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# Global Estimates for Generalized Double Bernstein Operators

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**Abstract.** In this paper, we obtain quantitative estimates for generalized two dimensional Bernstein operators. We calculate global results for these operators using Lipschitz-type space and estimate the error using modulus of continuity.

**Key Words and Phrases**: Bernstein operators, Lipschitz-type space, modulus of continuity.

**2010 Mathematics Subject Classifications**: 41A10, 41A25, 41A30, 26A15

## 1. Introduction

In [2], P. L. Butzer introduced two dimensional Bernstein polynomials $B_n^*(f;x,y)$ on the square $\square := \{(x,y) : 0 \leq x, y \leq 1\}$ and defined as follows:

$$B_n^*(f;x,y) = \sum_{k,l=0}^{n} p_{n,k}(x)\, p_{n,l}(y) f\left(\frac{k}{n}, \frac{l}{n}\right), \quad (x,y) \in [0,1] \times [0,1], \tag{1}$$

where $p_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k}$ and $p_{n,l}(y) = \left(\begin{array}{c} n \\ l \end{array}\right) y^l(1-y)^{n-l}$ are the Bernstein basis with $0 \leq k \leq n,\ 0 \leq l \leq n$ and $f(x,y) \in C_B[0,1;0,1]$.

Deo and Bhardwaj [3], characterized the rate of approximation by means of $K-$functionals and estimate the order of convergence by means of a semi-norm $\phi(f)$ for the two dimensional Bernstein operators, which was introduced by Stancu [15] and its Durrmeyer variants studied by Zhou [17] on a simplex.

Many researchers have studied better estimate for the one dimensional operators like Bernstein, Szász, Baskakov operators and its variants (see [4]-[8], [12], [13]).

 

Approximation properties of $q$ and $(p, q)$-analogue of Bernstein operators and its variants are studied in [1], [9], [10] and [11]. Özarslan and Duman [13] have introduced a different approach in order to get a faster approximation without preserving the test functions. Özarslan and Aktuǵlu [14] have calculated quantitative global estimates for two dimensional Szász-Mirakjan operators. Motivated by these research work, we consider generalized two dimensional Bernstein operators and obtained the best error estimate.

The classical Bernstein operators are defined as:

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ 0 \le x \le 1. \tag{2}$$

Let $f_h(x) = x^h, h \in \{0,1,2\}$ then auxiliary properties of (2) are as follows:

$$B_n(f_0;x) = 1,\ B_n(f_1;x) = x,\ B_n(f_2;x) = \left(1 - \frac{1}{n}\right)x^2 + \frac{x}{n}.$$

Following the similar arguments as used in [13], the best error estimation among all the general two dimensional Bernstein operators can be obtained from the case by taking

$$a_n = 1,\ b_n = e_n = 0,\ c_n = 1 - \frac{1}{n},\ d_n = \frac{1}{n},$$

for all $n \in \mathbb{N}$ where $(a_n), (b_n), (c_n), (d_n)$ and $(e_n)$ are sequences of non-negative real numbers satisfying the conditions given in [13].

Now observe that

$$u_n^*(x) = \frac{2a_n x - d_n}{2c_n} = \frac{2nx - 1}{2(n-1)} \in [0,1],$$

if and only if $\frac{1}{2n} \le x \le 1 - \frac{1}{2n}$ for $n \ge 2$ where $u_n^*$ is a functional sequence, $u_n^* : I \to [0,1]$. Hence, choosing

$$I = \left[\frac{1}{4}, \frac{3}{4}\right] \subset [0,1].$$

The best error estimation among all the general two dimensional Bernstein operators can be obtained from the case

$$u_n^*(x) = \frac{2nx - 1}{2(n-1)}, v_n^*(y) = \frac{2ny - 1}{2(n-1)}; n \in \mathbb{N},$$

for all $f \in C_B([0,1]) \times C_B([0,1])$ and $x, y \in I$. Hence, (1) becomes

$$B_n^{**}(f; x, y) =$$

$$= \sum_{k,l=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} n \\ l \end{pmatrix} (u_n^*(x))^k (1 - u_n^*(x))^{n-k} (v_n^*(y))^l (1 - v_n^*(y))^{n-l} f\left(\frac{k}{n}, \frac{l}{n}\right), \tag{3}$$

where $f \in C_B([0,1]) \times C_B([0,1])$.

For the operators $B_n^{**}(f; x, y)$, we have following Lemma:

**Lemma 1.** *Let* $\boldsymbol{x} = (x, y), \boldsymbol{t} = (t, s); e_{i,j}(x) = x^i y^j, i, j = 0, 1, 2$ *and* $\psi_x^2(t) = \|t - x\|^2$. *Then, for each* $x, y \in I$ *and* $n \geq 2$, *we have*

(*i*) $B_n^{**}(e_{0,0}; x, y) = 1$;

(*ii*) $B_n^{**}(e_{1,0}; x, y) = u_n^*(x)$;

(*iii*) $B_n^{**}(e_{0,1}; x, y) = v_n^*(y)$;

(*iv*) $B_n^{**}(e_{2,0} + e_{0,2}; x, y) = \left(1 - \frac{1}{n}\right)\left((u_n^*(x))^2 + (v_n^*(y))^2\right) + \dfrac{u_n^*(x) + v_n^*(y)}{n}$;

(*v*) $B_n^{**}\left(\psi_x^2(t); x, y\right) = (u_n^*(x) - x)^2 + (v_n^*(y) - y)^2 - \frac{1}{n}\left((u_n^*(x))^2 + (v_n^*(y))^2\right) + \frac{1}{n}(u_n^*(x) + v_n^*(y))$.

## 2. Global Results

We have used following definitions in this paper for global results of the operators $B_n^{**}(f; x, y)$.

Szàsz [16] earlier considered this space of bivariate extension of Lipschitz-type space, given as:

$$Lip_M^*(\alpha) :=$$

$$\left\{ f \in C([0,\infty) \times [0,\infty)) : |f(t) - f(x)| \leq M \frac{\|\mathbf{t} - \mathbf{x}\|^\alpha}{(\|\mathbf{t}\| + x + y)^{\frac{\alpha}{2}}}; t, s; x, y \in (0, \infty) \right\}$$

where $\mathbf{t} = (t, s), \mathbf{x} = (x, y)$ and $M$ is any positive constant and $0 < \alpha \leq 1$.

For all $f \in C([0,\infty) \times [0,\infty))$, the modulus of $f$ denoted by $\omega(f; \delta)$ is defined as

$$\omega(f; \delta) :=$$

$$\sup\left\{|f(t,s)-f(x,y)| : \sqrt{(t-x)^2+(s-y)^2}<\delta, (t,s), (x,y)\in[0,\infty)\times[0,\infty)\right\}.$$

Now, for the space $Lip_M^*(\alpha)$ with $0<\alpha\le 1$, we have the following approximation result.

**Theorem 1.** *For any* $f\in Lip_M^*(\alpha), \alpha\in(0,1]$ *and for each* $x,y\in I, n\ge 2$, *we have*

$$|B_n^{**}(f;x,y)-f(x,y)|\le \frac{M}{(x+y)^{\frac{\alpha}{2}}}\left[(u_n^*(x)-x)^2+(v_n^*(y)-y)^2\right.$$

$$\left.-\frac{1}{n}\left((u_n^*(x))^2+(v_n^*(y))^2\right)+\frac{1}{n}(u_n^*(x)+v_n^*(y))\right]^{\frac{\alpha}{2}} \tag{4}$$

*Proof.* Let $\alpha=1$. For each $x,y\in(0,\infty)$ and for $f\in Lip_M^*(1)$, we have

$$\begin{aligned}|B_n^{**}(f;x,y)-f(x,y)| &\le B_n^{**}(|f(t,s)-f(x,y)|;x,y)\\ &\le MB_n^{**}\left(\frac{\|\mathbf{t}-\mathbf{x}\|}{(\|\mathbf{t}\|+x+y)^{1/2}};x,y\right)\\ &\le \frac{M}{(x+y)^{1/2}}B_n^{**}(\|\mathbf{t}-\mathbf{x}\|;x,y).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$|B_n^{**}(f;x,y)-f(x,y)|\le\frac{M}{(x+y)^{1/2}}\sqrt{B_n^{**}(\psi_x^2(\mathbf{t});x,y)}=\frac{M}{(x+y)^{1/2}}$$
$$\sqrt{(u_n^*(x)-x)^2+(v_n^*(y)-y)^2-\frac{1}{n}((u_n^*(x))^2+(v_n^*(y))^2)+\frac{1}{n}(u_n^*(x)+v_n^*(y))}.$$

Now, let $0<\alpha<1$. Then for each $x,y\in I$ and for $f\in Lip_M^*(\alpha)$, we obtain

$$\begin{aligned}|B_n^{**}(f;x,y)-f(x,y)| &\le B_n^{**}(|f(t,s)-f(x,y)|;x,y)\\ &\le MB_n^{**}\left(\frac{\|\mathbf{t}-\mathbf{x}\|^\alpha}{(\|\mathbf{t}\|+x+y)^{\alpha/2}};x,y\right)\\ &\le \frac{M}{(x+y)^{\alpha/2}}B_n^{**}(\|\mathbf{t}-\mathbf{x}\|^\alpha;x,y).\end{aligned}$$

For Hölder inequality with $p = \dfrac{2}{\alpha}$ and $q = \dfrac{2}{2-\alpha}$, for any $f \in Lip_M^*(\alpha)$, we have

$$|B_n^{**}(f;x,y) - f(x,y)| \leq \frac{M}{(x+y)^{\alpha/2}} \left[B_n^{**}\left(\psi_x^2(\mathbf{t});x,y\right)\right]^{\alpha/2} = \frac{M}{(x+y)^{\alpha/2}}$$

$$\left[(u_n^*(x)-x)^2 + (v_n^*(y)-y)^2 - \frac{1}{n}\left((u_n^*(x))^2 + (v_n^*(y))^2\right) + \frac{1}{n}\left(u_n^*(x) + v_n^*(y)\right)\right]^{\alpha/2},$$

which is the required result. ◄

**Lemma 2.** *For each $x, y > 0$,*

$$B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right) \tag{5}$$
$$\leq \frac{1}{\sqrt{x}}\sqrt{(u_n^*(x)-x)^2 - \frac{(u_n^*(x))^2 - u_n^*(x)}{n}}$$
$$+\frac{1}{\sqrt{y}}\sqrt{(v_n^*(y)-y)^2 - \frac{(v_n^*(y))^2 - v_n^*(y)}{n}}.$$

*Proof.* We have $\sqrt{c+d} \leq \sqrt{c} + \sqrt{d}\,(c,d \geq 0)$, therefore

$$B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$
$$= \sum_{k,l=0}^{n} \binom{n}{k}\binom{n}{l}\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2 + \left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}$$
$$(u_n^*(x))^k(1-u_n^*(x))^{(n-k)}(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}$$
$$\leq \sum_{k=0}^{n}\binom{n}{k}\left|\sqrt{\frac{k}{n}}-\sqrt{x}\right|(u_n^*(x))^k(1-u_n^*(x))^{(n-k)}$$
$$+\sum_{l=0}^{n}\binom{n}{l}\left|\sqrt{\frac{l}{n}}-\sqrt{y}\right|(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}$$
$$= \sum_{k=0}^{n}\binom{n}{k}\frac{\left|\frac{k}{n}-x\right|}{\sqrt{\frac{k}{n}}+\sqrt{x}}(u_n^*(x))^k(1-u_n^*(x))^{(n-k)}$$
$$+\sum_{l=0}^{n}\binom{n}{l}\frac{\left|\frac{l}{n}-y\right|}{\sqrt{\frac{l}{n}}+\sqrt{y}}(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}$$

$$\leq \frac{1}{\sqrt{x}}\sum_{k=0}^{n}\binom{n}{k}\left|\frac{k}{n}-x\right|(u_n^*(x))^k(1-u_n^*(x))^{(n-k)}$$

$$+\frac{1}{\sqrt{y}}\sum_{l=0}^{n}\binom{n}{l}\left|\frac{l}{n}-y\right|(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}.$$

Using the Cauchy-Schwarz inequality,

$$B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$

$$\leq \frac{1}{\sqrt{x}}\sqrt{\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2(u_n^*(x))^k(1-u_n^*(x))^{(n-k)}}$$

$$+\frac{1}{\sqrt{y}}\sqrt{\sum_{l=0}^{n}\binom{n}{l}\left(\frac{l}{n}-y\right)^2(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}}.$$

Using Lemma 1,

$$B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$

$$\leq \frac{1}{\sqrt{x}}\sqrt{(u_n^*(x)-x)^2-\frac{(u_n^*(x))^2-u_n^*(x)}{n}}$$

$$+\frac{1}{\sqrt{y}}\sqrt{(v_n^*(y)-y)^2-\frac{(v_n^*(y))^2-v_n^*(y)}{n}},$$

which is the desired result. ◀

**Theorem 2.** *Let $g(x,y)=f\left(x^2,y^2\right)$. Then we have for each $x,y\in I$,*

$$\left|B_n^{**}(f;x,y)-f(x,y)\right|\leq 2\omega\left(g;\delta_n(x,y)\right),$$

*where*

$$\delta_n(x,y)=\frac{1}{\sqrt{x}}\sqrt{(u_n^*(x)-x)^2-\frac{(u_n^*(x))^2-u_n^*(x)}{n}}$$

$$+\frac{1}{\sqrt{y}}\sqrt{(v_n^*(y)-y)^2-\frac{(v_n^*(y))^2-v_n^*(y)}{n}}.$$

*Proof.* We have

$$|B_n^{**}(f;x,y)-f(x,y)| \le B_n^{**}(|f(t,s)-f(x,y)|;x,y)$$

$$= B_n^{**}\left(\left|g\left(\sqrt{t},\sqrt{s}\right)-g\left(\sqrt{x},\sqrt{y}\right)\right|;x,y\right)$$

$$\le B_n^{**}\left(\omega\left(g;\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2}\right);x,y\right)$$

$$= \sum_{k,l=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} n \\ l \end{array}\right)\omega\left(g;\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2};x,y\right)$$

$$(u_n^*(x))^k(1-u_n^*(x))^{(n-k)}(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}$$

$$= \sum_{k,l=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} n \\ l \end{array}\right)$$

$$\omega\left(g;\frac{\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}}{B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)}\right.$$

$$\left.B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right);x,y\right).$$

Now, we have

$$\omega(f;\lambda\delta) \le (1+\lambda)\,\omega(f;\delta).$$

Therefore,

$$|B_n^{**}(f;x,y)-f(x,y)| \le \omega\left(g;B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)\right)$$

$$\times\sum_{k,l=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} n \\ l \end{array}\right)\left[1+\frac{\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}}{B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)}\right]$$

$$(u_n^*(x))^k(1-u_n^*(x))^{(n-k)}(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}$$

$$\leq 2\omega\left(g; B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2}; x, y\right)\right).$$

Now, using Lemma 2, completes the proof. ◄

**Theorem 3.** *Let* $g(x,y) = f\left(x^2, y^2\right)$. *Let*

$$g \in Lip_M(\alpha) := \left\{g \in C_{\mathbf{B}}([0,1]\times[0,1]) : |g(t) - g(x)| \leq M\|t-x\|^{\alpha}; t, s; x, y \in I\right\},$$

*where* $\boldsymbol{t} = (t,s)$, $\boldsymbol{x} = (x,y)$ *and* $M$ *is any positive constant and* $0 < \alpha \leq 1$. *Then,*

$$\left|B_n^{**}(f; x, y) - f(x,y)\right| \leq M\delta_n^{\alpha}(x,y), \tag{6}$$

*where* $\delta_n(x,y)$ *is the same as in Theorem 2.*

*Proof.* We have

$$\begin{aligned}\left|B_n^{**}(f; x, y) - f(x,y)\right| &\leq B_n^{**}\left(\left|f(t,s) - f(x,y)\right|; x, y\right)\\ &= B_n^{**}\left(\left|g\left(\sqrt{t}, \sqrt{s}\right) - g\left(\sqrt{x}, \sqrt{y}\right)\right|; x, y\right)\\ &\leq M B_n^{**}\left(\left(\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2\right)^{\alpha/2}; x, y\right)\\ &= M\sum_{k,l=0}^{n}\binom{n}{k}\binom{n}{l}\left(\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2\right)^{\alpha/2}\\ &\quad (u_n^*(x))^k(1-u_n^*(x))^{(n-k)}(v_n^*(y))^l(1-v_n^*(y))^{(n-l)}.\end{aligned}$$

For Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\left|B_n^{**}(f; x, y) - f(x,y)\right| \leq M\left[B_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2}; x, y\right)\right]^{\alpha}.$$

By using Lemma 2, completes the proof. ◄

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