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B^*-Continuity for Multifunctions Based on Clustering

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Abstract. In the present paper we are going to introduce the concept of upper (lower) $\mathfrak{UV}$- B^*-continuous multifunction and study some properties and relationship among various generalizations of B^*-continuous multifunctions.

Key Words and Phrases: $\mathfrak{UV}$- B^*-continuous multifunction, product multifunction, $\mathfrak{U}$-compact, $\mathfrak{U}$-Hausdorff, $\mathfrak{U}$-normal.

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1. Introduction

The subject of generalized continuities of functions and multifunctions has been the interest of many authors during the recent past. Matejdes [11], [12], introduced and studied the concept of $\mathcal{E}$-cluster continuity. Later Ganguly and Mallick [6], [5], [4], further studied the convergence and existence of selection for $\mathcal{E}$-continuous multifunctions. In 2006, Noiri and Popa [14], studied slightly m-continuous multifunction. Recently, in 2014 Mitra and Ganguly [13] studied the concept of $\mathcal{E}_{\mathcal{P}}$-cluster multifunction. In our previous paper, [7], [8], we have introduced various generalized B^*-continuities, namely, contra B^*-contiuity, B^*-clopen continuity, B^*-irresolute, slightly B^*-continuity.

The aim of the present paper is to unify above mentioned generalized B^*-continuities. Here, we define and study a new class of cluster multifunction, namely, $\mathfrak{UV}$- B^*-continuous multifunction. The paper is organized as follows: In Section 2, we give the notion of upper and lower $\mathfrak{UV}$- B^*-continuity for multifunctions and investigate various properties. In Section 3, we study upper and lower $\mathfrak{UV}$- B^*-continuity for product multifunctions. Finally in Section 4, we give an interrelation diagram showing the relationship among

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various generalizations of B^*-continuity and give some conditions to prove some equivalence among the various members of the $\mathfrak{UW}$- B^*-continuous multifunctions.

Let us first give some preliminaries, see, e.g., [2], [3], [11], which are required for the subsequent sections.

Throughout the paper, X, Y and Z will denote topological spaces, unless specified otherwise. By $\operatorname{int}(A)$ and $\operatorname{cl}(A)$ we shall denote the interior and closure of the set A.

By $F : X \to Y$, we shall mean that F is a multifunction with domain X and co-domain $P(Y) \setminus \emptyset$, the power set of Y excluding the empty set.

If $F : X \to Y$ is a multifunction then for $A \subset Y$, we denote

$$F^+(A) = \{x \in X : F(x) \subset A\}$$

and

$$F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}.$$

A set B is said to be a B^*-set if it is not nowhere dense and having the property of Baire.

Let $P \subset X$. A point $x \in X$ is said to be a B^*-cluster point of P if for every B^*-set B containing x, $P \cap B \neq \emptyset$. The set of all B^*-cluster points of P is called B^*-cluster derived set of P and is denoted by B^*-cls-d- P.

A set P is said to be B^*-closed if B^*-cls-d- $P \subset P$. The B^*-closure of $P = P \cup$ (B^*-cls-d- P) is denoted by $B^*_{\mathrm{cl}}(P)$. The complement of a B^*-closed set is known as B^*-open set.

Matejdes [11] introduced the notion of an $\mathcal{E}$-cluster point, lower $\mathcal{E}$-continuity and upper $\mathcal{E}$-continuity as basic tools for investigating many properties of multifunctions.

Any nonempty system $\mathcal{E} \subseteq 2^X - \{\phi\}$ will be called a $\mathcal{E}$-cluster system. Some examples of such system are the following:

$\mathcal{O} = \{O : O$ is non-empty open set $\}$,

$\mathcal{B}_r = \{B : B$ is of second category with the Baire property $\}$,

$\mathcal{E}^0 = 2^X - \{\phi\}$ and

$\mathcal{B}^* = \{E \subseteq X : E$ is not nowhere dense with the Baire property $\}$.

A single-valued function $f : X \longrightarrow Y$ is said to be $\mathcal{E}$-continuous at a point $x \in X$, if for every pair of open sets U, V such that $x \in U$ and $f(x) \in V$, there exists a set $E \in \mathcal{E}$, $E \subseteq U$ such that $f(E) \subseteq V$. The function f will be $\mathcal{E}$-continuous on X if it is so at every $x \in X$.

A multi-valued function $F : X \longrightarrow Y$ is called lower (respectively, upper) $\mathcal{E}$-continuous at a point $x \in X$, if for any open sets U, V such that $x \in U$ and

$V \cap F(x) \neq \phi$ (respectively, $F(x) \subseteq V$), there exists a set $E \in \mathcal{E}$, $E \subseteq U$ such that $V \cap F(e) \neq \phi$ (respectively, $F(e) \subseteq V$) for any $e \in E$.

The lower (upper) $\mathcal{B}_r$-continuity is precisely the lower (upper) Baire continuity and lower (upper) $\mathcal{O}$-continuity is precisely the lower (upper) quasi-continuity.

2. $\mathfrak{UV}$- B^*-Continuity: Notion and Properties

Throughout, $\mathfrak{U}$ and $\mathfrak{V}$ will denote cluster systems. We begin by defining the following:

Definition 1. *A multifunction $F : X \to Y$ is said to be*

(a) upper $\mathfrak{UV}$- B^*-continuous at a point x, if for every set $U \in \mathfrak{U}$ containing x and for every set $V \in \mathfrak{V}$ such that $F(x) \subset V$, there exists a B^*-set B such that

$$B \subset F^+(V) \cap U.$$

(b) lower $\mathfrak{UV}$- B^*-continuous at a point x, if for every set $U \in \mathfrak{U}$ containing x and for every set $V \in \mathfrak{V}$ such that $F(x) \cap V \neq \emptyset$, there exists a B^*-set B such that

$$B \subset F^-(V) \cap U.$$

(c) $\mathfrak{UV}$- B^*-continuous if it is both upper and lower $\mathfrak{UV}$- B^*-continuous.

Remark 1. *Depending on the nature of the cluster systems $\mathfrak{U}$ and $\mathfrak{V}$, we get different generalized B^*-continuities as follows:*

(a) if $\mathfrak{U}$ and $\mathfrak{V}$ are both collection of open sets then $\mathfrak{UV}$- B^*-continuity corresponds to B^*-continuity.

(b) if $\mathfrak{U}$ is a collection of open sets and $\mathfrak{V}$ is a collection of clopen sets then $\mathfrak{UV}$- B^*-continuity corresponds to slightly- B^*-continuity.

(c) if $\mathfrak{U}$ is a collection of open sets and $\mathfrak{V}$ is a collection of closure of open sets then $\mathfrak{UV}$- B^*-continuity corresponds to weakly- B^*-continuity.

(d) if $\mathfrak{U}$ is a collection of open sets and $\mathfrak{V}$ is a collection of interior of closure of open sets then $\mathfrak{UV}$- B^*-continuity corresponds to almost- B^*-continuity.

(e) if $\mathfrak{U}$ is a collection of open sets and $\mathfrak{V}$ is a collection of B^*-sets then $\mathfrak{UV}$- B^*-continuity corresponds to B^*-irresolute.

(f) if $\mathfrak{U}$ is a collection of open sets and $\mathfrak{V}$ is a collection of closed sets then $\mathfrak{U}\mathfrak{V}$-B^*-continuity corresponds to contra-B^*-continuity.

(g) if $\mathfrak{U}$ is a collection of clopen sets and $\mathfrak{V}$ is a collection of open sets then $\mathfrak{U}\mathfrak{V}$-B^*-continuity corresponds to B^*-clopen continuity.

Definition 2. (a) *A cover of $A \subseteq X$ by members of the cluster system $\mathfrak{U}$ is said to be a $\mathfrak{U}$-cover of A.*

(b) *$A \subseteq X$ is said to be $\mathfrak{U}$-compact if every $\mathfrak{U}$-cover of A has a finite subcover.*

(c) *$A \subseteq X$ is said to be $\mathfrak{U}$-Lindelöf if every $\mathfrak{U}$-cover of A has a countable subcover.*

Definition 3. *A space X is said to be $\mathfrak{U}$-Hausdorff if for every two distinct points $x_1, x_2 \in X$, there exist $U_1, U_2 \in \mathfrak{U}$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.*

Definition 4. *A space X is said to be $\mathfrak{U}$-regular if for any point $x \in X$ and for any closed set V not containing x, there exist disjoint sets $U_1, U_2 \in \mathfrak{U}$ such that $x_1 \in U_1$ and $V \subseteq U_2$.*

Definition 5. *A space X is said to be $\mathfrak{U}$-normal if for any pair of disjoint closed sets V_1 and V_2 in X, there exist disjoint sets $U_1, U_2 \in \mathfrak{U}$ such that $V_1 \subseteq U_1$, and $V_2 \subseteq U_2$.*

Remark 2. *If $\mathfrak{U}$ be the collection of B^*-sets then the notions in Definitions 2, 3, 4 and 5 coincide with, respectively, B^*-compact, B^*-Lindelöf, B^*-Hausdorff, B^*-regular and B^*-normal as defined in [7].*

Theorem 1. *Let X be B^*-compact (B^*-Lindelöf) and $F : X \to Y$ be an upper $\mathfrak{U}\mathfrak{V}$-B^*-continuous multifunction. Then $F(X)$ is $\mathfrak{V}$-compact ($\mathfrak{V}$-Lindelöf).*

Proof. Let $x \in X$, $U_x \in \mathfrak{U}$ such that $x \in U_x$ and $V_x \in \mathfrak{V}$ with $F(x) \subseteq V_x$. Then $\{V_x : x \in X\}$ forms a $\mathfrak{V}$-cover for $F(X)$. Since, F is upper $\mathfrak{U}\mathfrak{V}$-B^*-continuous at x, there exists a B^*-set B_x such that

$$x \in B_x \subseteq U_x \qquad \text{and} \qquad F(B_x) \subseteq V_x.$$

Now, $X = \bigcup\limits_{x \in X} B_x$. So, $\{B_x\}$ is a B^*-cover of X and since X is B^*-compact, it admits a finite subcover $\{B_{x_1}, B_{x_2}, \ldots, B_{x_n}\}$, i.e., $X \subseteq \bigcup\limits_{i=1}^{n} B_{x_i}$. Consequently,

$$F(X) \subseteq F\Big(\bigcup_{i=1}^{n} B_{x_i}\Big) \subseteq \bigcup_{i=1}^{n} F(B_{x_i}) \subseteq \bigcup_{i=1}^{n} V_{x_i}$$

and we are done. The case of B^*-Lindelöff can be disposed of similarly. ◀

Let us recall (see [1]) that a multifunction $F : X \to Y$ is said to be punctually closed if, for each $x \in X$, $F(x)$ is closed.

The following theorem gives sufficient conditions for the space X to be B^*-Hausdorff.

Theorem 2. *Let $F : X \to Y$ be a punctually closed upper $\mathfrak{U}\mathfrak{V}$-B^*-continuous multifunction. If Y is $\mathfrak{V}$-normal and for any two distinct points $x, y \in X$, $F(x) \cap F(y) = \emptyset$, then X is B^*-Hausdorff.*

Proof. Let x and y be any two distinct point in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is $\mathfrak{V}$-normal and F is punctually closed there exist disjoint sets $V_1, V_2 \in \mathfrak{V}$ containing, respectively, $F(x)$ and $F(y)$. F being upper $\mathfrak{U}\mathfrak{V}$-B^*-continuous, for $U_1, U_2 \in \mathfrak{U}$ containing, respectively, x and y, there exist B^*-sets B_1 and B_2 such that

$$x \in B_1 \subseteq U_1 \qquad \text{and} \qquad y \in B_2 \subseteq U_2$$

and

$$F(B_1) \subseteq V_1 \qquad \text{and} \qquad F(B_2) \subseteq V_2.$$

Hence $B_1 \cap B_2 = \emptyset$ and the assertion follows. ◀

Remark 3. *In general, the class of B^*-sets is not closed under arbitrary union. We say that a space X has Property P if arbitrary union of B^*-sets in X is a B^*-set. In Theorem 2, if the space X possesses Property P, then it will become B^*-regular as well as B^*-normal.*

3. Product Multifunction

The following notion of product of two multifunctions is known:

Definition 6. *[1] Let $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to Y_2$ be two multifunctions. The product multifunction $F_1 \times F_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by*

$$(F_1 \times F_2)(x_1, x_2) = F_1(x_1) \times F_2(x_2)$$

for every $x_1 \in X_1$ and $x_2 \in X_2$.

Remark 4. *In the product space $X_1 \times X_2$, the cluster systems that we consider is denoted by $\mathfrak{U}_{X_1 \times X_2}$, which consists of all $U := U_1 \times U_2$ such that $U_1 \in \mathfrak{U}_{X_1}$ in X_1 and $U_2 \in \mathfrak{U}_{X_2}$ in X_2.*

Theorem 3. *Let* $F_i : X_i \to Y_i, i = 1, 2$ *be* $\mathfrak{U}\mathfrak{V}$*-* B^**-continuous, then the product multifunction* $F_1 \times F_2$ *is also* $\mathfrak{U}\mathfrak{V}$*-* B^**-continuous.*

Proof. Here, we prove only the upper $\mathfrak{U}\mathfrak{V}$- B^*-continuity. The other case is similar. Let $(x_1, x_2) \in X_1 \times X_2$, $U \in \mathfrak{U}_{X_1 \times X_2}$ containing (x_1, x_2) and $V \in \mathfrak{V}_{Y_1 \times Y_2}$ such that

$$(F_1 \times F_2)(x_1, x_2) \subset V.$$

Then there exist, $U_i \in \mathfrak{U}_{X_i}$ in X_i containing x_i and $V_i \in \mathfrak{V}_{Y_i}$ in Y_i, such that

$$F_i(x_i) \subset V_i \qquad i = 1, 2.$$

Since F_i is upper $\mathfrak{U}\mathfrak{V}$- B^*-continuous there exist a B^*-set $B_i \subset U_i$ containing x_i such that $F(B_i) \subset V_i$. Then $B := B_1 \times B_2$ is a B^*-set in $X_1 \times X_2$ containing (x_1, x_2) such that

$$(F_1 \times F_2)(B) \subset V$$

and we are done. ◀

Towards the converse of Theorem 3, we prove the following:

Theorem 4. *Let* $F_i : X_i \to Y_i, i = 1, 2.$ *If* $F_1 \times F_2$ *is* $\mathfrak{U}\mathfrak{V}$*-* B^**-continuous then either* F_1 *or* F_2 *is* $\mathfrak{U}\mathfrak{V}$*-* B^**-continuous.*

Proof. As before, we prove only the upper $\mathfrak{U}\mathfrak{V}$- B^*-continuity. Let $x_i \in X_i$, $U_i \in \mathfrak{U}_{X_i}$ containing x_i and $V_i \in \mathfrak{V}_{Yi}$ such that

$$F_i(x_i) \subset V_i.$$

Then, $(x_1, x_2) \in X_1 \times X_2$, $U_1 \times U_2 \in \mathfrak{U}_{X_1 \times X_2}$ and $V_1 \times V_2 \in \mathfrak{V}_{Y_1 \times Y_2}$ such that $F_1(x_1) \times F_2(x_2) \subset V_1 \times V_2$. Since, $F_1 \times F_2$ is upper $\mathfrak{U}\mathfrak{V}$- B^*-continuous there exists a B^*-set $B_1 \times B_2 \subset U_1 \times U_2$ such that

$$F_1(B_1) \times F_2(B_2) \subset V_1 \times V_2$$

i.e.,

$$\begin{aligned} B_1 \times B_2 &\subset (U_1 \times U_2) \bigcap (F_1 \times F_2)^+(V_1 \times V_2) \\ &\subset \left[U_1 \cap F_1^+(V_1)\right] \bigcap \left[U_2 \cap F_2^+(V_2)\right]. \end{aligned}$$

Then, either B_1 or B_2 is a B^*-set [10]. Hence, either F_1 or F_2 is upper $\mathfrak{U}\mathfrak{V}$- B^*-continuous. ◀

Remark 5. *In Theorem 4, if domain of $F_1 \times F_2$ is a single space X instead of the product space $X_1 \times X_2$, then both F_1 and F_2 will be $\mathfrak{UV}$-B^*-continuous. Precisely, we prove the following:*

Theorem 5. *Let $F_1 \times F_2 : X \to Y_1 \times Y_2$ be a product multifunction defined by $\big(F_1 \times F_2\big)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is $\mathfrak{UV}$-B^*-continuous, then both $F_1 : X \to Y_1$ and $F_2 : X \to Y_2$ are also $\mathfrak{UV}$-B^*-continuous.*

Proof. Let $x \in X$, $U \in \mathfrak{U}$ containing x, $V_i \in \mathfrak{V}_{Y_i}$ be such that

$$F_i(x) \subset V_i.$$

Then $V_1 \times V_2 \in \mathfrak{V}_{Y_1 \times Y_2}$. Since $F_1 \times F_2$ is a upper $\mathfrak{UV}$-B^*-continuous multifunction, it follows that there exists a B^*-set B containing x such that

$$B \subset U \cap (F_1 \times F_2)^+(V_1 \times V_2).$$

We obtain that $B \subset U \cap F_1^+(V_1)$ and $B \subset U \cap F_2^+(V_2)$. Thus, F_1 and F_2 are both upper $\mathfrak{UV}$-B^*-continuous. The assertion now follows. ◄

Theorem 6. *If Y is a $\mathfrak{V}$-normal space and $F_i : X_i \to Y$ is upper $\mathfrak{UV}$-B^*-continuous multifunction such that F_i is punctually closed for $i = 1, 2$, then a set $\{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is a B^*-closed set in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ and $(x_1, x_2) \in (X_1 \times X_2) \setminus A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is $\mathfrak{V}$-normal and F_i's are punctually closed for $i = 1, 2$, there exist disjoint $\mathfrak{V}$-sets V_1 and V_2 such that

$$F_i(x_i) \subset V_i.$$

Since F_i's are upper $\mathfrak{UV}$-B^*-continuous, for $U_1 \in \mathfrak{U}_{X_1}$ and $U_2 \in \mathfrak{U}_{X_2}$ containing, respectively, x_1 and x_2, there exist B^*-sets B_1 and B_2 such that

$$x_1 \in B_1 \subset U_1 \cap F_1^+(V_1) \qquad \text{and} \qquad x_2 \in B_2 \subset U_2 \cap F_2^+(V_2).$$

Then $B := B_1 \times B_2$ is a B^*-set and $(x_1, x_2) \in B \subset (X_1 \times X_2) \setminus A$. Hence, (x_1, x_2) is not a B^*-cluster point of A and so A is B^*-closed. ◄

4. Interrelation

In [7, 8, 9], various classes of $\mathfrak{UV}$-B^*-continuities were studied and discussed. The interrelationship among those continuities can be described by the following diagram:

$$
\begin{array}{ccccc}
 & B^*-\text{Irresolute} \Longrightarrow & B^*-\text{Continuity} & \Longrightarrow B^*-\text{Clopen Continuity} \\
 & & \Downarrow & \\
 & & \text{Almost } B^*-\text{Continuity} & \\
 & & \Downarrow & \\
\text{Contra } B^*-\text{Continuity} & \Longrightarrow & \text{Weakly } B^*-\text{Continuity} & \\
 & & \Downarrow & \\
 & & \text{Slightly } B^*-\text{Continuity}. &
\end{array}
$$

It was also shown, through various examples, that, in general, the reverse implications in the above diagram do not hold. For some of the reverse implications, additional conditions were provided. In the present section, we suppliment the above mentioned work by formulating conditions so as to make some of these continuities equivalent.

Theorem 7. *Let $F : X \to Y$ be weakly B^*-continuous and $F(x)$ is open in Y for each $x \in X$, then F is almost B^*-continuous.*

Proof. Proof follows easily from the definition. ◀

Theorem 8. *Let $F : X \to Y$ be a closed valued multifunction and Y be normal T_1 space. Then the following are equivalent:*

(a) *F is B^*-continuous.*

(b) *F is almost B^*-continuous.*

(c) *F is weakly B^*-continuous.*

Proof. It is sufficient to prove that (c) implies (a). Here, we prove only for upper continuities. The other case is similar. Let $x \in X$, U and V be open sets containing, respectively, x and $F(x)$. Since F is closed valued, $F(x)$ is closed in Y. Then by normality of Y, there exists an open set $D \subset Y$ such that

$$F(x) \subset D \subset \mathrm{cl}(D) \subset V.$$

F being upper weakly B^*-continuous at x there exists, a B^*-set $B \subset U$ containing x such that

$$F(B) \subset \mathrm{cl}(D) \subset V.$$

Hence, F is upper B^*-continuous, and we are done. ◀

Definition 7. *[15] Let $A \subset X$ then A is said to be*

(a) α-*regular if for each* $a \in A$ *and each open set* U *containing* a, *there exists an open set* $G \subset X$ *such that* $a \in G \subset cl(G) \subset U$.

(b) α-*paracompact if for every* X-*open cover of* A *has an* X-*open refinement which covers* A *and is locally finite for each points of* X.

Theorem 9. *Let* $F : X \to Y$ *be a multifunction,* $F(x)$ *be* α-*regular and* α-*paracompact for each* $x \in X$. *Then following are equivalent:*

(a) F *is* B^*-*continuous.*

(b) F *is almost* B^*-*continuous.*

(c) F *is weakly* B^*-*continuous.*

Proof. It is sufficient to prove that (c) implies (a). Here, we prove only for upper case. The other case is similar. Let $x \in X$, $U \subset X$ and $V \subset Y$ be open sets containing, respectively, $x \in U$ and $F(x) \subset V$. Since $F(x)$ is α-regular, α-paracompact then by Lemma in [15] there exists an open set $G \subset Y$ such that

$$F(x) \subset G \subset \mathrm{cl}(G) \subset V.$$

Now, F being weakly B^*-continuous there exists a B^*-set $B \subset U$, $x \in B$ such that for all $b \in B$,

$$F(b) \subset \mathrm{cl}(G) \subset V.$$

This implies F is upper B^*-continuous and we are done. ◄

Definition 8. *[14] A space* X *is said to be 0-dimensional, if each point of* X *has a neighbourhood base containing of clopen sets. Equivalently, for each point* $x \in X$ *and each closed set* G *not containing* x, *there exists a clopen set containing* x *disjoint from* G.

Theorem 10. *Let* $F : X \to Y$ *be a multifunction,* Y *be 0-dimensional and* $F(x)$ *be mildly compact for each* $x \in X$, *then following are equivalent.*

(a) F *is* B^*-*continuous,*

(b) F *is almost* B^*-*continuous,*

(c) F *is weakly* B^*-*continuous,*

(d) F *is slightly* B^*-*continuous.*

Proof. It is sufficient to prove (d) implies (a). Here, we prove only for upper continuities. The other case is similar. Let $x \in X$, U be an open set containing x and V be an open set containing $F(x)$. Then by 0-dimensionality of Y, for each $y \in F(x)$ there exists a clopen set $W_y \subset Y$ such that

$$y \in W_y \subset V.$$

Since $F(x)$ is mildly compact relative to Y, there exists a finite number of clopen sets $W_{y_1}, W_{y_2}, \dots W_{y_n}$ such that

$$F(x) \subset \bigcup_{i=1}^{n} W_{y_i} \subset V.$$

Now, put $W = \bigcup\limits_{i=1}^{n} W_{y_i}$. Then W be a clopen set such that

$$F(x) \subset W \subset V.$$

Since F is upper slightly B^*-continuous, there exists a B^*-set $B \subset U$ containing x such that for all $b \in B$

$$F(b) \subset W \subset V.$$

This implies F is B^*-continuous and we are done. ◄

Definition 9. *[6] A topological space is said to be categorically closed if every first category set is closed.*

Proposition A. *[6] In a categorically closed topological space the followings hold:*

(i) *If G is open and P is first category then $G \setminus P$ is open.*

(ii) *If E is a second category set with Baire property then E contains a nonempty open set.*

(iii) *If E is a B^*-set then E contains a nonempty open set.*

In addition in Theorem 4, let X be categorically closed then using the Proposition A we have the following corollary:

Corollary 1. *Let $F : X \to Y$ be a multifunction, Y be 0-dimensional, X be categorically closed and $F(x)$ be mildly compact for each $x \in X$, then following are equivalent.*

(a) *F is quasi continuous.*

(b) F *is* B^**-continuous,*

(c) F *is almost* B^**-continuous,*

(d) F *is weakly* B^**-continuous,*

(e) F *is slightly* B^**-continuous.*

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References

[1] E. Ekici, *Slightly* β*-continuous multifunctions*, Demon. Math., **XXXVIII**, 2005, 469–484.

[2] D.K. Ganguly, C. Mitra, B^**-continuity and other generalized continuities*, Rev. Acad. Canar. Cienc., **XII**, 2000, 9–17.

[3] D.K. Ganguly, C. Mitra, *On some weaker forms of* B^**-continuity for multifunctions*, Soochow J. Math., **30**, 2006, 59–69.

[4] D.K. Ganguly, P. Mallick, *On a generalized continuous multifunction and their selections*, Real. Anal. Exch., **34**, 2008, 449–456.

[5] D.K. Ganguly, P. Mallick, *On convergence preserving generalized continuous multifunctions*, Q and A in Gen. Top., **27**, 2009, 125-132.

[6] D.K. Ganguly, P. Mallick, *Note on some classes of generalized continuous functions*, Bull. Pure Appl. Math., **4**, 2010, 184–188.

[7] P. Jain, C. Basu, V. Panwar, *On some generalized* B^**-continuity,* B^**-coverings and* B^**-separations*, Eurasian Math. J., **10**, 2019, 28–39.

[8] P. Jain, C. Basu, V. Panwar, *On* B^**-clopen continuity, oscillation and convergence*, Tbilisi Mathematical Journal, **13**, 2020, 129-140

[9] P. Jain, C. Basu, V. Panwar, *On slightly* B^**-continuity for multifunctions.* (communicated)

[10] K. Kuratowski, *Topology*, **I**, Academic Press, New York, 1966.

[11] M. Matejdes, *Sur les selecteurs des multifonctions*, Math. Slovaca, **37**, 1987, 111–124.

[12] M. Matejdes, *Continuity properties of cluster multifunction and closed graph theorems*, Tatra Mt. Math. Publ., **40**, 2008, 81–89.

[13] C. Mitra, D.K. Ganguly, *Some results on the $\mathcal{E}_{\mathcal{P}}$-cluster system*, Bull. Cal. Math. Soc., **106**, 2014, 453–458.

[14] T. Noiri, V. Popa, *Slightly m-montinuous multifunctions*, Bull. Ins. Math. Aca. Sin., **1**, 2006, 485–505.

[15] V. Popa, T. Noiri, *On some weak forms of continuity for multifunctions*, Istanbul Univ. Fen. Fak. Mat. Der., **60**, 2001, 55–72.

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