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Improvements of Some Numerical Radius Inequalities

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Abstract. In this work, we improve and refine some numerical radius inequalities. In particular, for all Hilbert space operators T, the famous Kittaneh inequality reads:

$$\frac{1}{4}\left\|T^*T+TT^*\right\| \leq w^2\left(T\right) \leq \frac{1}{2}\left\|T^*T+TT^*\right\|.$$

In this work we provide some important refinements for the upper bound of the Kittaneh inequality. Namely, we establish

$$w^2\left(T\right) \leq \frac{1}{2}\left\|T^*T+TT^*\right\| - \frac{1}{4}\inf_{\|x\|=1}\left(\left\langle \left|T\right| x, x\right\rangle - \left\langle \left|T^*\right| x, x\right\rangle\right)^2,$$

which is also refined and improved as

$$w^2\left(T\right) \leq \frac{1}{2}\left\|T^*T+TT^*\right\| - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle \left|T\right| x, x\right\rangle - \left\langle \left|T^*\right| x, x\right\rangle\right)^2,$$

with the third improvement

$$w^2\left(T\right) \leq \frac{1}{4}\left\|\left|T\right|+\left|T^*\right|\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left(\left\langle \left|T\right| x, x\right\rangle - \left\langle \left|T^*\right| x, x\right\rangle\right)^2.$$

Other related results are also obtained.
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1. Introduction

Let $\mathscr{B}\left(\mathscr{H}\right)$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $\left(\mathscr{H};\left\langle \cdot,\cdot\right\rangle\right)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}\left(\mathscr{H}\right)$. A bounded linear operator A defined on $\mathscr{H}$ is selfadjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in \mathscr{H}$. Consider the real vector space $\mathscr{B}(\mathscr{H})_{sa}$ of self-adjoint operators on $\mathscr{H}$ and its positive cone $\mathscr{B}(\mathscr{H})^{+}$ of positive operators on $\mathscr{H}$. A partial order is naturally equipped on $\mathscr{B}(\mathscr{H})_{sa}$ by defining $A \le B$ if and only if $B-A \in \mathscr{B}(\mathscr{H})^{+}$. We write $A > 0$ to mean that A is a strictly positive operator, or equivalently, $A \ge 0$ and A is invertible.

The Schwarz inequality for positive operators reads that if A is a positive operator in $\mathscr{B}(\mathscr{H})$, then

$$|\langle Ax, y\rangle|^2 \le \langle Ax, x\rangle \langle Ay, y\rangle \tag{1}$$

for any vectors $x, y \in \mathscr{H}$.

In 1951, Reid [13] proved an inequality which in some sense was a variant of the Schwarz inequality. In fact, he proved that for all operators $A \in \mathscr{B}(\mathscr{H})$ such that A is positive and AB is selfadjoint the relation

$$|\langle ABx, y\rangle| \le \|B\| \langle Ax, x\rangle \tag{2}$$

holds for all $x \in \mathscr{H}$. In [7], Halmos presented the stronger version of the Reid inequality (2) by replacing $\|B\|$ with $r(B)$.

In 1952, Kato [11] introduced a companion inequality of (1), called the mixed Schwarz inequality, which asserts

$$|\langle Ax, y\rangle|^2 \le \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle, \qquad 0 \le \alpha \le 1. \tag{3}$$

for every operator $A \in \mathscr{B}(\mathscr{H})$ and any vectors $x, y \in \mathscr{H}$, where $|A| = (A^*A)^{1/2}$.

In 1988, Kittaneh [10] proved a very interesting extension combining both the Halmos–Reid inequality (2) and the mixed Schwarz inequality (3). His result reads that

$$|\langle ABx, y\rangle| \le r(B) \|f(|A|)x\| \|g(|A^*|)y\| \tag{4}$$

for any vectors $x, y \in \mathscr{H}$, where $A, B \in \mathscr{B}(\mathscr{H})$ are such that $|A|B = B^*|A|$ and f, g are nonnegative continuous functions defined on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \ge 0)$. For instance, if we set $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$ $(0 \le \alpha \le 1)$ with $B = 1$ in (4), we recapture Kato's inequality (3). Also, as noticed in [12], if in the inequality (4) A is assumed to be positive, then the condition $AB = B^*A$ is equivalent to saying that AB is self-adjoint. In this case, letting $f(t) = g(t) = t^{1/2}$ and $x = y$, we obtain the generalized Reid inequality (2) as a special case. A non-trivial improvement of (4) was established very recently by the author of this paper in [1]. The Cartesian decomposition form of (4) was also recently proved by Alomari in [2].

In 1994, Furuta [6] proved the following generalization of Kato's inequality (3):

$$\left|\left\langle T\left|T\right|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \le \left\langle \left|T\right|^{2\alpha}x,x\right\rangle\left\langle \left|T\right|^{2\beta}y,y\right\rangle \tag{5}$$

for any $x, y \in \mathscr{H}$ and $\alpha, \beta \in [0,1]$ with $\alpha+\beta \ge 1$.

The inequality (5) was generalized for any $\alpha, \beta \ge 0$ with $\alpha+\beta \ge 1$ by Dragomir in [5]. As noted by Dragomir, the condition $\alpha, \beta \in [0,1]$ was assumed by Furuta to fit with the Heinz–Kato inequality, which reads:

$$|\langle Tx, y\rangle| \le \|A^\alpha x\| \left\|B^{1-\alpha}y\right\|$$

for any $x, y \in \mathscr{H}$ and $\alpha \in [0,1]$, where A and B are prositive operators such that $\|Tx\| \le \|Ax\|$ and $\|T^*y\| \le \|By\|$ for any $x, y \in \mathscr{H}$.

For a bounded linear operator T on a Hilbert space $\mathscr{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to \langle Tx, x\rangle$ associated with the operator. More precisely,

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\}.$$

Also, the numerical radius is defined to be

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

The spectral radius of an operator T is defined to be

$$r(T) = \sup\{|\lambda| : \lambda \in \operatorname{sp}(T)\}.$$

We recall that the usual operator norm of an operator T is defined to be

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ defines an operator norm on $\mathscr{B}(\mathscr{H})$ which is equivalent to operator norm $\|\cdot\|$, and for every $T \in \mathscr{B}(T)$, we have

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|. \tag{6}$$

Thus, the usual operator norm and the numerical radius norm are equivalent. The inequalities in (6) are sharp: if $T^2 = 0$, then the first inequality becomes an equality, while the second inequality becomes an equality if T is normal.

In 2003, Kittaneh [10] refined the right-hand side of (6), by proving that

$$w(T) \le \frac{1}{2}\left(\|T\| + \|T^2\|^{1/2}\right) \tag{7}$$

for any $T \in \mathscr{B}(\mathscr{H})$.

After that in 2005, the same author in [8] proved that

$$\frac{1}{4}\|A^*A + AA^*\| \le w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|. \tag{8}$$

The inequality is sharp.

In 2007, Yamazaki [16] improved (8) by proving that

$$w(T) \le \frac{1}{2}\left(\|T\| + w\left(\widetilde{T}\right)\right) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right),$$

where $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ and U is the unitary operator in the polar decomposition T of the form $T = U|T|$.

In 2008, Dragomir [4] used Buzano inequality to improve (1), by proving that

$$w^2(T) \le \frac{1}{2}\left(\|T\| + w\left(T^2\right)\right).$$

This result was also recently generalized by Sattari et al. in [14] and Alomari in [2]. For more recent results about the numerical radius see the recent monograph [3].

In this work, we improve and refine some numerical radius inequalities. In particular, for all Hilbert space operators T, the famous Kittaneh inequality reads:

$$\frac{1}{4}\|T^*T + TT^*\| \le w^2(T) \le \frac{1}{2}\|T^*T + TT^*\|.$$

In this work we provide some important refinements for the upper bound of the Kittaneh inequality. Namely, we establish

$$w^2(T) \le \frac{1}{2}\|T^*T + TT^*\| - \frac{1}{4}\inf_{\|x\|=1}\left(\langle |T|\,x, x\rangle - \langle |T^*|\,x, x\rangle\right)^2,$$

which is also refined and improved as

$$w^2(T) \le \frac{1}{2}\|T^*T + TT^*\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|\,x, x\rangle - \langle |T^*|\,x, x\rangle\right)^2,$$

and

$$w^2(T) \le \frac{1}{2}\|T^*T + TT^*\| - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2\,x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^2\,x, x\right\rangle^{\frac{1}{2}}\right)^2,$$

with the third improvement

$$w^2(T) \le \frac{1}{4}\||T| + |T^*|\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left(\langle |T|\,x, x\rangle - \langle |T^*|\,x, x\rangle\right)^2.$$

Other related results are also obtained.

2. Numerical Radius Inequalities

In order to prove our main result we need the following lemmas:

Lemma 1. *Let $S \in \mathscr{B}(\mathscr{H})$, $S \geq 0$ and $x \in \mathscr{H}$ be a unit vector. Then, the Jensen's operator inequality*

$$\langle Sx, x\rangle^r \leq \langle S^r x, x\rangle, \qquad r \geq 1 \tag{9}$$

and

$$\langle S^r x, x\rangle \leq \langle Sx, x\rangle^r, \qquad r \in [0,1]\,. \tag{10}$$

Kittaneh and Manasrah [9] obtained the following result which is a refinement of the scalar Young inequality.

Lemma 2. *Let $a, b \geq 0$, and $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab + min\left\{\frac{1}{p}, \frac{1}{q}\right\}(a^{\frac{p}{2}} - b^{\frac{q}{2}})^2 \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{11}$$

Recently, Sheikhhosseini *et al.* [15] have obtained the following generalization of (11).

Lemma 3. *If $a, b > 0$, and $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $m = 1, 2, 3, \ldots$,*

$$(a^{\frac{1}{p}} b^{\frac{1}{q}})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{\frac{m}{r}}, \ r \geq 1, \tag{12}$$

where $r_0 = min\left\{\frac{1}{p}, \frac{1}{q}\right\}$. In particular, if $p = q = 2$, then

$$(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + \frac{1}{2^m}(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq 2^{\frac{-m}{r}} (a^r + b^r)^{\frac{m}{r}}\,.$$

For $m = 1$

$$(a^{\frac{1}{2}} b^{\frac{1}{2}}) + \frac{1}{2}(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 \leq 2^{\frac{-1}{r}} (a^r + b^r)^{\frac{1}{r}}\,.$$

In what follows, we establish some numerical radius inequalities by providing some refinements of well-known numerical radius inequalities. Let us begin with the following result.

Theorem 1. *Let $T \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then*

$$w^m\left(T|T|^{\alpha+\beta-1}\right) \leq \frac{1}{2^{\frac{m}{r}}}\left\||T|^{2r\alpha}+|T^*|^{2r\beta}\right\|^{\frac{m}{r}} - \frac{1}{2^m}\inf_{\|x\|=1}\left(\left\langle |T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}} - \left\langle |T^*|^{2\beta}x,x\right\rangle^{\frac{m}{2}}\right)^2 \tag{13}$$

Proof. Let $y = x$ in (5). Then for all $m \geq 1$ we have

$$\begin{aligned}
\left|\left\langle T|T|^{\alpha+\beta-1}x,x\right\rangle\right|^m &\leq \left\langle |T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}}\left\langle |T^*|^{2\beta}x,x\right\rangle^{\frac{m}{2}} \\
&\leq \left(\frac{\left\langle |T|^{2\alpha}x,x\right\rangle^r + \left\langle |T^*|^{2\beta}x,x\right\rangle^r}{2}\right)^{\frac{m}{r}} \qquad \text{(by (12))} \\
&\quad - \frac{1}{2^m}\left(\left\langle |T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}} - \left\langle |T^*|^{2\beta}x,x\right\rangle^{\frac{m}{2}}\right)^2 \\
&\leq \left(\frac{\left\langle |T|^{2r\alpha}x,x\right\rangle + \left\langle |T^*|^{2r\beta}x,x\right\rangle}{2}\right)^{\frac{m}{r}} \qquad \text{(by Lemma 1)} \\
&\quad - \frac{1}{2^m}\left(\left\langle |T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}} - \left\langle |T^*|^{2\beta}x,x\right\rangle^{\frac{m}{2}}\right)^2.
\end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we get the desired result. ◀

Corollary 1. *Let $T \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then*

$$w^2\left(T|T|^{\alpha+\beta-1}\right) \leq \frac{1}{2^{\frac{2}{r}}}\left\||T|^{2r\alpha}+|T^*|^{2r\beta}\right\|^{\frac{2}{r}} - \frac{1}{4}\inf_{\|x\|=1}\left(\left\langle |T|^{2\alpha}x,x\right\rangle - \left\langle |T^*|^{2\beta}x,x\right\rangle\right)^2 \tag{14}$$

Proof. Setting $m = 1$ in (13) we get the desired result. ◀

Remark 1. *Setting $r = 1$ in (14), we get*

$$w^2\left(T|T|^{\alpha+\beta-1}\right) \leq \frac{1}{4}\left\||T|^{2\alpha}+|T^*|^{2\beta}\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left(\left\langle |T|^{2\alpha}x,x\right\rangle - \left\langle |T^*|^{2\beta}x,x\right\rangle\right)^2$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$.

Choosing $\alpha = \beta = \frac{1}{2}$, we get

$$w^2(T) \le \frac{1}{4}\left\| |T| + |T^*| \right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left(\langle |T|\,x,x\rangle - \langle |T^*|\,x,x\rangle\right)^2.$$

However, if we choose $\alpha = \beta = 1$, we get

$$w^2(T\,|T|) \le \frac{1}{4}\left\| |T|^2 + |T^*|^2 \right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left(\left\langle |T|^2\,x,x\right\rangle - \left\langle |T^*|^2\,x,x\right\rangle\right)^2,$$

or it can be rewritten as

$$w^2(T\,|T|) \le \frac{1}{4}\left\| T^*T + TT^* \right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left\langle [T^*T - TT^*]\,x,x\right\rangle^2.$$

.

A generalization of the above results could be embodied as follows:

Theorem 2. *Let $T \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta \ge 0$ be such that $\alpha + \beta \ge 1$. Then*

$$w^{2s}\left(T\,|T|^{\alpha+\beta-1}\right) \le 2^{-\frac{2}{r}}\left\| |T|^{2rs\alpha} + |T^*|^{2rs\beta} \right\|^{\frac{2}{r}} - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle |T|^{2sr\alpha}\,x,x\right\rangle - \left\langle |T^*|^{2rs\beta}\,y,y\right\rangle\right] \tag{15}$$

for all $r, s \ge 1$.

Proof. Let $y = x$ in (5). By applying Lemma 3 with $p = q = 2$ and $m = 2$, we get

$$\begin{aligned}
&\left|\left\langle T\,|T|^{\alpha+\beta-1}\,x,x\right\rangle\right|^{2s} \\
&\le \left\langle |T|^{2\alpha}\,x,x\right\rangle^{s}\left\langle |T^*|^{2\beta}\,x,x\right\rangle^{s} && (t^s \text{ increasing}) \\
&\le \left\langle |T|^{2s\alpha}\,x,x\right\rangle\left\langle |T^*|^{2s\beta}\,x,x\right\rangle && (\text{by convexity of } t^s) \\
&\le 2^{-\frac{2}{r}}\left(\left\langle |T|^{2s\alpha}\,x,x\right\rangle^{r} + \left\langle |T^*|^{2s\beta}\,x,x\right\rangle^{r}\right)^{\frac{2}{r}} && (\text{by Lemma 3}) \\
&\quad - \frac{1}{4}\left[\left\langle |T|^{2sr\alpha}\,x,x\right\rangle - \left\langle |T^*|^{2rs\beta}\,x,x\right\rangle\right] \\
&\le 2^{-\frac{2}{r}}\left(\left\langle |T|^{2rs\alpha}\,x,x\right\rangle + \left\langle |T^*|^{2rs\beta}\,x,x\right\rangle\right)^{\frac{2}{r}} && (\text{by Lemma1}) \\
&\quad - \frac{1}{4}\left[\left\langle |T|^{2sr\alpha}\,x,x\right\rangle - \left\langle |T^*|^{2rs\beta}\,x,x\right\rangle\right].
\end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we get the desired result. ◄

Corollary 2. *Let $T \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then*

$$w^{2s}\left(T\left|T\right|^{\alpha+\beta-1}\right) \leq \frac{1}{4}\left\|\left|T\right|^{2s\alpha} + \left|T^*\right|^{2s\beta}\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|^{2s\alpha}x, x\right\rangle - \left\langle\left|T^*\right|^{2s\beta}x, x\right\rangle\right] \quad (16)$$

for all $s \geq 1$.

Proof. Setting $r = 1$ in (15). ◄

Remark 2. *Setting $\alpha = \beta = \frac{1}{2}$ in (16), we get*

$$w^{2s}(T) \leq \frac{1}{4}\left\|\left|T\right|^s + \left|T^*\right|^s\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|^s x, x\right\rangle - \left\langle\left|T^*\right|^s x, x\right\rangle\right]$$

for all $s \geq 1$. In particular case, choosing $s = 1$ we get

$$w^2(T) \leq \frac{1}{4}\left\|\left|T\right| + \left|T^*\right|\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|x, x\right\rangle - \left\langle\left|T^*\right|x, x\right\rangle\right].$$

.

Remark 3. *Setting $\alpha = \beta = \frac{1}{s}$, $s \geq 1$, we get*

$$w^{2s}\left(T\left|T\right|^{\frac{2}{s}-1}\right) \leq \frac{1}{4}\left\|\left|T\right|^2 + \left|T^*\right|^2\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|^2 x, x\right\rangle - \left\langle\left|T^*\right|^2 x, x\right\rangle\right]. \quad (17)$$

In particular case, choosing $s = 1$ in (17), we get

$$w^2(T\left|T\right|) \leq \frac{1}{4}\left\|\left|T\right|^2 + \left|T^*\right|^2\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|^2 x, x\right\rangle - \left\langle\left|T^*\right|^2 x, x\right\rangle\right], \quad (18)$$

which can be rewritten as

$$w^2(T\left|T\right|) \leq \frac{1}{4}\left\|T^*T + TT^*\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|^2 x, x\right\rangle - \left\langle\left|T^*\right|^2 x, x\right\rangle\right],$$

Remark 4. *Setting $\alpha = \beta = \frac{1}{2}$, $s = 1$, $r = 2$, we get*

$$w^2(T) \leq \frac{1}{2}\left\|\left|T\right|^2 + \left|T^*\right|^2\right\| - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|^2 x, x\right\rangle - \left\langle\left|T^*\right|^2 x, x\right\rangle\right],$$

or

$$w^2(T) \leq \frac{1}{2}\left\|T^*T + TT^*\right\| - \frac{1}{4}\inf_{\|x\|=1}\left[\left\langle\left|T\right|^2 x, x\right\rangle - \left\langle\left|T^*\right|^2 x, x\right\rangle\right], \quad (19)$$

and this refines the upper bound in the Kittaneh inequality (7).

Theorem 3. *Let $T \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then*

$$w^{2s}\left(T\,|T|^{\alpha+\beta-1}\right) \leq \left\| \frac{1}{p}\,|T|^{2sp\alpha} + \frac{1}{q}\,|T^*|^{2sq\beta} \right\| - r_0 \inf_{\|x\|=1} \left(\left\langle |T|^{2s\alpha} x, x \right\rangle^{\frac{p}{2}} - \left\langle |T^*|^{2s\beta} x, x \right\rangle^{\frac{q}{2}} \right)^2 \tag{20}$$

for all $s \geq 1$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $r_0 := \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

In particular case, we have

$$w^{2s}\left(T\,|T|^{\alpha+\beta-1}\right) \leq \frac{1}{2}\left\| |T|^{4s\alpha} + |T^*|^{4s\beta} \right\| - \frac{1}{2} \inf_{\|x\|=1} \left(\left\langle |T|^{2s\alpha} x, x \right\rangle - \left\langle |T^*|^{2s\beta} x, x \right\rangle \right)^2. \tag{21}$$

Proof. Let $s \geq 1$. Setting $y = x$ in (5), we get

$$\begin{aligned}
\left| \left\langle T\,|T|^{\alpha+\beta-1} x, x \right\rangle \right|^{2s} &\leq \left\langle |T|^{2\alpha} x, x \right\rangle^{s} \left\langle |T^*|^{2\beta} x, x \right\rangle^{s} && \text{(by (5))} \\
&\leq \left\langle |T|^{2s\alpha} x, x \right\rangle \left\langle |T^*|^{2s\beta} x, x \right\rangle && \text{(by convexity of } t^s\text{)} \\
&\leq \frac{1}{p}\left\langle |T|^{2s\alpha} x, x \right\rangle^{p} + \frac{1}{q}\left\langle |T^*|^{2s\beta} x, x \right\rangle^{q} && \text{(by Lemma 2)} \\
&\quad - r_0 \left(\left\langle |T|^{2s\alpha} x, x \right\rangle^{\frac{p}{2}} - \left\langle |T^*|^{2s\beta} x, x \right\rangle^{\frac{q}{2}} \right)^2 \\
&\leq \frac{1}{p}\left\langle |T|^{2sp\alpha} x, x \right\rangle + \frac{1}{q}\left\langle |T^*|^{2sq\beta} x, x \right\rangle && \text{(by Lemma1)} \\
&\quad - r_0 \left(\left\langle |T|^{2s\alpha} x, x \right\rangle^{\frac{p}{2}} - \left\langle |T^*|^{2s\beta} x, x \right\rangle^{\frac{q}{2}} \right)^2
\end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we get the required result. The particular case follows by setting $p = q = 2$. ◄

Various interesting special cases could be deduced from (13). In what follows, we give some of these cases in remarks.

Remark 5. *Setting $\alpha = \beta = \frac{1}{2}$ in (14), we have*

$$w^{2s}(T) \leq \frac{1}{2}\left\| |T|^{2s} + |T^*|^{2s} \right\| - \frac{1}{2} \inf_{\|x\|=1} \left(\langle |T|^{s} x, x \rangle - \langle |T^*|^{s} x, x \rangle \right)^2$$

for all $s \geq 1$. In particular, for $s = 1$ we get

$$w^{2}(T) \leq \frac{1}{2}\left\| |T|^{2} + |T^*|^{2} \right\| - \frac{1}{2} \inf_{\|x\|=1} \left(\langle |T|\, x, x \rangle - \langle |T^*|\, x, x \rangle \right)^2,$$

which can be rewritten as

$$w^2(T) \le \frac{1}{2}\|T^*T + TT^*\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|x,x\rangle - \langle |T^*|x,x\rangle\right)^2. \tag{22}$$

This refines the upper bound of the refinement of Kittaneh inequality (19). *Clearly,* (22) *is better than* (19), *which, in turn, is better than* (7).

Remark 6. *Setting $\alpha = \beta = 1$ in* (20), *we have*

$$\begin{aligned} w^{2s}(T|T|) \le &\left\|\frac{1}{p}|T|^{2sp} + \frac{1}{q}|T^*|^{2sq}\right\| \\ &- r_0 \inf_{\|x\|=1}\left(\left\langle |T|^{2s}x,x\right\rangle^{\frac{p}{2}} - \left\langle |T^*|^{2s}x,x\right\rangle^{\frac{q}{2}}\right)^2 \end{aligned}$$

for all $s \ge 1$ and $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $r_0 := \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

In particular case, choosing $s = 1$ and $p = q = 2$ in the previous inequality, we get

$$w^2(T|T|) \le \frac{1}{2}\left\||T|^4 + |T^*|^4\right\| - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2x,x\right\rangle - \left\langle |T^*|^2x,x\right\rangle\right)^2.$$

Numerical radius inequality for a special type of Hilbert space operators for commutators can be established as follows:

Theorem 4. *Let $T, S \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta, \gamma, \delta \ge 0$ be such that $\alpha + \beta \ge 1$ and $\gamma + \delta \ge 1$. Then*

$$\begin{aligned} &w\left(T|T|^{\alpha+\beta-1} + S|S|^{\gamma+\delta-1}\right) \qquad (23)\\ &\le 2^{-\frac{1}{r}}\left\||T|^{2r\alpha} + |T^*|^{2r\beta}\right\|^{\frac{1}{r}} + 2^{-\frac{1}{r}}\left\||S|^{2r\gamma} + |S^*|^{2r\delta}\right\|^{\frac{1}{r}} \\ &\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^{2\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x,x\right\rangle^{\frac{1}{2}}\right)^2 \\ &\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |S|^{2\gamma}x,x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x,x\right\rangle^{\frac{1}{2}}\right)^2 \end{aligned}$$

for all $r \ge 1$.

Proof. Employing the triangle inequality, we have

$$\left|\left\langle\left(T|T|^{\alpha+\beta-1} + S|S|^{\gamma+\delta-1}\right)x,x\right\rangle\right|$$

$$
\begin{aligned}
&\leq \left|\left\langle T\left|T\right|^{\alpha+\beta-1}x,x\right\rangle\right| + \left|\left\langle S\left|S\right|^{\gamma+\delta-1}x,x\right\rangle\right| \\
&\leq \left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{\frac{1}{2}}\left\langle \left|T^*\right|^{2\beta}x,x\right\rangle^{\frac{1}{2}} + \left\langle \left|S\right|^{2\gamma}x,x\right\rangle^{\frac{1}{2}}\left\langle \left|S^*\right|^{2\delta}x,x\right\rangle^{\frac{1}{2}} \qquad \text{(by (5))} \\
&\leq 2^{-\frac{1}{r}}\left(\left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{r} + \left\langle \left|T^*\right|^{2\beta}x,x\right\rangle^{r}\right)^{\frac{1}{r}} \qquad \text{(by Lemma 3)} \\
&\qquad - \frac{1}{2}\left(\left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle \left|T^*\right|^{2\beta}x,x\right\rangle^{\frac{1}{2}}\right)^2 \\
&\quad + 2^{-\frac{1}{r}}\left(\left\langle \left|S\right|^{2\gamma}x,x\right\rangle^{r} + \left\langle \left|S^*\right|^{2\delta}x,x\right\rangle^{r}\right)^{\frac{1}{r}} \\
&\qquad - \frac{1}{2}\left(\left\langle \left|S\right|^{2\gamma}x,x\right\rangle^{\frac{1}{2}} - \left\langle \left|S^*\right|^{2\delta}x,x\right\rangle^{\frac{1}{2}}\right)^2 \\
&\leq 2^{-\frac{1}{r}}\left(\left\langle \left|T\right|^{2r\alpha}x,x\right\rangle + \left\langle \left|T^*\right|^{2r\beta}x,x\right\rangle\right)^{\frac{1}{r}} \qquad \text{(by Lemma 1)} \\
&\qquad - \frac{1}{2}\left(\left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle \left|T^*\right|^{2\beta}x,x\right\rangle^{\frac{1}{2}}\right)^2 \\
&\quad + 2^{-\frac{1}{r}}\left(\left\langle \left|S\right|^{2r\gamma}x,x\right\rangle + \left\langle \left|S^*\right|^{2r\delta}x,x\right\rangle\right)^{\frac{1}{r}} \\
&\quad - \frac{1}{2}\left(\left\langle \left|S\right|^{2\gamma}x,x\right\rangle^{\frac{1}{2}} - \left\langle \left|S^*\right|^{2\delta}x,x\right\rangle^{\frac{1}{2}}\right)^2.
\end{aligned}
$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we get the desired result. ◀

Corollary 3. *Let $T, S \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta, \gamma, \delta \geq 0$ be such that $\alpha + \beta \geq 1$ and $\gamma + \delta \geq 1$. Then*

$$
\begin{aligned}
w\left(T\left|T\right|^{\alpha+\beta-1} + S\left|S\right|^{\gamma+\delta-1}\right) \leq \frac{1}{2}\left\|\left|T\right|^{2\alpha} + \left|T^*\right|^{2\beta} + \left|S\right|^{2\gamma} + \left|S^*\right|^{2\delta}\right\| \qquad (24) \\
- \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle \left|T^*\right|^{2\beta}x,x\right\rangle^{\frac{1}{2}}\right)^2 \\
- \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle \left|S\right|^{2\gamma}x,x\right\rangle^{\frac{1}{2}} - \left\langle \left|S^*\right|^{2\delta}x,x\right\rangle^{\frac{1}{2}}\right)^2.
\end{aligned}
$$

Proof. Setting $r = 1$ in the proof of Theorem 4, and then taking the supremum over all unit vectors $x \in \mathscr{H}$, we get the desired result. ◀

Remark 7. *Setting $\alpha = \beta = \gamma = \delta = \frac{1}{2}$ in (24), we get*

$$
w(T+S) \leq \frac{1}{2}\left\|\left|T\right| + \left|T^*\right| + \left|S\right| + \left|S^*\right|\right\| - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle \left|T\right|x,x\right\rangle^{\frac{1}{2}} - \left\langle \left|T^*\right|x,x\right\rangle^{\frac{1}{2}}\right)^2
$$

$$-\frac{1}{2}\inf_{\|x\|=1}\left(\langle |S|\,x,x\rangle^{\frac{1}{2}}-\langle |S^*|\,x,x\rangle^{\frac{1}{2}}\right)^2$$

In particular, taking $S=T$ we get

$$w(T)\leq \frac{1}{2}\left\||T|+|T^*|+\right\|-\frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|\,x,x\rangle^{\frac{1}{2}}-\langle |T^*|\,x,x\rangle^{\frac{1}{2}}\right)^2.$$

Remark 8. *Setting $\alpha=\beta=\gamma=\delta=1$ in (24), we get*

$$\begin{aligned}w(T\,|T|+S\,|S|)\leq{}& \frac{1}{2}\left\||T|^2+|T^*|^2+|S|^2+|S^*|^2\right\|\\ &-\frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2\,x,x\right\rangle^{\frac{1}{2}}-\left\langle |T^*|^2\,x,x\right\rangle^{\frac{1}{2}}\right)^2\\ &-\frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |S|^2\,x,x\right\rangle^{\frac{1}{2}}-\left\langle |S^*|^2\,x,x\right\rangle^{\frac{1}{2}}\right)^2\end{aligned}$$

In particular, taking $S=T$, we get

$$\begin{aligned}w(T\,|T|)&\leq \frac{1}{2}\left\||T|^2+|T^*|^2\right\|-\frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2\,x,x\right\rangle^{\frac{1}{2}}-\left\langle |T^*|^2\,x,x\right\rangle^{\frac{1}{2}}\right)^2\\ &=\frac{1}{2}\left\|T^*T+TT^*\right\|-\frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2\,x,x\right\rangle^{\frac{1}{2}}-\left\langle |T^*|^2\,x,x\right\rangle^{\frac{1}{2}}\right)^2\end{aligned}$$

References

[1] M.W. Alomari, *On the generalized mixed Schwarz inequality*, Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, **46(1)**, 2020, 3–15.

[2] M.W. Alomari, *Refinements of some numerical radius inequalities for Hilbert space operators*, Linear and Multilinear Algebra, **69(7)**, 2021, 1208–1223.

[3] S.S. Dragomir, *Inequalities for the numerical radius of linear operators in Hilbert spaces*, Springer Briefs in Mathematics, 2013.

[4] S.S. Dragomir, *Some inequalities for the norm and the numerical radius of linear operator in Hilbert spaces*, Tamkang J. Math., **39(1)**, 2008, 1–7.

[5] S.S. Dragomir, *Some Inequalities generalizing Kato's and Furuta's results*, Filomat, **28(1)**, 2014, 179–195.

[6] T. Furuta, *An extension of the Heinz–Kato theorem*, Proc. Amer. Math. Soc., **120(3)**, 1994, 785–787.

[7] P.R. Halmos, *A Hilbert space problem book*, Van Nostrand Company, Inc., Princeton, N.J., 1967.

[8] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168(1)**, 2005, 73–80.

[9] F. Kittaneh, Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., **361**, 2010, 262–269.

[10] F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158**, 2003, 11–17.

[11] T. Kato, Notes on some inequalities for linear operators, Math. Ann., **125**, 1952, 208–212.

[12] C.-S. Lin, S.S. Dragomir, *On High-power Operator inequalities and spectral radii of operators*, Publ. RIMS, Kyoto Univ., **42**, 2006, 391–397.

[13] W. Reid, *Symmetrizable completely continuous linear tarnsformations in Hilbert space*, Duke Math., **18**, 1951, 41–56.

[14] M. Sattari, M.S. Moslehian, T. Yamazaki, *Some genaralized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl., **470**, 2014, 1–12.

[15] A. Sheikhhosseini, M.S. Moslehian, K. Shebrawi, *Inequalities for generalized Euclidean operator radius via Young's inequality*, J. Math. Anal. Appl., **445**, 2017, 1516–1529.

[16] T. Yamazaki, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math., **178**, 2007, 83–89.

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