epsilon \mathcal{S}_x \widetilde{U}_{n,j} \\ \epsilon \mathcal{S}_x \widetilde{V}_{n,j} \end{bmatrix} = h_n^2 \begin{bmatrix} \beta_1 \tilde{\psi}_{n+1,\ j} + \beta_0 \tilde{\tilde{\psi}}_{n,\ j} + \beta_2 \tilde{\psi}_{n-1,\ j} \\ \beta_1 \tilde{\varphi}_{n+1,\ j} + \beta_0 \tilde{\tilde{\varphi}}_{n,\ j} + \beta_2 \tilde{\varphi}_{n-1,\ j} \end{bmatrix} + \begin{bmatrix} O(h_n^2 k^2 + h_n^4 k + h_n^6) \\ O(h_n^2 k^2 + h_n^4 k + h_n^6) \end{bmatrix}, \tag{19}$$

where $n = 1(1)N, j = 0, 1, 2, \cdots, \vartheta = \dfrac{1 + \mu h_n}{10\epsilon(2 + \mu h_n)}, \zeta = \dfrac{1}{2}$,

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} \dfrac{\mu}{1+\mu h_n} & \dfrac{1}{(1+\mu h_n)(2+\mu h_n)h_n} & -\dfrac{1+\mu h_n}{(2+\mu h_n)h_n} \\ -\dfrac{2+\mu h_n}{(1+\mu h_n)h_n} & \dfrac{3+2\mu h_n}{(1+\mu h_n)(2+\mu h_n)h_n} & \dfrac{1+\mu h_n}{(2+\mu h_n)h_n} \\ \dfrac{2+\mu h_n}{(1+\mu h_n)h_n} & -\dfrac{1}{(1+\mu h_n)(2+\mu h_n)h_n} & -\dfrac{3+2\mu h_n}{(2+\mu h_n)h_n} \end{bmatrix},$$

$$\beta_0 = \frac{2\mu^2 h_n^2 + 5\mu h_n + 5}{6(1+\mu h_n)}, \beta_1 = \frac{3\mu h_n + 1}{6(1+\mu h_n)(2+\mu h_n)}, \beta_2 = \frac{(1+\mu h_n)(1-2\mu h_n)}{6(2+\mu h_n)}, \tag{20}$$

and

$$\mathcal{S}_x \widetilde{U}_{n,j} = \frac{2}{(2+\mu h_n)(1+\mu h_n)}\widetilde{U}_{n+1,\, j} - \frac{2}{1+\mu h_n}\widetilde{U}_{n,\, j} + \frac{2}{\mu h_n + 2}\widetilde{U}_{n-1,\, j}, \tag{21}$$

$$\mathcal{S}_x \widetilde{V}_{n,j} = \frac{2}{(2+\mu h_n)(1+\mu h_n)}\widetilde{V}_{n+1,\, j} - \frac{2}{\mu h_n + 1}\widetilde{V}_{n,\, j} + \frac{2}{\mu h_n + 2}\widetilde{V}_{n-1,\, j}. \tag{22}$$

The new quasi-variable mesh discretization method (19) yields a scheme of accuracy four in space and two along the temporal axis. If λ_n denotes the mesh-ratio parameter, then the choice of time step-size $k \approx \lambda_n h_n^2$ yields the local truncation error $LTE \approx O(h_n^6)$. Consequently, the order of the scheme (19) is $h_n^{-2} LTE \approx O(h_n^4)$ for μ to be zero or non-zero. In other words, we have obtained the fourth-order accurate difference formula on a uniform mesh-network and quasi-variable mesh-network as well. The consistency of the difference scheme (19) can be seen as $LTE \to 0$, when $\max_n h_n \to 0$. For the algorithmic implementations, we omit the truncation error and combine it with the initial and boundary data

$$\begin{aligned} &U_{n,0} = \phi_1(x_n), U_{0,j} = F_1(t_j),\ U_{N+1,j} = F_2(t_j), \\ &V_{n,0} = \phi_2(x_n), V_{0,j} = G_1(t_j),\ V_{N+1,j} = G_2(t_j), n = 0(1)N+1,\ j = 0(1)J. \end{aligned}$$

On neglecting higher-order terms, the system of nonlinear equations (19) is numerically solved by Newton's block iterative procedure. Since the values $U_{n+\tau,j}$, $V_{n+\tau,j}$, $\tau = 0, \pm 1$ are known from the initial data or previous iterations, it is feasible to express the system of equation (19) as follows

$$\mathcal{P}_n \equiv \mathcal{P}_n(U_{n-1,\, j}, U_{n,\, j}, U_{n+1,\, j}, V_{n-1,\, j}, V_{n,\, j}, V_{n+1,\, j}) = 0, \tag{23}$$

$$\mathcal{Q}_n \equiv \mathcal{Q}_n(U_{n-1,\, j}, U_{n,\, j}, U_{n+1,\, j}, V_{n-1,\, j}, V_{n,\, j}, V_{n+1,\, j}) = 0, \tag{24}$$

where

$$\mathcal{P}_n = -\epsilon \mathcal{S}_x \widetilde{U}_{n,\ j} + h_n^2(\beta_1 \tilde{\psi}_{n+1,\ j} + \beta_0 \tilde{\tilde{\psi}}_{n,\ j} + \beta_2 \tilde{\psi}_{n-1,\ j}), \tag{25}$$

$$\mathcal{Q}_n = -\epsilon \mathcal{S}_x \widetilde{V}_{n,\ j} + h_n^2(\beta_1 \tilde{\varphi}_{n+1,\ j} + \beta_0 \tilde{\tilde{\varphi}}_{n,\ j} + \beta_2 \tilde{\varphi}_{n-1,\ j}). \tag{26}$$

Let $\boldsymbol{U}_{j+1} = [U_{1,j+1}, U_{2,j+1}, \cdots, U_{N,j+1}]^T, \boldsymbol{V}_{j+1} = [V_{1,j+1}, V_{2,j+1}, \cdots, V_{N,j+1}]^T$ be the solution vectors at $(j+1)^{th}$-time level and denote

$$\boldsymbol{F}(\boldsymbol{U}_{j+1},\ \boldsymbol{V}_{j+1}) = [F_1, F_2, \cdots, F_N]^T, \quad \boldsymbol{G}(\boldsymbol{U}_{j+1},\ \boldsymbol{V}_{j+1}) = [G_1, G_2, \cdots, G_N]^T.$$

Then, the Jacobian of $\boldsymbol{F}$ and $\boldsymbol{G}$ is given by $\boldsymbol{J} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}_{2N\times 2N}$, where $\boldsymbol{A} = [a_{l,m}], \boldsymbol{B} = [b_{l,m}], \boldsymbol{C} = [c_{l,m}]$ and $\boldsymbol{D} = [d_{l,m}]$, are tri-diagonal matrices of order N having the following form

$$[a_{l,m}] = \begin{pmatrix} a_{1,1} & a_{1,2} & & 0 \\ a_{2,1} & \ddots & \ddots & \\ & \ddots & \ddots & a_{N-1,N} \\ 0 & & a_{N,N-1} & a_{N,N} \end{pmatrix},$$

and $a_{l,m} = \dfrac{\partial F_l}{\partial U_{m,j+1}}, b_{l,m} = \dfrac{\partial F_l}{\partial V_{m,j+1}}, c_{l,m} = \dfrac{\partial G_l}{\partial U_{m,j+1}}, d_{l,m} = \dfrac{\partial G_l}{\partial V_{m,j+1}}$.

Now, Newton's block iterative procedure is given by

$$\begin{bmatrix} \boldsymbol{A}\ \boldsymbol{B} \\ \boldsymbol{C}\ \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{U}_{j+1} \\ \delta\boldsymbol{V}_{j+1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{F}(\boldsymbol{U}_{j+1},\ \boldsymbol{V}_{j+1}) \\ \boldsymbol{G}(\boldsymbol{U}_{j+1},\ \boldsymbol{V}_{j+1}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \tag{27}$$

where $\delta\boldsymbol{U}_{j+1}$ and $\delta\boldsymbol{V}_{j+1}$ acts as liaise solution vectors with some finite starting guess. Let us define

$$\begin{bmatrix} \boldsymbol{U}_{j+1}^{(l+1)} \\ \boldsymbol{V}_{j+1}^{(l+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_{j+1}^{(l)} \\ \boldsymbol{V}_{j+1}^{(l)} \end{bmatrix} + \begin{bmatrix} \delta\boldsymbol{U}_{j+1}^{(l)} \\ \delta\boldsymbol{V}_{j+1}^{(l)} \end{bmatrix}, l = 0, 1, 2 \cdots. \tag{28}$$

We can solve the system of equations (27) for $\delta\boldsymbol{U}_{j+1}^{(l)}$ and $\delta\boldsymbol{V}_{j+1}^{(l)}$ by applying block inner iteration in the following manner:

$$\begin{bmatrix} \boldsymbol{A}\delta\boldsymbol{U}_{j+1}^{(m+1)} + \boldsymbol{F}(\boldsymbol{U}_{j+1}^{(m)},\ \boldsymbol{V}_{j+1}^{(m)}) + \boldsymbol{B}\delta\boldsymbol{V}_{j+1}^{(m+1)} \\ \boldsymbol{D}\delta\boldsymbol{V}_{j+1}^{(m+1)} + \boldsymbol{G}(\boldsymbol{U}_{j+1}^{(m)},\ \boldsymbol{V}_{j+1}^{(m)}) + \boldsymbol{C}\delta\boldsymbol{U}_{j+1}^{(m+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, m = 0, 1, 2 \cdots. \tag{29}$$

This coupled form of nonlinear equations may be computed to obtain $\delta\boldsymbol{U}_{j+1}^{(l)}$ and $\delta\boldsymbol{V}_{j+1}^{(l)}$ by means of Thomas algorithm and, we shall compute $\delta\boldsymbol{U}_{j+1}^{(l+1)}$ and $\delta\boldsymbol{V}_{j+1}^{(l+1)}, l = 0, 1, 2 \cdots$. The Newton's iteration converges with the initial solution vector taken to be close to the exact solution values [16, 18].

4. Computational experiments

To illustrate the proposed scheme, we have computed accuracies in approximate solution values and exact solution values using the metrics maximum absolute (MA), root-mean-squared (RMS) errors and numerical convergence order Θ_∞ and Θ_2. They are defined in the following manner for $W = U, V, w = u, v$:

$$\| \boldsymbol{\epsilon} \|_\infty^{(N)} = \max_{j=1(1)J, n=1(1)N} | W_{n,j} - w_{n,j} |, \quad \Theta_\infty = \log_2(\| \boldsymbol{\epsilon} \|_\infty^{(N)} \, / \, \| \boldsymbol{\epsilon} \|_\infty^{(2N+1)}),$$

$$\| \boldsymbol{\epsilon} \|_2^{(N)} = \sqrt{\frac{1}{NJ} \sum_{n=1}^{N} \sum_{j=1}^{J} | W_{n,j} - w_{n,j} |^2}, \quad \Theta_2 = \log_2(\| \boldsymbol{\epsilon} \|_2^{(N)} \, / \, \| \boldsymbol{\epsilon} \|_2^{(2N+1)}).$$

The norm values are calculated for both the mesh spacing; quasi-variable meshes ($\mu \neq 0$) and uniform mesh step sizes ($\mu = 0$). As a test procedure, the initial and end data are determined from the known solution. Newton's iterative method solve the nonlinear difference equations using the error tolerance value 10^{-12} along with zero vector as an initial guess [16]. In all the computations, we have chosen mesh ratio parameter $\lambda_n = 1.6$ and the number of temporal steps $J = (N+1)^2/\lambda_n$, so that $k \approx \lambda_n {h_n}^2$ and hence, it is practical to verify the fourth-order exactness of convergence to the numerical scheme. Maple's *CodeGeneration* is utilized to generate difference relations, and numerical computations are performed in C on the Mac operating system.

Example 1: We do experiment with the coupled viscous Burgers' equation (8)-(11) that possesses following traveling wave solutions [17]
for $\alpha \neq 1, \beta \neq 1$:

$$U(x,t) = 2\frac{\alpha - 1}{\alpha\beta - 1}[1 - \epsilon \tanh(8t - 2x)], \;\; V(x,t) = 2\frac{\beta - 1}{\alpha\beta - 1}[1 - \epsilon \tanh(8t - 2x)],$$

and for $\alpha = 1, \beta = 1$:

$$U(x,t) = 1 - \epsilon \tanh(2x - 8t), \;\; V(x,t) = 1 - \epsilon \tanh(2x - 8t).$$

Boundary and initial data are taken from the true values of solutions as a test case. Experiments with equal-spaced meshes compact scheme (19) for $\epsilon = 10^{-2}, 10^{-3}$ yield unstable solutions while the quasi-variable meshes high-order compact scheme (19) for $\mu \neq 0$ gives rise to the accurate solution in conformity with order and accuracies. If $\alpha < \beta \ll \epsilon$, the solution behavior is regular; therefore we shall perform experiments for the following three cases: $\epsilon \ll \alpha < \beta$; $\epsilon \ll \alpha = \beta$, and $\epsilon \ll \beta < \alpha$. Table 1 adduces maximum errors (absolute), root-mean-squared errors along with computational order for the equal values of α, and β at the temporal level $t = 1$, for various unequal mesh spacing. In Table 2 and Table 3, we assessed the maximum errors in absolute, root-mean-squared errors along with the order of convergence for $\alpha \neq \beta$, committed in the solution data $U(x,t)$ and $V(x,t)$ at the time level $t = 1$ for various unequal mesh spacing.

Table 1: MA and RMS errors and computational order in example 1.

$N+1$	μ	$\|\|\epsilon\|\|_\infty^{(N)}$	$\|\|\epsilon\|\|_2^{(N)}$	Θ_∞	Θ_2
$\epsilon = 10^{-2}, \alpha = \beta = 1$					
4	0.600	$3.86e-04$	$2.34e-04$	$--$	$--$
8	0.339	$2.43e-05$	$1.71e-05$	4.0	3.8
16	0.102	$1.43e-06$	$9.94e-07$	4.1	4.1
32	0.040	$8.61e-08$	$5.73e-08$	4.1	4.1
$\epsilon = 10^{-3}, \alpha = \beta = 1$					
4	0.990	$1.17e-04$	$7.17e-05$	$--$	$--$
8	0.428	$9.78e-06$	$6.78e-06$	3.6	3.6
16	0.111	$4.91e-07$	$2.91e-07$	4.3	4.5
32	0.041	$3.13e-08$	$2.13e-08$	4.0	3.8
$\epsilon = 10^{-2}, \alpha = \beta = 10$					
4	0.600	$7.01e-05$	$4.28e-05$	$--$	$--$
8	0.339	$4.41e-06$	$3.10e-06$	4.0	3.8
16	0.103	$2.67e-07$	$1.87e-07$	4.0	4.1
32	0.040	$1.56e-08$	$7.89e-09$	4.1	4.6
$\epsilon = 10^{-3}, \alpha = \beta = 10$					
4	0.9900	$2.13e-05$	$1.44e-05$	$--$	$--$
8	0.4320	$5.30e-06$	$3.44e-06$	2.0	2.1
16	0.1121	$2.78e-07$	$7.81e-07$	4.2	4.2
32	0.0420	$1.83e-08$	$6.66e-09$	4.1	4.8

5. Remarks and conclusions

Based on a quasi-variable meshes network, a new compact scheme of accuracy $O(h_n^4)$ in the spatial and $O(k^2)$ in time direction has been obtained for the numerical computation of coupled viscous Burger's equation. The new scheme utilizes the minimum number of stencils for the discretization purpose. It refers to both quasi-variable meshes and uniform meshes high-order method of the same order of solution accuracy, similar to a *supra*-convergent discretization that has a lower order of truncation errors. For a particular choice of mesh parameter ($\mu = 0$), it is a fourth-order uniform mesh scheme, and for ($\mu \neq 0$), it results

Table 2: MA and RMS errors and computational order in example 1 for $U(x,t)$.

$N+1$	μ	$\|\|\epsilon\|\|_\infty^{(N)}$	$\|\|\epsilon\|\|_2^{(N)}$	Θ_∞	Θ_2
$\epsilon=10^{-2}, \alpha=3, \beta=8$					
4	0.380	$6.61e-05$	$4.78e-05$	$--$	$--$
8	0.341	$4.15e-06$	$2.96e-06$	4.0	4.0
16	0.103	$2.61e-07$	$1.83e-07$	4.0	4.0
32	0.040	$1.44e-08$	$1.04e-09$	4.2	4.1
$\epsilon=10^{-3}, \alpha=3, \beta=8$					
4	0.0000	$1.07e-04$	$1.10e-05$	$--$	$--$
8	0.4300	$8.86e-06$	$5.69e-06$	3.6	4.3
16	0.1121	$4.79e-07$	$3.06e-07$	4.2	4.2
32	0.0412	$2.92e-08$	$1.16e-09$	4.0	4.7
$\epsilon=10^{-2}, \alpha=7, \beta=4$					
4	0.580	$1.73e-04$	$1.10e-04$	$--$	$--$
8	0.342	$1.30e-05$	$9.13e-06$	3.7	3.6
16	0.103	$6.92e-07$	$4.85e-07$	4.2	4.2
32	0.040	$4.06e-08$	$3.78e-08$	4.1	3.7

in a fourth-order quasi-variable mesh scheme. The truncation error depends on mesh spacing and derivative of the variable. Thus, the uniform distribution of discretization error is possible only through the variable spacing between mesh points. Experiments with various values of parameters that appeared in the governing equations have been described in detail and obtained convergent solution values. The error estimates on the new scheme prove superior as compared with existing results.

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Table 3: MA and RMS errors and computational order in example 1 for $V(x,t)$.

$N+1$	μ	$\|\|\epsilon\|\|_\infty^{(N)}$	$\|\|\epsilon\|\|_2^{(N)}$	Θ_∞	Θ_2
$\epsilon = 10^{-2}, \alpha = 3, \beta = 8$					
4	0.380	$1.36e-04$	$1.15e-04$	$--$	$--$
8	0.341	$1.97e-05$	$1.39e-05$	2.8	3.0
16	0.103	$8.52e-07$	$5.97e-07$	4.5	4.5
32	0.040	$4.92e-08$	$3.07e-08$	4.1	4.3
$\epsilon = 10^{-3}, \alpha = 3, \beta = 8$					
4	0.0000	$4.66e-04$	$3.15e-04$	$--$	$--$
8	0.4300	$1.83e-05$	$1.18e-05$	4.7	4.7
16	0.1121	$1.26e-06$	$7.79e-07$	3.9	3.9
32	0.0412	$1.24e-07$	$4.91e-08$	3.3	4.0
$\epsilon = 10^{-2}, \alpha = 7, \beta = 4$					
4	0.580	$1.20e-04$	$7.32e-05$	$--$	$--$
8	0.342	$6.83e-06$	$4.80e-06$	4.1	3.9
16	0.103	$3.58e-07$	$2.51e-07$	4.3	4.3
32	0.040	$1.65e-08$	$8.31e-09$	4.4	4.9

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