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f-Aggregation Operators on a Bounded Lattice

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Abstract. In this paper, we introduce and study the notion of aggregation operator with respect to a given function f (f-aggregation operator, for short) on a bounded lattice. This new notion is a natural generalization of the aggregation operators on bounded lattices. More precisely, we show some new properties of binary operations based on a given function on a lattice, and study their composition with respect to a given aggregation operator. Also, we investigate the transformation of f-aggregation operators based on a lattice-automorphism and a strong negation. Moreover, under some conditions on a given function f, we give the smallest (resp. the greatest) f-aggregation operator on a bounded lattice.

Key Words and Phrases: bounded lattice, binary operation, aggregation operator, f-aggregation operator.
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1. Introduction

Binary operations are among the oldest fundamental concepts in algebraic structures. A binary operation on a non-empty set X is a map from the Cartesian product $X \times X$ into X. Binary operations have become essential tools in lattice theory and its applications, several notions and properties and the notion of the lattice itself can be expressed in terms of binary operations [6, 11, 18]. Further, binary operations with specific properties appear in various theoretical and application domains. For instance, aggregation operators (also known as aggregation functions) as generalizations of the meet and the join operations on the unit interval $[0, 1]$, or on a bounded lattice have been used in the fuzzy set theory [3, 9, 23]. Growing interest in the field of aggregation operators [7, 10, 12, 13, 16, 20] has led to the development of several new mathematical techniques, which have subsequently fostered the creation and analysis of new families of aggregation operators. The importance of aggregation operators is made apparent by their

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wide use, not only in pure mathematics, but also in several applied areas such as operations research, computer and information sciences, economics and social sciences [3, 5, 8, 9]. Several classes of aggregation operators on bounded partially ordered sets are introduced and discussed (see e.g. [13, 16, 17]). Recently, the relaxation of monotonicity has become a trend in the theory of aggregation operators [15, 19]. In general, the notion of monotonicity and its different types and extensions are witnessing attention and appear in various studies. Two of these types of monotonicity are especially relevant:

1. The idea of weak monotonicity (Wilkin and Beliakov [22]), which, for a function of n variables, considers increasingness or decreasingness only along the ray defined by the vector $(1, \ldots, 1)$. It turns out that this notion allows to cover some relevant statistical operators which are not monotone, as, for instance, the mode, Gini means and mixture functions (see e.g. [2, 21]).

2. The idea of directional monotonicity (Bustince et al [4]), which generalizes weak monotonicity considering increasingness or decreasingness along a ray which can be defined by any vector. When a function is directionally increasing with respect to a set of vectors that form a cone, it is called cone-monotone (Beliakov, Calvo and Wilkin [1]). This idea has led to the notion of pre-aggregation operator, which is a function fulfilling the same boundary conditions as an aggregation operator, but it is just directionally increasing along some ray. Pre-aggregation operators have shown themselves a very powerful tool in classification problems (Lucca et al [14]).

Given the importance of the above ideas and the different generalized notions of monotonicity of aggregation operators, in the present paper we introduce the notion of f-aggregation operator on a bounded lattice as a new natural generalization of aggregation operator, based on an arbitrary function f (not necessary the identity function). To that end, we introduce the notions of f-monotonicity and f-boundary conditions of binary operations on a bounded lattice with respect to a given function f. Some new generalized properties of binary operations with respect to a given function f on a lattice such as f-increasing (resp. f-decreasing), f-conjunctive (resp. f-disjunctive), f-commutative, f-idempotent and f-neutral element are discussed. Also, we study the relationships between some interesting properties of binary operations on a lattice and their extensions with respect to a given function on that lattice. We provide some properties of a binary operation based on an arbitrary function on a lattice in order that it can be represented by the meet and the join operations of that lattice. Further, we study the composition of f-aggregation operators on a bounded lattice with respect to an aggregation operator, and investigate their transformations based on a lattice-automorphism and a strong negation. Finally, we provide some conditions that have to be satisfied to provide the existence of the smallest (resp. the greatest) f-aggregation operator on a bounded lattice.

The rest of the paper is organized as follows. In Section 2, we recall the necessary concepts that will be needed throughout this paper. In Section 3, we introduce the notions of f-increasing, f-conjunctive, f-disjunctive and f-idempotent binary operations on a lattice and investigate their various properties. In Section 4, we extend the notion of aggregation operator on a bounded lattice to f-aggregation operator with respect to an arbitrary function f and investigate its various properties. In Section 5, we show a relationship between f-aggregation operator transforming a given f-aggregation operator on a bounded lattice based on a given lattice-automorphism (resp. a strong negation). We investigate the existence of the smallest and the greatest f-aggregation operators on a bounded lattice in Section 6. Finally, we present some conclusions and discuss future research in Section 7.

2. Basic concepts

In this section, we recall some basic definitions and properties of lattices, functions and binary operations on a lattice that will be needed throughout this paper. Further information can be found in [6, 18].

A partially ordered set (poset, for short) $(L, \leq)$ is called a *lattice* if any two elements x and y have a *greatest lower bound*, denoted $x \wedge y$ and called the *meet* (*infimum*) of x and y, as well as a *least upper bound*, denoted $x \vee y$ and called the *join* (*supremum*) of x and y. A lattice can also be defined as an algebraic structure: a set L equipped with two binary operations $\wedge$ and $\vee$ that are idempotent, commutative, associative and satisfy the absorption laws ($x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$, for any $x, y \in L$). The order relation and the meet and join operations are then related as follows: $x \leq y$ if and only if $x \wedge y = x$; $x \leq y$ if and only if $x \vee y = y$. A bounded lattice is a lattice $(L, \leq, \wedge, \vee)$ that additionally has a *least element* denoted by 0 and a *greatest element* denoted by 1 satisfying $0 \leq x \leq 1$, for any $x \in L$. For a bounded lattice, the notation $(L, \leq, \wedge, \vee, 0, 1)$ is used.
Throughout this paper, for a given function $f : L \longrightarrow L$, we shortly write fx instead of $f(x)$.

Definition 1. *Let $(L, \leq)$ be a poset and f a function on L. Then f is called isotone (resp. antitone) if $x \leq y$ implies $fx \leq fy$ (resp. $fy \leq fx$), for any $x, y \in L$.*

Definition 2. *Let $(L, \leq, \wedge, \vee)$ be a lattice and f a function from L into L. f is called a lattice-endomorphism, if it satisfies $f(x \wedge y) = fx \wedge fy$ and $f(x \vee y) = fx \vee fy$, for any $x, y \in L$. A lattice-automorphism is a bijective lattice-endomorphism.*

Next, we need the following result.

Proposition 1. *[6] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. If f is a lattice-automorphism, then $f(0) = f^{-1}(0) = 0$ and $f(1) = f^{-1}(1) = 1$.*

Definition 3. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A function N on L is called a negation on L if it satisfies the following conditions:*

1. *$N(0) = 1$ and $N(1) = 0$;*
2. *N is antitone.*

Additionally, N is called a strong negation, if it is also involutive (i.e., $N(Nx) = x$, for any $x \in L$).

Definition 4. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $A : L^2 \longrightarrow L$ a binary operation on L.*

1. *A is called idempotent, if $A(x, x) = x$, for any $x \in L$;*
2. *A is called increasing, if $x_1 \leq x_2$ and $y_1 \leq y_2$ implies $A(x_1, y_1) \leq A(x_2, y_2)$, for any $x_1, y_1, x_2, y_2 \in L$;*
3. *A is called conjunctive (resp. disjunctive), if $A(x, y) \leq x \wedge y$ (resp. $x \vee y \leq A(x, y)$), for any $x, y \in X$;*
4. *A is called averaging, if $x \wedge y \leq A(x, y) \leq x \vee y$, for any $x, y \in L$;*
5. *An element $e \in L$ is called a neutral element of A, if $A(e, x) = A(x, e) = x$, for any $x \in L$.*

3. Function-based new generalized properties of binary operations on a lattice

In this section, we discuss some new generalized properties of binary operations on a lattice with respect to a given function on that lattice. These new properties are generalization of known properties of binary operations on a lattice L with respect to a function f on L, and coincide with them when f is the identity function. As application, with respect to a given function we study the relationships between some interesting properties of binary operations on a lattice and their extensions.

3.1. New generalized properties of binary operations on a lattice

In this subsection, we introduce some new properties of binary operations on a lattice with respect to a given function and we present an illustrative example. More precisely, we introduce the notions of f-increasing (resp. f-decreasing), f-conjunctive (resp. f-disjunctive) and f-idempotent binary operations on a lattice and investigate their properties.

Definition 5. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. Then A is called:*

1. *left increasing (resp. left decreasing) with respect to f (left f-increasing (resp. left f-decreasing), for short), if $x \leq y$ implies $A(fx, z) \leq A(fy, z)$ (resp. $A(fy, z) \leq A(fx, z)$) , for any $x, y, z \in L$;*

2. *right increasing (resp. right decreasing) with respect to f (right f-increasing (resp. right f-decreasing), for short), if $x \leq y$ implies $A(z, fx) \leq A(z, fy)$ (resp. $A(z, fy) \leq A(z, fx)$) , for any $x, y, z \in L$;*

3. *increasing (resp. decreasing) with respect to f (f-increasing (resp. f-decreasing), for short), if A is both left and right f-increasing (resp. f-decreasing).*

Definition 6. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. Then A is called:*

1. *left (resp. right) f-conjunctive if it satisfies $A(fx, y) \leq x$ (resp. $A(x, fy) \leq y$), for any $x, y \in L$;*

2. *f-conjunctive if it is both left and right f-conjunctive;*

3. *left (resp. right) f-disjunctive if it satisfies $x \leq A(fx, y)$ (resp. $y \leq A(x, fy)$), for any $x, y \in L$;*

4. *f-disjunctive if it is both left and right f-disjunctive.*

In the following, we give an illustrative example of the above new generalized properties of binary operations on a lattice.

Example 1. *Let $(D(12), |, gcd, lcm)$ be the bounded lattice of positive divisors of 12 ordered by the divisibility order. Let $f : D(12) \longrightarrow D(12)$ be a function and A, B two binary operations on $D(12)$ defined as follows:*

x	1	2	3	4	6	12
fx	6	2	1	12	3	4

,

$A(x,y)$	1	2	3	4	6	12
1	3	1	3	3	1	1
2	1	2	2	2	1	2
3	3	2	6	6	1	2
4	3	2	6	12	1	4
6	1	1	1	1	1	1
12	1	2	2	4	1	4

and

$B(x,y)$	1	2	3	4	6	12
1	3	6	6	12	3	12
2	6	2	6	12	2	4
3	6	6	6	12	6	12
4	12	12	12	12	12	12
6	3	2	6	12	1	4
12	12	4	12	12	4	4

One easily verifies that A and B are f-increasing, but they are not increasing. Indeed, let $x, y \in D(12)$ be such that $x \mid y$. Setting $x = 3$, $y = 6$ and $z = 1$, we obtain $x \mid y$, but $A(x,z) = A(3,1) = 3 \nmid 1 = A(6,1) = A(y,z)$ and $B(x,z) = B(3,1) = 6 \nmid 3 = B(6,1) = B(y,z)$. Hence, A and B are not increasing. Therefore, A and B are not aggregation operators on $D(12)$.

Furthermore, it is not difficult to check that A is f-conjunctive and B is f-disjunctive on $D(12)$. Notice that A (resp. B) is not conjunctive (resp. disjunctive) on $D(12)$.

Definition 7. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. An element $e \in L$ is called:*

1. *left (resp. right) f-neutral element of A, if $A(e, fx) = x$ (resp. $A(fx, e) = x$), for any $x \in L$;*

2. *f-neutral element of A if it is both a left and a right f-neutral element of A.*

Example 2. *Let $(L = \{0, a, b, c, 1\}, \leq, \wedge, \vee)$ be the lattice given by the Hasse diagram in Figure 1 and f (resp. A) a function (resp. binary operation) on L defined as follows:*

x	0	a	b	c	1
fx	1	b	c	a	0

and

$A(x,y)$	0	a	b	c	1
0	1	1	0	0	0
a	1	c	a	b	0
b	0	a	b	c	1
c	0	b	c	a	1
1	0	0	1	1	1

It is not difficult to show that the element $a \in L$ is an f-neutral element of A, but it is not a neutral element of A.

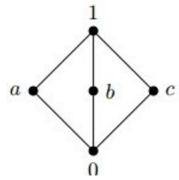

Figure 1: The Hasse diagram of the lattice $(L = \{0, a, b, c, 1\}, \leq)$.

The following proposition show that if an element is both a neutral and an f-neutral element of a binary operation on a lattice, then f is the identity function. The proof is straightforward.

Proposition 2. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) be a binary operation (resp. function) on L. If $e \in L$ is both a neutral and an f-neutral element of A, then f is the identity function of L.*

Definition 8. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. A is called f-commutative, if $A(fx, y) = A(x, fy)$, for any $x, y \in L$.*

Example 3. *Let A (resp. f) be a binary operation (resp. function) on L given in Example 2. One easily verifies that A is f-commutative.*

Definition 9. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. A is called f-idempotent, if $A(fx, fx) = x$, for any $x \in L$.*

Example 4. *Consider the binary operation A and the function f given in Example 1. One easily verifies that A is f-idempotent, and not idempotent.*

3.2. Relationships between properties of binary operations on a lattice and their f-extensions

In this subsection, we study the relationships between some interesting properties of binary operations on a lattice and their extensions with respect to a given function on that lattice. Moreover, we provide some properties of a binary operation based on an arbitrary function on a lattice in order that it can be represented by the meet and the join operations of that lattice.

Proposition 3. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A a binary operation on L. Then the following assertions are true:*

1. *If A is increasing, then A is f-increasing, for any isotone function f on L;*

2. If A is increasing, then A is f-decreasing, for any antitone function f on L.

The following proposition shows the interaction of the notion of f-increasing with the function composition.

Proposition 4. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f, g two functions on L such that g is isotone. If A is f-increasing on L, then A is $f \circ g$-increasing on L.*

Proof. Let $x, y \in L$ be such that $x \leq y$. Since g is isotone, we have $gx \leq gy$. The fact that A is f-increasing implies that $A(f(gx), z) \leq A(f(gy), z)$, for any $z \in L$. Thus, $A(f \circ g(x), z) \leq A(f \circ g(y), z)$, for any $z \in L$. Therefore, A is $f \circ g$-increasing. ◄

The above propositions lead to the following corollary.

Corollary 1. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A an f-increasing binary operation on L. The following statements hold:*

1. *If f is isotone, then A is f^n-increasing, for any $n \in \mathbb{N}^*$;*
2. *If f is antitone, then A is f^{2n+1}-increasing, for any $n \in \mathbb{N}$;*

Remark 1. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. The following implications hold:*

1. *If A is conjunctive and $fx \leq x$, for any $x \in L$, then A is f-conjunctive;*
2. *If A is disjunctive and $x \leq fx$, for any $x \in L$, then A is f-disjunctive.*

The following proposition shows that a given binary operation on a lattice has at most one f-neutral element.

Proposition 5. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. If A is f-commutative, then A has at most one f-neutral element.*

Proof. Let A be an f-commutative binary operation on L having two f-neutral elements $e_1, e_2 \in L$. Then $e_1 = A(fe_1, e_2) = A(e_1, fe_2) = e_2$. Therefore, $e_1 = e_2$. ◄

In the following, we give an example to explain the result of Theorem 3 .

Example 5. *Let A (resp. f) be an f-commutative binary operation (resp. function) on L given in Example 2. One easily verifies that $a \in L$ is the only f-neutral element of A.*

In the following theorem, we characterize f-conjunctive (resp. f-disjunctive) binary operation on a bounded lattice in terms of f-neutral element. This characterization is an extension to that known in [24] (Proposition 5.3).

Theorem 1. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A a binary operation on L having an f-neutral element $e \in L$ such that A is f-commutative and f-increasing. The following equivalences hold:*

1. *A is f-conjunctive if and only if $e = 1$;*
2. *A is f-disjunctive if and only if $e = 0$.*

Proof.

1. The fact that A is f-commutative and f-conjunctive implies that $1 = A(f1, e) = A(1, fe) \leq e$. Hence, $e = 1$. Conversely, suppose that $e = 1$ and let $x, y \in L$. Since A is f-commutative and f-increasing, it follows that $A(fx, y) = A(x, fy) \leq A(x, f1) = A(fx, 1) = x$. In similar way, we obtain $A(x, fy) \leq y$. Hence, A is left and right f-conjunctive. Thus, A is f-conjunctive.
2. The proof is dual to that of (i). ◀

The following result provides some properties of a binary operation based on an arbitrary function on a lattice in order that it can be represented by the meet and the join operations of that lattice.

Proposition 6. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. If f is surjective, then the following implications hold:*

1. *if A is f-idempotent, f-increasing and f-conjunctive, then for any $a, b \in L$, there exist $x, y \in L$ such that $A(a, b) = x \wedge y$;*
2. *if A is f-idempotent, f-increasing and f-disjunctive, then for any $a, b \in L$, there exist $x, y \in L$ such that $A(a, b) = x \vee y$.*

Proof.

1. Let $a, b \in L$. Then there exist $x, y \in L$ such that $fx = a$ and $fy = b$ due to the fact that f is surjective. Since A is f-increasing and f-idempotent, it follows that $x \wedge y = A(f(x \wedge y), f(x \wedge y)) \leq A(fx, fy) = A(a, b)$. The fact that A is f-conjunctive implies that $A(a, b) = A(fx, fy) \leq x \wedge y$. Thus, $A(a, b) = x \wedge y$.
2. The proof is dual to that of (i). ◀

Proposition 6 leads to the following result.

Proposition 7. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A (resp. f) a binary operation (resp. function) on L. If f is surjective, then the following equivalences hold:*

1. *A is f-idempotent, f-increasing and f-conjunctive if and only if for any $x, y \in L$ there is $A(fx, fy) = x \wedge y$;*
2. *A is f-idempotent, f-increasing and f-disjunctive if and only if for any $x, y \in L$ there is $A(fx, fy) = x \vee y$.*

Proof.

1. The proof of the direct implication follows from Propositions 6. Next, we prove the converse implication. Let $x, y, z \in L$ be such that $x \leq y$. The fact that f is surjective means that there exists $t \in L$ such that $z = ft$. Thus, $A(fx, z) = A(fx, ft) = x \wedge t \leq y \wedge t = A(fy, ft) = A(fy, z)$. Hence, A is left f-increasing. In similar way, we can show that A is right f-increasing. Now, we prove that A is f-conjunctive. Let $x, y \in L$. Since f is surjective, there exists $s \in L$ such that $y = fs$. Then $A(fx, y) = A(fx, fs) = x \wedge s \leq x$. In similar way, we can show that $A(x, fy) \leq y$. Hence A is f-conjunctive. It is obvious that A is f-idempotent.
2. The proof is dual to that of (i). ◄

4. f-aggregation operators on a bounded lattice

In this section, we extend the notion of aggregation operator on a bounded lattice introduced by Mesiar and Komorníková [16] to f-aggregation operator, where f is an arbitrary function on that bounded lattice. Furthermore, various properties of this notion and its links with the notion of aggregation operator on a bounded lattice are discussed.

4.1. Definitions and examples

In this subsection, we introduce the notion of f-aggregation operator on a bounded lattice and we give some illustrative examples for the clarity. First, we recall the definition of aggregation operator on a bounded lattice.

Definition 10. *[16] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. An aggregation operator on L is a binary operation A on L which is increasing and satisfies the boundary conditions $A(0, 0) = 0$ and $A(1, 1) = 1$.*

Next, we extend this definition by using a given function f on a bounded lattice as follows.

Definition 11. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. An aggregation operator with respect to the function f (f-aggregation operator, for short) on L is a binary operation A on L which is f-increasing and satisfies the f-boundary conditions $A(f(0), f(0)) = 0$ and $A(f(1), f(1)) = 1$.*

Remark 2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A an f-aggregation operator on L. If f is a constant function on L, then $L = \{0\}$. Indeed, let A be an f-aggregation operator on L. Since f is constant, we have $fx = fy$, for any $x, y \in L$. Then $0 = A(f(0), f(0)) = A(f(1), f(1)) = 1$. Hence, $0 = 1$. Therefore, $L = \{0\}$.*

For the rest of the paper we will assume that f is not a constant function.

Example 6. *Let $f : [0, 1] \longrightarrow [0, 1]$ be a function and A a binary operation on $[0, 1]$ defined by*

$$fx = \frac{1}{2}x \text{ and } A(x, y) = \begin{cases} 1 & \text{if } x = y = \frac{1}{2}; \\ \frac{1}{(1+x)(1+y)} & \text{if } (x, y) \in]\frac{1}{2}, 1]\times]\frac{1}{2}, 1]; \\ xy & \text{otherwise.} \end{cases}$$

One easily verifies that A is an f-aggregation operator. But, A is not an aggregation operator on $[0, 1]$. Indeed, let $x, y \in [0, 1[$ be such that $x \leq y$. Setting $x = \frac{2}{3}$, $y = \frac{3}{4}$ and $z = 1$, we obtain $x \leq y$, but $A(x, z) = \frac{1}{(1+x)(1+z)} = \frac{3}{10} \geq \frac{2}{7} = \frac{1}{(1+y)(1+z)} = A(y, z)$. Hence, A is not increasing. However, A is not an aggregation operator.

Example 7. *Let A, B (resp. f) be the binary operations (resp. the function) on $D(12)$ defined in Example 1. One easily verifies that A and B are f-increasing and satisfy the f-boundary conditions. Thus, A and B are f-aggregation operators on $D(12)$.*

Remark 3. *In general, we use aggregation operators (increasing binary operations) on a given universe to aggregate objects on that universe. While the notion of f-aggregation operators (f-increasing operations) allows the use of non-increasing operations to aggregate objects with respect to specific functions on that universe.*

4.2. Properties of f-aggregation operators on a bounded lattice

In this subsection, we investigate basic properties of f-aggregation operators on a bounded lattice. First, we show that any aggregation operator is an f-aggregation operator, for any isotone function on that lattice and not conversely.

Proposition 8. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A a binary operation on L. If A is an aggregation operator on L, then A is an f-aggregation operator, for any isotone function f on L satisfying $f(0) = 0$ and $f(1) = 1$.*

Proof. Since A is increasing and f is isotone, Proposition 3 guarantees that A is f-increasing. The fact that $f(0) = 0$ and $f(1) = 1$ implies that $A(f(0), f(0)) = 0$ and $A(f(1), f(1)) = 1$. Thus, A is an f-aggregation operator on L. ◄

The following counter example shows that the converse implication of Proposition 8 does not necessarily hold.

Example 8. *Let $(D(30), |, gcd, lcm)$ be the lattice of the positive divisors of 30 and f (resp. A) an isotone function (resp. a binary operation) on $D(30)$ defined as follows:*

x	1	2	3	5	6	10	15	30
fx	1	5	5	3	10	15	30	30

and

$$A(x, y) = \begin{cases} lcm(x, y) & \text{if } (x, y) \in \{2, 6\} \times \{2, 6\}; \\ gcd(x, y) & \text{otherwise.} \end{cases}$$

One easily verifies that $A(f(1), f(1)) = 1$ and $A(f(30), f(30)) = 30$. Next, we prove that A is f-increasing. Let $x, y \in D(30)$ be such that $x \mid y$. Since f is isotone, we have $fx \mid fy$. The fact that $fx, fy \in D(30) \setminus \{2, 6\}$ implies $A(fx, z) = gcd(fx, z)$ and $A(fy, z) = gcd(fy, z)$. Then $A(fx, z) \mid A(fy, z)$, for any $z \in D(30)$. Hence, A is left f-increasing. Similarly, we prove that A is right f-increasing. Thus, A is an f-aggregation operator on $D(30)$. On the other hand, setting $x = 2$, $y = 10$ and $z = 6$, we obtain $x \mid y$, but $A(x, z) = A(2, 6) = lcm(2, 6) = 6$ and $A(y, z) = A(10, 6) = gcd(10, 6) = 2$. Hence, $6 \nmid 2$, i.e., A is not increasing. Consequently, A is not an aggregation operator.

In the same line, the following example gives an f-aggregation operator A such that f is not an isotone function and A is not an aggregation operator.

Example 9. *Let A (resp. f) be a binary operation (resp. function) on $D(12)$ given in Example 1. Then A is an f-aggregation operator, f is not an isotone function and A is not an aggregation operator on $D(12)$.*

Theorem 2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, f a lattice-automorphism and A a binary operation on L. Then A is an f-aggregation operator if and only if A is an f^{-1}-aggregation operator.*

Proof. Since f is a lattice-automorphism, then Proposition 1 guarantees that $A(f^{-1}(0), f^{-1}(0)) = A(f(0), f(0)) = 0$ and $A(f^{-1}(1), f^{-1}(1)) = A(f(1), f(1)) = 1$. Assume $x, y \in L$ are such that $x \leq y$. Then there exist $s, t \in L$ such that $x = f^2(s)$ and $y = f^2(t)$. The fact that f^{-1} is isotone implies $f^{-2}(x) \leq f^{-2}(y)$, i.e., $s \leq t$. Since A is an f-aggregation operator on L, it follows that $A(fs, z) \leq A(ft, z)$, for any $z \in L$. This is equivalent to $A(f^{-1}(f^2(s)), z) \leq A(f^{-1}(f^2(t)), z)$, for any $z \in L$. Hence, $A(f^{-1}(x), z) \leq A(f^{-1}(y), z)$, for any $z \in L$. Thus, A is left f^{-1}-increasing. Similarly, we can show that A is right f^{-1}-increasing. Therefore, A is an f^{-1}-aggregation operator on L. The proof of the converse implication follows from the fact that $(f^{-1})^{-1} = f$. ◄

Proposition 9. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, f a lattice-automorphism and A a binary operation on L. The following statements are equivalent:*

1. *A is an aggregation operator;*
2. *A is an f-aggregation operator;*
3. *A is an f^{-1}-aggregation operator.*

Proof. (i) $\Rightarrow$ (ii): Follows from Propositions 1 and 8.
(ii) $\Rightarrow$ (iii): The proof is a direct application of Theorem 2.
(iii) $\Rightarrow$ (i): Obvious that A satisfies the f-boundary conditions. Next, let $x, y \in L$ be such that $x \leq y$. Then $fx \leq fy$. Since A is an f^{-1}-aggregation operator, we have $A(f^{-1}(fx), z) \leq A(f^{-1}(fy), z)$, for any $z \in L$. Thus, $A(x, z) \leq A(y, z)$, for any $z \in L$. Therefore, A is an aggregation operator on L. ◄

Proposition 10. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ a lattice-epimorphism on L. If A is an idempotent f-aggregation operator on L, then A is averaging.*

Proof. Let $x, y \in L$ be such that $ft = x$ and $fs = y$. The fact that A is f-increasing implies $A(x, y) = A(ft, y) \leq A(f(t \vee s), y) = A(f(t \vee s), fs) \leq A(f(t \vee s), f(t \vee s))$. Thus $A(x, y) \leq A(f(t \vee s), f(t \vee s))$. Since A is idempotent and f is a homomorphism, it follows that $A(x, y) \leq f(t \vee s) = x \vee y$. Analogously, we show that $x \wedge y \leq A(x, y)$. Hence, $x \wedge y \leq A(x, y) \leq x \vee y$. Therefore, A is averaging. ◄

4.3. Composition of f-aggregation operators on a bounded lattice

In this subsection, we study the composition of f-aggregation operators on a bounded lattice. First, we show that the aggregation of two f-aggregation operators on a bounded lattice is also an f-aggregation operator.

Proposition 11. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A, F_1, F_2 three binary operations on L. If A is an aggregation operator and F_1, F_2 are two f-aggregation operators on L, then the aggregation of F_1 and F_2 based on A defined for any $x, y \in L$ as*

$$\mathcal{A}(F_1, F_2)(x, y) = A(F_1(x, y), F_2(x, y)),$$

is also an f-aggregation operator on L.

Proof. Let $x, y \in L$ be such that $x \leq y$. Since F_1, F_2 are two f-aggregation operators and A is an aggregation operator on L, it follows that $\mathcal{A}(F_1, F_2)(fx, z) = A(F_1(fx, z), F_2(fx, z)) \leq A(F_1(fy, z), F_2(fy, z)) = \mathcal{A}(F_1, F_2)(fy, z)$, for any $z \in L$. Thus, $\mathcal{A}(F_1, F_2)$ is left f-increasing on L. Similarly, we can show that $\mathcal{A}(F_1, F_2)$ is right f-increasing. Therefore, $\mathcal{A}(F_1, F_2)$ is f-increasing. Next, it is obvious that $\mathcal{A}(F_1, F_2)$ satisfies the f-boundary conditions. Thus, $\mathcal{A}(F_1, F_2)$ is an f-aggregation operator on L. ◄

Proposition 12. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A, F_1, F_2 three binary operations on L. If A is an f-aggregation operator and F_1, F_2 are two idempotent aggregation operators on L such that $f(F_1(x, y)) = F_1(fx, y) = F_1(x, fy)$ and $f(F_2(x, y)) = F_2(fx, y) = F_2(x, fy)$, then $\mathcal{A}(F_1, F_2)$ is an f-aggregation operator on L.*

Proof. Let $x, y \in L$ be such that $x \leq y$. Since F_1, F_2 are two aggregation operators and A is an f-aggregation operator on L, it follows that $\mathcal{A}(F_1, F_2)(fx, z) = A(F_1(fx, z), F_2(fx, z)) = A(f(F_1(x, z)), f(F_2(x, z))) \leq A(f(F_1(y, z)), f(F_2(y, z))) = \mathcal{A}(F_1, F_2)(fy, z)$, for any $z \in L$. Thus, $\mathcal{A}(F_1, F_2)$ is left f-increasing on L. In similar way, we can show that $\mathcal{A}(F_1, F_2)$ is right f-increasing. Therefore, $\mathcal{A}(F_1, F_2)$ is f-increasing. Next, since F_1, F_2 are idempotent, then $\mathcal{A}(F_1, F_2)(f(0), f(0)) = A(F_1(f(0), f(0)), F_2(f(0), f(0))) = A(f(0), f(0)) = 0$. Similarly, we can show that $\mathcal{A}(F_1, F_2)(f(1), f(1)) = 1$. Consequently, $\mathcal{A}(F_1, F_2)$ is an f-aggregation operator on L. ◄

Example 10. *Let F_1, F_2 be two binary operations on L such that $F_1 = F_2 = \wedge$ and f a meet-translation on L (i.e., $f(x \wedge y) = x \wedge fy$, for any $x, y \in L$). One easily verifies that $f(F_1(x, y)) = F_1(fx, y) = F_1(x, fy)$ and $f(F_2(x, y)) = F_2(fx, y) = F_2(x, fy)$. Since $F_1 = F_2 = \wedge$ are idempotent aggregation operators, $\mathcal{A}(F_1, F_2)$ is an f-aggregation operator on L, for any f-aggregation operator A on L.*

5. Transformations of an f-aggregation operator on a bounded lattice

In this section, we investigate the transformations of a given f-aggregation operator on a bounded lattice by a lattice-automorphism and a strong negation.

Theorem 3. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a lattice-automorphism φ and f a function which satisfies $f \circ \varphi = \varphi \circ f$. Then A is an f-aggregation operator on L if and only if A_φ is an f-aggregation operator on L, where A_φ is a binary operation on L given by:*

$$A_\varphi(x, y) = \varphi^{-1}(A(\varphi x, \varphi y)), \text{ for any } x, y \in L.$$

Proof. Suppose that A is an f-aggregation operator on a bounded lattice L and show that A_φ is also an f-aggregation operator on L. First, we prove that $A_\varphi(f(0), f(0)) = 0$ and $A_\varphi(f(1), f(1)) = 1$. Since φ is a lattice-automorphism, it follows from Proposition 1 that $\varphi(0) = \varphi^{-1}(0) = 0$ and $\varphi(1) = \varphi^{-1}(1) = 1$. From $f \circ \varphi = \varphi \circ f$ it follows that $A_\varphi(f(0), f(0)) = \varphi^{-1}(A(\varphi(f(0)), \varphi(f(0)))) = \varphi^{-1}(A(f(\varphi(0)), f(\varphi(0))))$ and $A_\varphi(f(1), f(1)) = \varphi^{-1}(A(\varphi(f(1)), \varphi(f(1)))) = \varphi^{-1}(A(f(\varphi(1)), f(\varphi(1))))$. Thus, $A_\varphi(f(0), f(0)) = \varphi^{-1}(A(f(0), f(0))) = \varphi^{-1}(0) = 0$ and $A_\varphi(f(1), f(1)) = \varphi^{-1}(A(f(1), f(1))) = \varphi^{-1}(1) = 1$, Next, we prove that A_φ is f-increasing. Let $x, y \in L$ be such that $x \leq y$. Then $\varphi x \leq \varphi y$. Since A is an f-aggregation operator on L, it follows that $A(f(\varphi x), \varphi z) \leq A(f(\varphi y), \varphi z)$, for any $z \in L$. The fact that φ is a lattice-automorphism guarantees that φ^{-1} is isotone, implies $\varphi^{-1}(A(f(\varphi x), \varphi z)) \leq \varphi^{-1}(A(f(\varphi y), \varphi z))$, for any $z \in L$. The equality $f \circ \varphi = \varphi \circ f$ implies that $\varphi^{-1}(A(\varphi(fx), \varphi z)) \leq \varphi^{-1}(A(\varphi(fy), \varphi z))$, for any $z \in L$. Hence, $A_\varphi(fx, z) \leq A_\varphi(fy, z)$, for any $z \in L$. Thus, A_φ is left f-increasing. Similarly, we can show that A_φ is also right f-increasing. Therefore, A_φ is an f-aggregation operator on L. The proof of the converse implication follows from the fact that $A = (A_\varphi)_{\varphi^{-1}}$. ◀

Theorem 2, Theorem 3 and Proposition 4 lead to the following corollary.

Corollary 2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f, φ two lattice-automorphisms on L such that $f \circ \varphi = \varphi \circ f$. Let A be a binary operation on L. The following statements are equivalent:*

1. *A is an f-aggregation operator;*
2. *A is an f^{-1}-aggregation operator;*
3. *A_φ is an f-aggregation operator;*

4. A_φ is an f^{-1}-aggregation operator.

For a given binary operation A on a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$ with a negation N, we denote by A_N its dual, i.e., $A_N(x, y) = N^{-1}(A(Nx, Ny))$, for any $x, y \in L$. One easily observes that if N is a strong negation, then $N^{-1} = N$, $(A_N)_N = A$ and $A_N(x, y) = N(A(Nx, Ny))$, for any $x, y \in L$.

In the same line, the following theorem shows that the transformation of an f-aggregation operator on bounded lattice by a strong negation is also an f-aggregation operator. The proof is analogous to that of Theorem 3.

Theorem 4. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a strong negation N and a function f such that $f \circ N = N \circ f$. Then A is an f-aggregation operator on L if and only if its dual operation A_N is an f-aggregation operator on L, where $A_N(x, y) = N(A(Nx, Ny))$, for any $x, y \in L$.*

6. Smallest and greatest f-aggregation operators on a bounded lattice

In this section, we provide some conditions on a given function f to define the smallest and the greatest f-aggregation operators on a bounded lattice.
For a given function f on a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$, we define the binary operations $A_\perp$ and $A_\top$ as:

$$A_\perp(x, y) = \begin{cases} 1 & \text{if } x = y = f(1); \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad A_\top(x, y) = \begin{cases} 0 & \text{if } x = y = f(0); \\ 1 & \text{otherwise.} \end{cases}$$

Remark 4. *One can observe that:*

1. *$A_\perp$ and $A_\top$ are not aggregation operators on L, in general. However, if $f(0) = 0$ and $f(1) = 1$, then $A_\perp$ (resp. $A_\top$) is an aggregation operator on L.*

2. *$A_\perp$ and $A_\top$ are not f-aggregation operators on L, in general. Indeed, let f be a function on $D(12)$ defined as follows:*

x	1	2	3	4	6	12
fx	6	2	4	12	6	4

.

It is not difficult to see that $A_\perp$ (resp. $A_\top$) is not f-increasing on $D(12)$ ($3 \mid 6$, but $A_\perp(f(3), 4) \nmid A_\perp(f(6), 4)$) (resp. $2 \mid 6$, but $A_\top(f(2), 6) \nmid A_\top(f(6), 6)$. Thus, $A_\perp$ and $A_\top$ are not f-aggregation operators on $D(12)$.

The following propositions provide some conditions on the function f under which $A_\perp$ and $A_\top$ are f-aggregation operators.

Proposition 13. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. The following equivalences hold:*

1. *$A_\perp$ is an f-aggregation operator on L if and only if for any $x \leq y$ the relation $(fx = f(1) \Rightarrow fy = f(1))$ holds;*
2. *$A_\top$ is an f-aggregation operator on L if and only if for any $x \leq y$ the relation $(fy = f(0) \Rightarrow fx = f(0))$ holds.*

Proof. We only give the proof of (1), as (2) can be proved analogously. Let $x, y \in L$ be such that $x \leq y$ and $fx = f(1)$. The fact that $A_\perp$ is f-increasing implies $A_\perp(fx, z) \leq A_\perp(fy, z)$, for any $z \in L$. For $z = f(1)$, we get $A_\perp(fy, f(1)) = 1$. Hence, $fy = f(1)$. Next, for the converse implication, let $x, y, z \in L$ be such that $x \leq y$. If $A_\perp(fx, z) = 0$, then $A_\perp(fx, z) \leq A_\perp(fy, z)$, for any $z \in L$. If $A_\perp(fx, z) = 1$, then $fx = f(1)$ and $z = f(1)$. Hence, $fy = f(1)$. Thus, $A_\perp(fy, z) = 1$. Then $A_\perp(fx, z) \leq A_\perp(fy, z)$, for any $z \in L$. Therefore, $A_\perp$ is left f-increasing. Similarly, we can show that $A_\perp$ is right f-increasing. Next, since f is a non-constant function, we have $f(0) \neq f(1)$. Thus, $A_\perp(f(0), f(0)) = 0$. Obviously, $A_\perp(f(1), f(1)) = 1$. Hence, $A_\perp$ satisfies the f-boundary conditions. Therefore, $A_\perp$ is an f-aggregation operator on L. ◀

Proposition 13 leads to the following corollaries.

Corollary 3. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. If f is injective, then $A_\perp$ and $A_\top$ are f-aggregation operators on L.*

Corollary 4. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. If f satisfies $fx = f(1)$ implies $x = 1$ (resp. $fx = f(0)$ implies $x = 0$), then $A_\perp$ (resp. $A_\top$) is an f-aggregation operator on L.*

The following result shows that if $A_\perp$ and $A_\top$ are f-aggregation operators on a given bounded lattice, then $A_\perp$ (resp. $A_\top$) is the smallest (resp. the greatest) f-aggregation operator on that lattice.

Proposition 14. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. If $A_\perp$ and $A_\top$ are f-aggregation operators on L, then $A_\perp$ (resp. $A_\top$) is the smallest (resp. the greatest) f-aggregation operator on L.*

Proof. The proof is straightforward. ◀

For a given function f on a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$, let us denote by $\mathcal{A}_f(L)$ the set of all f-aggregation operators on L. The following proposition provides a lattice structure of the set $\mathcal{A}_f(L)$.

Proposition 15. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* f *a function on* L. *If* $A_\perp$ *and* $A_\top$ *are* f*-aggregation operators on* L, *then* $(\mathcal{A}_f(L), \sqsubseteq, \sqcap, \sqcup, A_\perp, A_\top)$ *is a bounded lattice, where* $A \sqsubseteq B$ *if* $A(x, y) \leq B(x, y)$, $(A \sqcap B)(x, y) = A(x, y) \wedge B(x, y)$ *and* $(A \sqcup B)(x, y) = A(x, y) \vee B(x, y)$, *for any* $A, B \in \mathcal{A}_f(L)$ *and* $x, y \in L$.

Proof. Follows from Propositions 11 and 14. ◄

7. Conclusion and future research

In this paper, we have introduced the notion of f-aggregation operator with respect to a given function f on a bounded lattice. More precisely, we have given some new generalized properties of binary operations with respect to a given function on a lattice, and have studied the composition of f-aggregation operators on a bounded lattice with respect to an aggregation operator. Also, we have investigated the transformation of f-aggregation operators based on a lattice-automorphism (resp. a strong negation). Further, under some conditions on the arbitrary function f, we have provided the smallest and the greatest f-aggregation operators on a bounded lattice.

Finally, we intend to introduce the notions of some interesting aggregation operators (in particular, triangular norm and triangular conorm) with respect to an arbitrary function on a bounded lattice and investigate their possible properties.

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