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Lacunary d-Statistical Boundedness of Order α in Metric Spaces

H. Şengül Kandemir*, M. Et, H. Çakallı

Abstract. In this study, using a lacunary sequence we introduce the concepts of lacunary $d-$statistically convergent sequences of order α and lacunary $d-$statistically bounded sequences of order α in general metric spaces.

Key Words and Phrases: statistical convergence, lacunary sequence, metric space.

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1. Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers.

The idea of statistical convergence was given by Zygmund [40] in the first edition of his monograph published in Warsaw in 1935. The consept of statistical convergence was introduced by Steinhaus [39] and Fast [20] and later reintroduced by Schoenberg [33]. Over the years and under different names statistical convergence was discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Bhardwaj and Bala [2], Bilalov et al. ([3, 4, 5, 6]), Çakallı et al. ([7, 8, 9, 10]). Kočinac et al. ([11, 19]), Çınar et al. [12], Çolak [13], Connor [14], Et et al. ([15, 16, 17]), Fridy [22], Fridy and Orhan [23], Mursaleen et al. ([1, 29]) Salat [31], Savaş [32], Şengül et al. ([34, 35, 36, 37]) and many others.

*Corresponding author.

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The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k), \quad \text{provided that the limit exists.}$$

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0$.

The concept of statistical boundedness was given by Fridy and Orhan [24] as follows:

The real number sequence x is statistically bounded if there is a number B such that $\delta\left(\{k : |x_k| > B\}\right) = 0$.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([7, 8, 9, 18, 21, 23, 25, 30, 38, 41]).

Let X be a sequence space.

i) A sequence space X with metric d is said to be normal (or solid) if $(x_k) \in X$ and (y_k) is a sequence such that $d(y_k, a) \leq d(x_k, a)$ implies $(y_k) \in X$,

ii) *Monotone* if it contains the canonical preimages of all its stepspaces,

iii) *Symmetric*, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where π is a permutation of $\mathbb{N}$.

2. Lacunary $d-$Statistical Convergence of Order α and Lacunary $d-$Statistical Boundedness of Order α

The concepts of lacunary d-statistical convergence and lacunary strong pd-Cesaro summability of order α in metric spaces were introduced by Et and Karataş [18] and the concept of lacunary d-statistical boundedness in metric spaces was introduced by Sengul et al. [37]. In this study, we introduce lacunary d-statistical boundedness of order α in metric spaces. The results which we obtained in this study are much more general than those obtained by Et and Karataş [18] and Sengul et al. [37].

Definition 1. *[18] Let (X, d) be a metric space, $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. A metric valued sequence $x = (x_k)$ is said to be lacunary $d-$statistically convergent of order α (or $S_\theta^{d,\alpha}-$convergent) if there is a real number $a \in X$ such that*

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \left|\{k \in I_r : d(x_k, a) \geq \varepsilon\}\right| = 0, \tag{1}$$

where $I_r = (k_{r-1}, k_r]$ *and* h_r^α *denotes the* α*th power* $(h_r)^\alpha$ *of* h_r*, that is* $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, ..., h_r^\alpha, ...)$. *In this case we can find a real number* N_ε *such that*

$$|\{k \in I_r : x_k \notin B_\epsilon(a) \geq \varepsilon\}| \leq N_\varepsilon,$$

where $B_\epsilon(a) = \{x \in X : d(x,a) < \varepsilon\}$ *is the open ball of radius* ε *and center* a*. In this case we write* $S_\theta^{d,\alpha} - \lim x_k = a$*. The set of all lacunary* $d-$*statistically convergent sequences of order* α *will be denoted by* $S_\theta^{d,\alpha}$*. If* $\theta = (2^r)$*, then lacunary* $d-$*statistical convergence of order* α *coincides with* $d-$*statistical convergence of order* α *of sequences of real numbers introduced by Kayan et al. [26]. If* $\theta = (2^r)$ *and* $\alpha = 1$*, then lacunary* $d-$*statistical convergence of order* α *reduces to* $d-$*statistical convergence in a metric space introduced by Küçükaslan et al. [27].*

The lacunary $d-$statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general.

Definition 2. *Let* (X,d) *be a metric space,* $\theta = (k_r)$ *be a lacunary sequence and* $0 < \alpha \leq 1$ *be given. A metric valued sequence* $x = (x_k)$ *is said to be lacunary* $d-$*statistically bounded of order* α *(or* $BS_\theta^{d,\alpha}-$*bounded) if there are real numbers* $a \in X$ *and* M *such that*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : d(x_k, a) \geq M\}| = 0.$$

The set of all lacunary $d-$*statistically bounded sequences of order* α *will be denoted by* $BS_\theta^{d,\alpha}$*. If* $\theta = (2^r)$*, then lacunary* $d-$*statistical boundedness of order* α *coincides with* $d-$*statistical boundedness of order* α *of sequences of real numbers introduced by Kayan et al. [26]. If* $\theta = (2^r)$ *and* $\alpha = 1$*, then lacunary* $d-$*statistical boundedness of order* α *reduces to* $d-$*statistical boundedness in a metric space introduced by Küçükaslan et al. [28].*

Theorem 1. *Every bounded sequence is lacunary* $d-$*statistically bounded of order* α *in a metric space, but the converse is not true.*

Proof. If $x = (x_k)$ is a bounded sequence, then for an arbitrary $x \in X$ there is $M > 0$ such that $d(x_k, a) < M$ for all $k \in \mathbb{N}$. Hence

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : d(x_k, a) \geq M\}| = 0,$$

since $\{k \in I_r : d(x_k, a) \geq M\} = \phi$, for all $k \in \mathbb{N}$ and $\alpha \in (0,1]$. To show the strictness of the inclusion, choose $\theta = (2^r)$, $X = \mathbb{R}$, $d(x,y) = |x-y|$, $\alpha = 1$ and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} k, & n = k^2, \\ (-1^n), & \text{otherwise} \end{cases}.$$

It is clear that $x = (x_k)$ is not bounded, but it is $d-$statistically bounded. ◄

Theorem 2. *Every lacunary $d-$statistically convergent sequence of order α is lacunary $d-$statistically bounded of order α, but the converse is not true.*

Proof. Let $x = (x_k)$ be a lacunary $d-$statistically convergent sequence of order α and $\varepsilon > 0$ be given. Then there exists $a \in X$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \left|\{k \in I_r : d(x_k, a) \geq \varepsilon\}\right| = 0.$$

For any $\varepsilon > 0$ and a real number M, we have

$$\{k \in I_r : d(x_k, a) \geq M\} \subset \{k \in I_r : d(x_k, a) \geq \varepsilon\}$$

and so

$$\left|\{k \in I_r : d(x_k, a) \geq M\}\right| \leq \left|\{k \in I_r : d(x_k, a) \geq \varepsilon\}\right|.$$

Taking the limit as $r \to \infty$, we get

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \left|\{k \in I_r : d(x_k, a) \geq M\}\right| = 0.$$

To show the strictness of the inclusion, choose $\theta = (2^r)$, $X = \mathbb{R}$, $d(x, y) = |x - y|$, $\alpha = 1$ and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & k = 2n \\ -1, & k \neq 2n \end{cases} \quad k, n \in \mathbb{N}.$$

It is clear that $x = (x_k)$ is not statistically convergent, but it is statistically bounded. ◄

Theorem 3. *Let (X, d) be a metric space, $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be two real numbers such that $0 < \alpha \leq \beta \leq 1$,*

(*i*) *Let*

$$\lim_{r\to\infty} \inf \frac{h_r^{\alpha}}{\ell_r^{\beta}} > 0. \tag{1}$$

If a sequence $x = (x_k)$ is $BS_{\theta'}^{d,\beta}-$bounded, then it is $BS_{\theta}^{d,\alpha}-$bounded.

(*ii*) *Suppose that the inequality* (1) *is satisfied. Then if a sequence $x = (x_k)$ is $S_{\theta'}^{d,\beta}-$convergent, then it is $BS_{\theta}^{d,\alpha}-$bounded.*

(*iii*) *Let*

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^{\beta}} = 1. \tag{2}$$

If a sequence $x = (x_k)$ is $BS_{\theta}^{d,\alpha}-$bounded, then it is $BS_{\theta'}^{d,\beta}-$bounded.

Proof. (i) Suppose that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let (1) be satisfied. For given $\varepsilon > 0$ we have

$$\{k \in J_r : d(x_k, a) \geq M\} \supseteq \{k \in I_r : d(x_k, a) \geq M\}$$

and so

$$\frac{1}{\ell_r^{\beta}} |\{k \in J_r : d(x_k, a) \geq M\}| \geq \frac{h_r^{\alpha}}{\ell_r^{\beta}} \frac{1}{h_r^{\alpha}} |\{k \in I_r : d(x_k, a) \geq M\}|$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$ and $\ell_r = s_r - s_{r-1}$. Now taking the limit as $r \to \infty$ in the last inequality and using (1) we get $x = (x_k)$ is $BS_{\theta}^{d,\alpha}-$bounded.

(ii) Proof is similar to that of (i).

(iii) Omitted. ◄

Theorem 4. *Let (X, d) be a metric space, $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be two real numbers such that $0 < \alpha \leq \beta \leq 1$,*

(i) Suppose that the inequality (1) is satisfied. Then if a sequence $x = (x_k)$ is $S_{\theta'}^{d,\beta}-$convergent, then it is $S_{\theta}^{d,\alpha}-$convergent.

(ii) Suppose that the relation (2) is satisfied. Then if a sequence $x = (x_k)$ is $S_{\theta}^{d,\alpha}-$convergent, then it is $S_{\theta'}^{d,\beta}-$convergent.

Proof. Proof is similar to that of Theorem 3. ◄

Theorem 5. *$i)$ $BS_{\theta}^{d,\alpha}$ is not symmetric,*

$ii)$ $BS_{\theta}^{d,\alpha}$ is normal and hence monotone.

Proof. $i)$ Let $x = (x_k) = (1, 0, 0, 2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 4, ...) \in BS_{\theta}^{d,\alpha}$. Let $y = (y_k)$ be a rearrangement of (x_k), which is defined as follows:

$$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...) =$$
$$= (1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, ...).$$

Clearly for any $M > 0$, $\delta_{\theta}(\{k : |y_k| > M\}) \neq 0$, in the special case $\theta = (2^r)$, $X = \mathbb{R}, d(x, y) = |x - y|$ and $\alpha = 1$.

$ii)$ Let $x = (x_k) \in BS_{\theta}^{d,\alpha}$ and $y = (y_k)$ be a sequence such that $d(y_k, a) \leq d(x_k, a)$ for all $k \in \mathbb{N}$. Since $x \in BS_{\theta}^{d,\alpha}$ there exists a number M such that $\delta_{\theta}^{\alpha}(\{k : d(x_k, a) > M\}) = 0$. Clearly $y \in BS_{\theta}^{d,\alpha}$ as $\{k : d(y_k, a) > M\} \subset \{k : d(x_k, a) > M\}$. So $BS_{\theta}^{d,\alpha}$ is normal. It is well known that every normal space is monotone, so $BS_{\theta}^{d,\alpha}$ is monotone. ◄

Remark 1. *Using the method of Theorem 5, it can be easily shown that $S_\theta^{d,\alpha}$ is normal and monotone, but not symmetric.*

Theorem 6. *Let (X,d) be a metric space, $\theta=(k_r)$ be a lacunary sequence and the parameters α and β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$. Then $BS_\theta^{d,\alpha}\subset BS_\theta^{d,\beta}$ and the inclusion is strict.*

Proof. Proof follows from the inequality

$$\frac{1}{h_r^\beta}\left|\{k\in I_r : d(x_k,a)\geq M\}\right|\leq\frac{1}{h_r^\alpha}\left|\{k\in I_r : d(x_k,a)\geq M\}\right|.$$

To show the strictness of the inclusion, choose $\theta=(2^r)$, $X=\mathbb{R}$, $d(x,y)=|x-y|$, $a=0$ and define a sequence $x=(x_k)$ by

$$x_k=\begin{cases}\left[\sqrt{h_r}\right], & k=1,2,3,...,\left[\sqrt{h_r}\right]\\ 0, & \text{otherwise}\end{cases}.$$

Then $x\in BS_\theta^{d,\beta}$ for $\beta\in\left(\frac{1}{2},1\right]$, but $x\notin BS_\theta^{d,\alpha}$ for $x\in\left(0,\frac{1}{2}\right]$. ◀

From Theorem 6 we have the following.

Corollary 1. *If a sequence is lacunary $d-$statistically convergent of order α, then it is lacunary $d-$statistically convergent.*

The proof of the following results are straightforward, so we choose to state these results without proof.

Theorem 7. *Let (X,d) be a metric space, $\theta=(k_r)$ be a lacunary sequence and $0<\alpha\leq 1$ be given. If $\liminf_r q_r>1$, then $BS^{d,\alpha}\subseteq BS_\theta^{d,\alpha}$.*

Theorem 8. *Let (X,d) be a metric space, $\theta=(k_r)$ be a lacunary sequence and $0<\alpha\leq 1$ be given. If $\limsup_r q_r<\infty$, then $BS_\theta^{d,\alpha}\subseteq BS^d$.*

3. Lacunary Strong $d-$Cesàro Convergence of Order α

In [18], Et and Karataş have introduced the concept of lacunary strong $d-$Cesàro summablility of order α in metric spaces and obtained some results on this concept. In this study we give some relations between lacunary statistical convergence of order α and lacunary strong $d-$Cesàro summablility of order α in metric spaces.

Definition 3. *[18] Let (X,d) be a metric space, $\theta=(k_r)$ be a lacunary sequence and $\alpha\in\mathbb{R}^+$ be given. A metric valued sequence $x=(x_k)$ is said to be strongly $N_\theta^{d,\alpha}-$summable (or lacunary strongly $d-$Cesàro summable of order α) if there is a real number $a\in X$ such that*

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}d(x_k,a)=0.$$

In this case we write $N_\theta^{d,\alpha}-\lim x_k=a$. The set of all lacunary strongly $d-$Cesàro summable sequences of order α will be denoted by $N_\theta^{d,\alpha}$. In case of $a=0$, we write $N_{\theta,0}^{d,\alpha}$ instead of $N_\theta^{d,\alpha}$. In the special case $\alpha=1$ we shall write N_θ^{d} instead of $N_\theta^{d,\alpha}$. If $\theta=(2^r)$, then lacunary strong $d-$Cesàro summability of order α coincides with strong $d-$Cesàro summability of order α of sequences of real numbers which were introduced by Kayan et al. [26] denoted by $w^{d,\alpha}$. If $a=0$, then we shall write $w_0^{d,\alpha}$ instead of $w^{d,\alpha}$.

Theorem 9. *Let (X,d) be a metric space, $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$, α and β be fixed real numbers such that $0<\alpha\le\beta\le 1$. Then we have*

(i) If (1) holds, then $N_{\theta'}^{d,\beta}\subset N_\theta^{d,\alpha}$,

(ii) Suppose (2) holds and $x=(x_k)$ is a bounded sequence. Then $N_\theta^{d,\alpha}\subset N_{\theta'}^{d,\beta}$.

Proof. Omitted. ◀

Theorem 10. *Let (X,d) be a metric space, $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$, α and β be fixed real numbers such that $0<\alpha\le\beta\le 1$. Then*

(i) Let (1) hold. If a sequence is strongly $N_{\theta'}^{d,\beta}-$summable to a, then it is $S_\theta^{d,\alpha}-$statistically convergent to a,

(ii) Let (2) hold. If a bounded sequence is $S_\theta^{d,\alpha}-$statistically convergent to a, then it is strongly $N_{\theta'}^{d,\beta}-$summable to a.

Proof. (i) Omitted.

(ii) Suppose that $S_\theta^{d,\alpha}-\lim x_k=a$ and $x=(x_k)$ is bounded. Then there exists some $M>0$ such that $d(x_k,a)\le M$ for all k, and for every $\varepsilon>0$ we may write

$$\begin{aligned}\frac{1}{\ell_r^\beta}\sum_{k\in J_r}d(x_k,a) &= \frac{1}{\ell_r^\beta}\sum_{k\in J_r-I_r}d(x_k,a)+\frac{1}{\ell_r^\beta}\sum_{k\in I_r}d(x_k,a)\\ &\le \left(\frac{\ell_r-h_r}{\ell_r^\beta}\right)M+\frac{1}{\ell_r^\beta}\sum_{k\in I_r}d(x_k,a)\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{\ell_r - h_r^\beta}{\ell_r^\beta}\right) M + \frac{1}{\ell_r^\beta}\sum_{k\in I_r} d(x_k,a) \\
&\leq \left(\frac{\ell_r}{h_r^\beta}-1\right) M + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ d(x_k,a)\geq\varepsilon}} d(x_k,a) + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ d(x_k,a)<\varepsilon}} d(x_k,a) \\
&\leq \left(\frac{\ell_r}{h_r^\beta}-1\right) M + \frac{M}{h_r^\beta}\left|\{k\in I_r : d(x_k,a)\geq\varepsilon\}\right| + \frac{h_r}{h_r^\beta}\varepsilon \\
&\leq \left(\frac{\ell_r}{h_r^\beta}-1\right) M + \frac{M}{h_r^\alpha}\left|\{k\in I_r : d(x_k,a)\geq\varepsilon\}\right| + \frac{\ell_r}{h_r^\beta}\varepsilon
\end{aligned}
$$

for all $r \in \mathbb{N}$. Using (2) we obtain $N_{\theta'}^{d,\beta} - \lim x_k = a$, whenever $S_\theta^{d,\alpha} - \lim x_k = a$.

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Hacer Şengül Kandemir
Faculty of Education, Harran University, Osmanbey Campus 63190, Şanlıurfa, Turkey
E-mail: hacer.sengul@hotmail.com

Mikail Et
Department of Mathematics, Fırat University 23119, Elazığ, Turkey
E-mail: mikailet68@gmail.com

Hüseyin Çakallı
Mathematics Division, Graduate School of Science and Engineering, Maltepe University, Maltepe, Istanbul, Turkey
E-mail: huseyincakalli@maltepe.edu.tr

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