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On the Crossing Numbers of Join Products of Four Graphs of Order Six With the Discrete Graph

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Abstract. The aim of this paper is to extend known results concerning crossing numbers of graphs by giving the crossing number for join product $G^* + D_n$ of the connected graph G^* of order six consisting of one 3-cycle and three leaves of which exactly two are adjacent with the same vertex of such 3-cycle, and D_n consists of n isolated vertices. The proofs rely on a partial classification of all subgraphs whose edges cross the edges of G^* just once. Due to the mentioned algebraic topological approach, we extend known results concerning crossing numbers for join products of new graphs. Finally, by adding new edges to the graph G^*, the crossing numbers of $G_i + D_n$ for three other graphs G_i of order six will be also established.
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1. Introduction

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane (for the definition of a *drawing*, see also Klešč [13]). One can easily verify that a drawing with the minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no two edges cross more than once, no edge crosses itself, and also no two edges incident with the same vertex cross. Let D be a good drawing of the graph G. We denote by $\mathrm{cr}_D(G)$ the number of crossings among edges of G in the drawing D.

Let G_i and G_j be two edge-disjoint subgraphs of G. We denote, by $\mathrm{cr}_D(G_i, G_j)$, the number of crossings between the edges of G_i and edges of G_j, and, by $\mathrm{cr}_D(G_i)$ and $\mathrm{cr}_D(G_j)$, the number of crossings among edges of G_i and of G_j in D, respectively. For any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G by [13], the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

The investigation on the crossing number of graphs is a classical and very difficult problem. Garey and Johnson [6] proved that determining $\mathrm{cr}(G)$ is an NP-complete problem. Nevertheless, many researchers are trying to solve this problem. Research of the problem of reducing the number of crossings in the graph is studied not only in the graph theory, but also by computer scientists. Note that the exact values of the crossing numbers are known for only a few families of graphs, see Clancy *et al.* [5]. The purpose of this article is to extend the known results concerning this topic. Some parts of proofs will be based on Kleitman's result [11] on the crossing numbers for some complete bipartite graphs. He showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{with} \min\{m, n\} \le 6.$$

The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let D_n and P_n be the *discrete graph* and the *path* on n vertices, respectively. The crossings numbers of the join products of the discrete graphs and paths with all graphs of order at most four have been well-known for a long time by Klešč [12], and Klešč and Schrötter [17], and therefore it is understandable that our immediate goal is to establish the exact values for the crossing numbers of $G + D_n$ and $G + P_n$ also for all graphs G of order five and six. Of course, the crossing numbers of $G + D_n$ are already known for a lot of graphs G of order five and six, see [1, 3, 8, 9, 10, 13, 15, 14, 16, 19, 20, 21]. In all these cases, the graph G is connected and contains usually at least one cycle. Note that the crossing numbers of the join product $G + D_n$ are known only for some disconnected graphs G on five or six vertices [4, 18, 22, 23].

In the paper, we will use definitions and notation of the crossing numbers of graphs presented by Klešč [12]. Let G^* be the connected graph of order six consisting of one 3-cycle (for brevity, we will write $C_3(G^*)$) and three leaves of which exactly two are adjacent with the same vertex of such $C_3(G^*)$. The main aim of the paper is to establish the crossing numbers of the join products of G^* with the discrete graph D_n. The required result of Theorem 1 is determined for all positive integers n mainly by using several auxiliary statements. The paper concludes by giving the crossing numbers of $G_i + D_n$ for three different graphs

G_i in Corollary 2. The methods presented in the paper are based on multiple combinatorial properties of the cyclic permutations. The similar methods were partially used first time by Hernández-Vélez *et al.* [7]. By Berežný and Staš [3, 4], the properties of cyclic permutations are also verified by the help of software. In our paper, certain parts of proofs can be also simplified with the help of software COGA generating all cyclic permutations of six elements and its description can be found in Berežný and Buša [2]. The list with the short names of $6!/6 = 120$ cyclic permutations of six elements are collected in Table 1 of [3]. Note that we were unable to determine the crossing number of the join product $G^* + D_n$ using the methods used by Klešč [13]. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

2. Cyclic Permutations and Configurations

We consider the join product of the graph G^* with the discrete graph D_n on n vertices. The graph $G^* + D_n$ consists of one copy of the graph G^* and n vertices $t_1, t_2, \ldots, t_n$, where any vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G^*. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex t_i. This means that the graph $T^1 \cup \cdots \cup T^n$ is isomorphic to the complete bipartite graph $K_{6,n}$ and

$$G^* + D_n = G^* \cup K_{6,n} = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1}$$

Let D be a drawing of the graph $G^* + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i has been defined by Hernández-Vélez *et al.* [7] or Woodall [24]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5, and t_iv_6. Recall that a rotation is a cyclic permutation; that is, (123456), (234561), (345612), (456123), (561234), and (612345) denote the same rotation. By $\overline{\mathrm{rot}}_D(t_i)$, we understand the inverse permutation of $\mathrm{rot}_D(t_i)$. We will separate all subgraphs T^i of the graph $G^* + D_n$ into three mutually-disjoint subsets depending on how many times the considered T^i crosses the edges of G^* in D. For $i = 1, \ldots, n$, $T^i \in R_D$ if $\mathrm{cr}_D(G^*, T^i) = 0$, and $T^i \in S_D$ if $\mathrm{cr}_D(G^*, T^i) = 1$. Every other subgraph T^i crosses the edges of G^* at least twice in D. Moreover, let F^i denote the subgraph $G^* \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \ldots, n\}$. Thus, for a given subdrawing of G^* in D, any subgraph F^i is exactly represented by $\mathrm{rot}_D(t_i)$.

According to the arguments in the proof of the main Theorem 1, if we would like to obtain an optimal drawing D of $G^* + D_n$, then the set R_D must be nonempty. Thus, we will only consider drawings of the graph G^* for which there is a possibility to obtain a subgraph T^i whose edges do not cross the edges of G^*. Since there is only one subdrawing of its subgraph $C_3(G^*)$, the remaining edges of G^* can cross the edges of $C_3(G^*)$ or they cross each other in the considered subdrawings. Hence, there are thirteen possible non isomorphic drawings of G^* which are presented in Figure 1, and their vertex notation will be justified later.

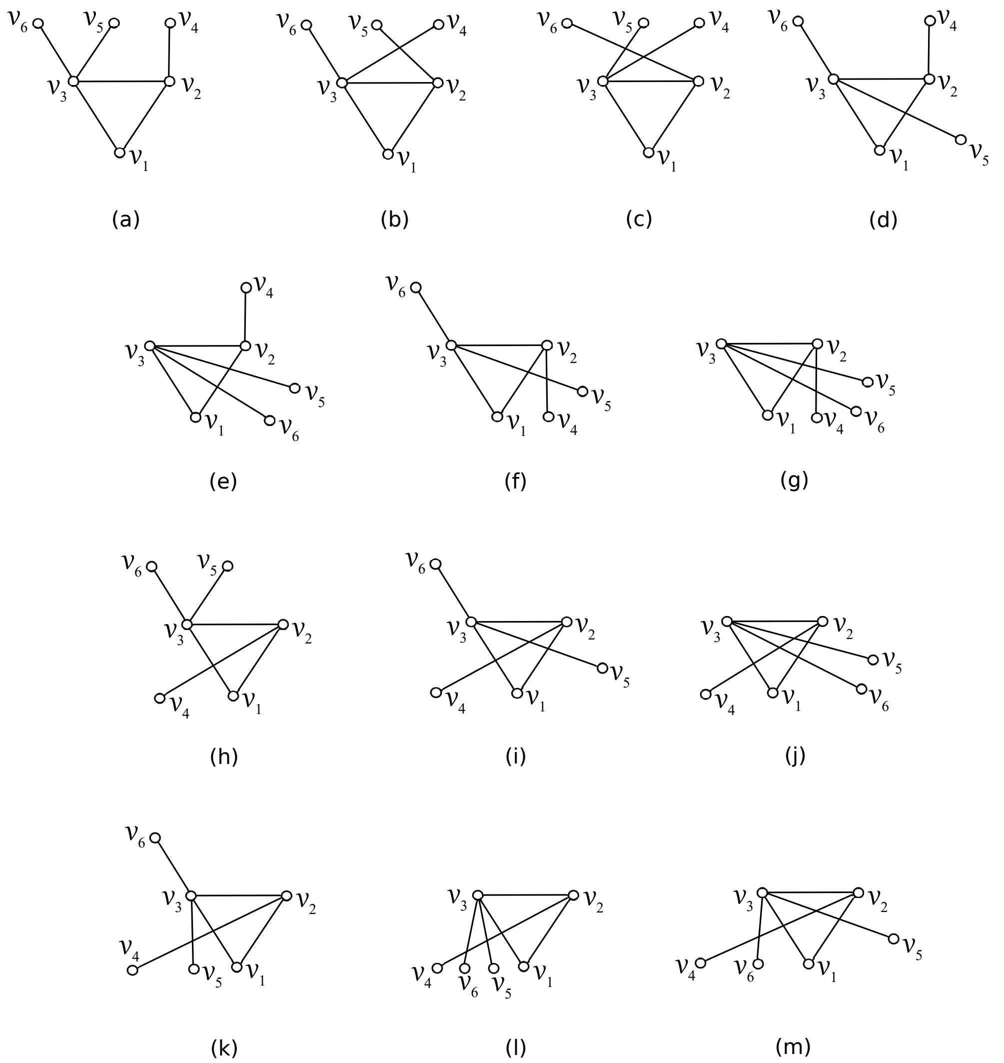

Figure 1: Thirteen possible non isomorphic drawings of the graph G^*.

Let us first assume a good drawing D of the graph $G^* + D_n$ in which the edges of G^* do not cross each other. In this case, without loss of generality, from the drawings in Figure 1 we can choose the vertex notation of the graph G^* as shown in Figure 1(a). Our aim shall be to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G^*. Since there is only one subdrawing of $F^i \setminus \{v_2, v_3\}$ represented by the rotation (1654), there are two and three possibilities for how to obtain the subdrawing of F^i depending on which region the edges t_iv_2 and t_iv_3 are placed in, respectively. These $2 \times 3 = 6$ possibilities under our consideration are denoted by $\mathcal{A}_p$, for $p = 1, \ldots, 6$, and we will call them by the *configurations* of corresponding subdrawings of the subgraph F^i in D. For our purposes, it does not matter which of the regions is unbounded, so we can assume that the drawings are as shown in Figure 2.

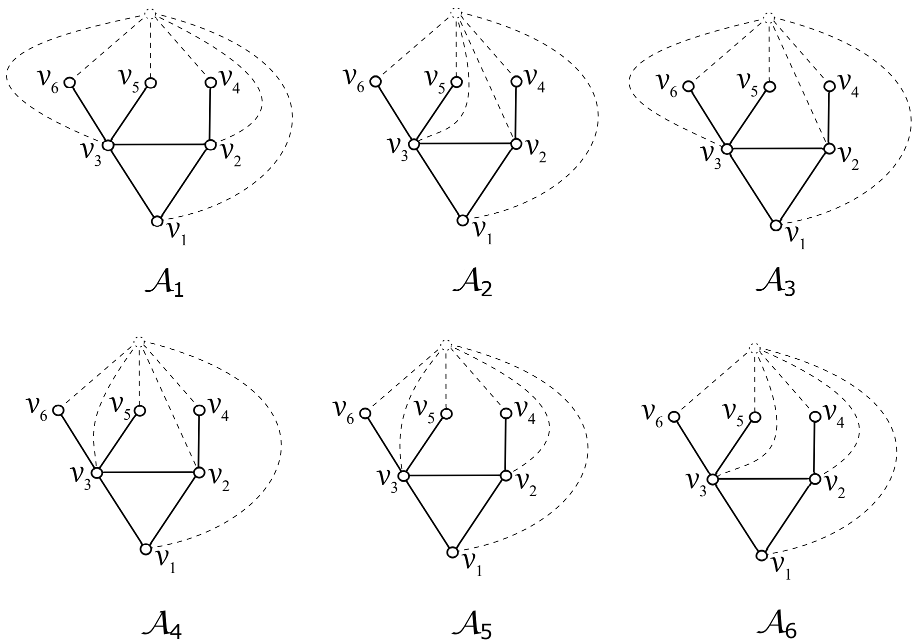

Figure 2: Drawings of six possible configurations from $\mathcal{M}$ of the subgraph F^i.

In the rest of the paper, we represent a cyclic permutation by the permutation with 1 in the first position. Thus the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$, and $\mathcal{A}_6$ are represented by the cyclic permutations (136542), (165324), (136524), (163524), (163542), and (165342), respectively. Of course, in a fixed drawing of the graph $G^* + D_n$, some configurations from $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\}$ need not appear. So we denote by $\mathcal{M}_D$ the set of all configurations for the drawing D belonging to $\mathcal{M}$.

We remark that if two different subgraphs F^i and F^j with configurations from $\mathcal{M}_D$ cross in a drawing D of G^*+D_n, then only the edges of T^i cross the edges of T^j. Let $\mathcal{X}$, $\mathcal{Y}$ be the configurations from $\mathcal{M}_D$. We denote by $\mathrm{cr}_D(\mathcal{X},\mathcal{Y})$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations $\mathcal{X}$, $\mathcal{Y}$, respectively. Finally, let $\mathrm{cr}(\mathcal{X},\mathcal{Y}) = \min\{\mathrm{cr}_D(\mathcal{X},\mathcal{Y})\}$ over all good drawings of the graph G^*+D_n with $\mathcal{X},\mathcal{Y} \in \mathcal{M}_D$. Our aim is to establish $\mathrm{cr}(\mathcal{X},\mathcal{Y})$ for all pairs $\mathcal{X},\mathcal{Y} \in \mathcal{M}$. Let $\overline{P_i}$ denote the inverse cyclic permutation to the permutation P_i, for $i = 1,\dots,120$, where the list with the short names of $6!/6 = 120$ cyclic permutations of six elements is presented in Table 1 of [3]. Woodall [24] defined the cyclic-ordered graph COG with the set of vertices $V = \{P_1, P_2, \dots, P_{120}\}$, and with the set of edges E, where two vertices are joined by the edge if the vertices correspond to the permutations P_i and P_j, which are formed by the exchange of exactly two adjacent elements of the 6-tuple (i. e., an ordered set with 6 elements). Hence, if $d_{COG}(\text{"}\mathrm{rot}_D(t_i)\text{"}, \text{"}\mathrm{rot}_D(t_j)\text{"})$ denotes the distance between two vertices which correspond to the cyclic permutations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$ in the graph COG, then

$$\mathrm{cr}_D(T^i,T^j) \geq Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) = d_{COG}(\text{"}\mathrm{rot}_D(t_i)\text{"}, \text{"}\overline{\mathrm{rot}_D(t_j)}\text{"}) \qquad (2)$$

for any two different subgraphs T^i and T^j, where $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ has been already defined in [3] as the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$. It turns out that the property (2) is a useful tool for the establishing lower bounds for several cases.

Now, we are ready to find the necessary numbers of crossings between subgraphs T^i and T^j for the corresponding configurations of F^i and F^j from $\mathcal{M}$. In particular, the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ are represented by the cyclic permutations $P_{111} = (136542)$ and $P_{72} = (165324)$, respectively. Since $\overline{P_{72}} = (142356) = P_4$, we have $\mathrm{cr}(\mathcal{A}_1,\mathcal{A}_2) \geq 3$ using $d_{COG}(\text{"}P_{111}\text{"}, \text{"}P_4\text{"}) = 3$. Details have been worked out by Woodall [24]. The same reason gives $\mathrm{cr}(\mathcal{A}_1,\mathcal{A}_3) \geq 5$, $\mathrm{cr}(\mathcal{A}_1,\mathcal{A}_4) \geq 4$, $\mathrm{cr}(\mathcal{A}_1,\mathcal{A}_5) \geq 5$, $\mathrm{cr}(\mathcal{A}_1,\mathcal{A}_6) \geq 4$, $\mathrm{cr}(\mathcal{A}_2,\mathcal{A}_3) \geq 4$, $\mathrm{cr}(\mathcal{A}_2,\mathcal{A}_4) \geq 5$, $\mathrm{cr}(\mathcal{A}_2,\mathcal{A}_5) \geq 4$, $\mathrm{cr}(\mathcal{A}_2,\mathcal{A}_6) \geq 5$, $\mathrm{cr}(\mathcal{A}_3,\mathcal{A}_4) \geq 5$, $\mathrm{cr}(\mathcal{A}_3,\mathcal{A}_5) \geq 4$, $\mathrm{cr}(\mathcal{A}_3,\mathcal{A}_6) \geq 3$, $\mathrm{cr}(\mathcal{A}_4,\mathcal{A}_5) \geq 5$, $\mathrm{cr}(\mathcal{A}_4,\mathcal{A}_6) \geq 4$, and $\mathrm{cr}(\mathcal{A}_5,\mathcal{A}_6) \geq 5$. Moreover, by a discussion of possible subdrawings, we can verify that $\mathrm{cr}(\mathcal{A}_3,\mathcal{A}_5) \geq 6$ and $\mathrm{cr}(\mathcal{A}_3,\mathcal{A}_6) \geq 4$. For any $T^i \in R_D$ with the configuration $\mathcal{A}_6$ of F^i, if there is a subgraph T^j, $j \neq i$ such that $\mathrm{cr}_D(T^i,T^j) \leq 3$, then the vertex t_j must be placed in one of three possible regions with three vertices of G^* on its boundary in the subdrawing $D(F^i)$. For all three possible placements of t_j, three edges of T^j enforce at least 4 crossings on the edges of T^i if the edges of the graph G^* cannot be crossed by T^j, and therefore, $\mathrm{cr}(\mathcal{A}_3,\mathcal{A}_6) \geq 4$. Similar arguments can be applied to show that $\mathrm{cr}(\mathcal{A}_3,\mathcal{A}_5) \geq 6$. Clearly, also $\mathrm{cr}(\mathcal{A}_p,\mathcal{A}_p) \geq 6$ for any $p = 1,\dots,6$. The resulting lower bounds for the number of crossings of configurations from $\mathcal{M}$ are summarized in the symmetric Table 1 (here, $\mathcal{A}_p$ and $\mathcal{A}_q$ are configurations of the subgraphs F^i and F^j, where $p, q \in \{1, \dots, 6\}$).

$-$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$
$\mathcal{A}_1$	6	3	5	4	5	4
$\mathcal{A}_2$	3	6	4	5	4	5
$\mathcal{A}_3$	5	4	6	5	6	4
$\mathcal{A}_4$	4	5	5	6	5	4
$\mathcal{A}_5$	5	4	6	5	6	5
$\mathcal{A}_6$	4	5	4	4	5	6

Table 1: The necessary number of crossings between T^i and T^j for the configurations $\mathcal{A}_p$ and $\mathcal{A}_q$.

For easier and more accurate labeling in the proofs of assertions, let us define notation of regions in some subdrawings of $G^* + D_n$. For $T^i \in R_D$, the unique drawing of F^i contains six regions with the vertex t_i on its boundary. For example, if F^i has the configuration $\mathcal{A}_1$, then let us denote these six regions by $\omega_{1,2}$, $\omega_{2,4}$, $\omega_{2,4,5,3}$, $\omega_{3,5,6}$, $\omega_{3,6}$, and $\omega_{1,3}$ depending on which of vertices are located on the boundary of the corresponding region.

3. The Crossing Number of $G^* + D_n$

Two vertices t_i and t_j of $G^* + D_n$ are *antipodal* in a drawing of $G^* + D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipode-free* if it has no antipodal vertices. The following statements related to some restricted drawings will be helpful in proving the main theorem.

Lemma 1 ([3], Lemma 3.1)**.** *Let D be a good and antipode-free drawing of $G^* + D_n$, $n > 2$. Let $2|R_D| + |S_D| > 2n - 2\lfloor \frac{n}{2} \rfloor$ and let $T^i, T^j \in R_D$ be two different subgraphs with $\mathrm{cr}_D(T^i \cup T^j) \geq 4$. If both conditions*

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 10 \qquad \textit{for any } T^l \in R_D \setminus \{T^i, T^j\}, \quad (3)$$

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 7 \qquad \textit{for any } T^l \in S_D \quad (4)$$

hold, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings in D.

Let us first note that if D is a good and antipode-free drawing of $G^* + D_n$ with the vertex notation of the graph G^* in a way shown in Figure 1(a), and $T^i \in R_D$ such that $F^i = G^* \cup T^i$ has some configuration $\mathcal{A}_p \in \mathcal{M}_D$, then

$\mathrm{cr}_D(G^* \cup T^i, T^l) \geq 2$ holds for any T^l, $l \neq i$, see Figure 2. Therewith, there are possibilities of an existence of a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$ only for the case of the configuration $\mathcal{A}_1$ of F^i.

Lemma 2. *Let D be a good and antipode-free drawing of $G^* + D_n$, $n > 2$, with the vertex notation of the graph G^* in a way shown in Figure 1(a). Let $T^i \in R_D$ be a subgraph such that F^i has configuration $\mathcal{A}_1 \in \mathcal{M}_D$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$, then*

a) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 8$ *for any subgraph $T^l \in S_D$, $l \neq k$ with respect to the restriction* $\mathrm{cr}_D(T^i, T^l) = 1$*;*

b) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 7$ *for any subgraph $T^l \in R_D$, $l \neq i$;*

c) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 7$ *for any subgraph $T^l \in S_D$ with* $\mathrm{cr}_D(T^i, T^l) \geq 2$*;*

d) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 7$ *for any subgraph $T^l \notin R_D \cup S_D$.*

Proof. Let us assume the configuration $\mathcal{A}_1$ of F^i, and remark that it is represented by the cyclic permutation $P_{111} = (136542)$.

a) The unique drawing of F^i contains six regions with the vertex t_i on their boundaries, see Figure 2. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$, then the vertex t_k must be placed in the pentagonal region of $D(F^i)$ with four vertices of G^* on its boundary, i.e., $t_k \in \omega_{2,4,5,3}$. This enforces that the edge v_2v_3 of the graph G^* must be crossed by the edge t_kv_1 and $\mathrm{cr}_D(T^i, T^k) = 1$ only for the subgraph T^k with $\mathrm{rot}_D(t_k) = (124653) = P_{79}$. Then, $\mathrm{cr}_D(T^k, T^l) \geq 6$ holds for any $T^l \in S_D$, $l \neq k$ with respect to the restriction $\mathrm{cr}_D(T^i, T^l) = 1$, because $\mathrm{rot}_D(t_k) = \mathrm{rot}_D(t_l)$, for more see also Woodall [24]. Hence, $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1 + 1 + 6 = 8$ is fulfilled for such a subgraph T^l.

b) Let us assume that $T^l \in R_D$, $l \neq i$, with the configuration $\mathcal{A}_p \in \mathcal{M}_D$ of F^l for some $p \in \{1, \ldots, 6\}$. Since any subgraph F^l is exactly represented by $\mathrm{rot}_D(t_l)$, then the lower bounds of $\mathrm{cr}_D(T^k, T^l)$ can be defined by the property (2). The resulting lower bounds of number of crossings for such a graph $T^k \cup T^l$ are given in the second column of Table 2. The values in the first column of Table 2 are given by the lower bounds from the first column of Table 1. The smallest value in the last column of Table 2 gives the required minimum number of crossings, because T^l does not cross the edges of G^*.

$\mathrm{conf}(F^l)$	$\mathrm{cr}_D(T^i, T^l)$	$\mathrm{cr}_D(T^k, T^l)$	$\mathrm{cr}_D(T^i \cup T^k, T^l)$
$\mathcal{A}_1$	6	1	7
$\mathcal{A}_2$	3	4	7
$\mathcal{A}_3$	5	2	7
$\mathcal{A}_4$	4	3	7
$\mathcal{A}_5$	5	2	7
$\mathcal{A}_6$	4	3	7

Table 2: All possibilities of F^l for $T^l \in R_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$, and $T^k \in S_D$.

c) Let $T^k \in S_D$ be a subgraph with $\mathrm{cr}_D(T^i, T^k) = 1$, i.e., the subdrawing of F^k is represented by $P_{79} = (124653)$. If we still assume a $T^l \in S_D$ such that $\mathrm{cr}_D(T^i, T^l) = 2$, then the vertex t_l is also placed in the region of $D(F^i)$ with four vertices of G^* on its boundary, i.e., $t_l \in \omega_{2,4,5,3}$. It is not hard to verify that $\mathrm{rot}_D(t_l)$ is either $P_{80} = (142653)$ or $P_{37} = (124635)$ or $P_{117} = (146532)$ or $P_{71} = (156324)$. Now, using $\overline{P_{79}} = (135642) = P_{109}$, $d_{COG}("P_{80}", "P_{109}") = d_{COG}("P_{37}", "P_{109}") = d_{COG}("P_{117}", "P_{109}") = 5$, and $d_{COG}("P_{71}", "P_{109}") = 3$, we see that $\mathrm{cr}_D(T^k, T^l) \geq 5$ trivially holds for the first three possibilities. The subdrawing of $G^* \cup T^i \cup T^k \cup T^l$ should be used for the last one, and therefore, $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1 + 2 + 5 = 8$. We can apply the same idea for the case of $\mathrm{cr}_D(T^k, T^l) \leq 2$. Clearly, if $\mathrm{cr}_D(T^i, T^l) \geq 3$ and $\mathrm{cr}_D(T^k, T^l) \geq 3$, we obtain the desired result $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1 + 3 + 3 = 7$.

d) Let T^l be any subgraph by which the edges of G^* are crossed at least twice. As $\mathrm{cr}_D(K_{6,3}) \geq 6$ and $\mathrm{cr}_D(T^i, T^k) = 1$, the edges of $T^i \cup T^k$ are crossed at least five times by the edges of T^l, which yields that $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 2 + 5 = 7$. ◀

Lemma 3. *Let D be a good and antipode-free drawing of $G^* + D_n$, $n > 2$, with the vertex notation of the graph G^* in a way shown in Figure 1(a). Let $T^i \in R_D$ be a subgraph such that F^i has configuration $\mathcal{A}_p \in \mathcal{M}_D$, $p \in \{1, 2, 3, 5\}$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then*

a) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 7$ for any subgraph $T^l \in S_D$, $l \neq k$ with respect to the restriction $\mathrm{cr}_D(T^i, T^l) = 2$;

b) $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 7$ for any subgraph $T^l \in S_D$ with $\mathrm{cr}_D(T^i, T^l) \geq 3$.

Proof. Let us consider the configuration $\mathcal{A}_2$ of F^i, and note that it is represented by the cyclic permutation $P_{72} = (165324)$.

a) We can follow the similar arguments as in the proof in Lemma 2. The unique drawing of F^i contains six regions with the vertex t_i on their boundaries, see Figure 2. If there is a $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then the vertex t_k must be placed in the quadrangular region of $D(F^i)$ with three vertices of G^* on its boundary, i.e., $t_k \in \omega_{1,3,6}$. This forces that no edge of the graph G^* can be crossed by the edge t_kv_4, and $\mathrm{cr}_D(T^i, T^k) = 2$ only for T^k with either $\mathrm{rot}_D(t_k) = (123654) = P_{55}$ or $\mathrm{rot}_D(t_k) = (135642) = P_{109}$ if the edge v_1v_3 or v_3v_6 is crossed by the edge t_kv_2 or t_kv_5, respectively. Using $\overline{P_{109}} = (124653) = P_{79}$, and $d_{COG}("P_{55}", "P_{79}") = 4$ we obtain $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1+2+4 = 7$ for any subgraph $T^l \in S_D$, $l \neq k$ with $\mathrm{cr}_D(T^i, T^l) = 2$.

b) Let $T^k \in S_D$ be a subgraph with $\mathrm{cr}_D(T^i, T^k) = 2$, i.e., the subdrawing of F^k is represented by either $P_{55} = (123654)$ or $P_{109} = (135642)$. Without loss of generality, let us assume that $\mathrm{rot}_D(t_k) = (123654)$. If there is a $T^l \in S_D$ with $\mathrm{cr}_D(T^k, T^l) = 1$, then the vertex t_l must be placed in the pentagonal region of $D(F^k)$ with four vertices of G^* on its boundary. Hence, the subgraph F^l must be represented by the cyclic permutation $P_{71} = (156324)$. Since $\overline{P_{71}} = (142365) = P_{28}$, the distance $d_{COG}("P_{28}", "P_{72}") = 5$ implies $\mathrm{cr}_D(T^i, T^l) \geq 5$. Thus, $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1+5+1 = 7$. Further, let us assume that $\mathrm{cr}_D(T^k, T^l) \geq 2$ for any $T^l \in S_D$. Since the case $\mathrm{cr}_D(T^i, T^l) \geq 4$ implies $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1+4+2 = 7$, let us consider a subgraph T^l with $\mathrm{cr}_D(T^i, T^l) = 3$. Hence, the vertex t_l must be placed in the region $\omega_{1,3,6}$ of the unique subdrawing of F^i. Consequently, we have $\mathrm{cr}_D(T^k, T^l) \geq 3$, i.e., $\mathrm{cr}_D(G^* \cup T^i \cup T^k, T^l) \geq 1+3+3 = 7$.

Since we are able to use the similar arguments for the remaining configurations $\mathcal{A}_1$, $\mathcal{A}_3$ and $\mathcal{A}_5$ of F^i, this completes the proof of Lemma 3. ◀

Corollary 1. *Let D be a good and antipode-free drawing of $G^* + D_n$, for $n > 2$, with the vertex notation of the graph G^* in a way shown in Figure 1(a). If T^i, $T^j \in R_D$ are different subgraphs such that F^i, F^j have different configurations from any of the sets $\{\mathcal{A}_2, \mathcal{A}_3\}$ and $\{\mathcal{A}_2, \mathcal{A}_5\}$, then*

$$\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 6 \qquad \text{for any } T^k \in S_D,$$

i.e.,

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 7 \qquad \text{for any } T^k \in S_D.$$

Moreover, if there is no $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$ for each $T^i \in R_D$ with the configuration $\mathcal{A}_1 \in \mathcal{M}_D$ of F^i, then the same result is true for the pair $\{\mathcal{A}_1, \mathcal{A}_2\}$.

Proof. Let us assume the configurations $\mathcal{A}_2$ of F^i and $\mathcal{A}_3$ of F^j. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then the subgraph F^k can be represented only by the cyclic permutation either $P_{55} = (123654)$ or $P_{109} = (135642)$. Note that the configuration $\mathcal{A}_3$ is represented by P_{69}. Using $\overline{P_{69}} = (142563) = P_{74}$, and $d_{COG}("P_{55}","P_{74}") = d_{COG}("P_{109}","P_{74}") = 4$ we obtain $\mathrm{cr}_D(T^j, T^k) \geq 4$. Hence, $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 1 + 2 + 4 = 7$. We can apply the same idea for the case where there is a $T^k \in S_D$ with $\mathrm{cr}_D(T^j, T^k) = 2$. Let us assume that $\mathrm{cr}_D(T^i, T^k) \geq 3$, and $\mathrm{cr}_D(T^j, T^k) \geq 3$ for any $T^k \in S_D$. This enforces that, $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 1+3+3 = 7$ trivially holds for any $T^k \in S_D$. The similar arguments can be used for the remaining pairs of configurations, and this completes the proof of Corollary 1. ◀

We have to emphasize that, in Corollary 1, the assumption $\mathrm{cr}_D(T^i, T^k) \geq 2$ for any $T^k \in S_D$ and each $T^i \in R_D$ with the configuration $\mathcal{A}_1 \in \mathcal{M}_D$ of F^i is inevitable. For $T^k \in S_D$ and for $T^j \in R_D$ with the configuration $\mathcal{A}_2$ of F^j, the reader can easily find a subdrawing of $G^* \cup T^i \cup T^j \cup T^k$ in which $\mathrm{cr}_D(T^i, T^k) = 1$ and $\mathrm{cr}_D(T^j, T^k) = 4$, that is, $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) = 6$.

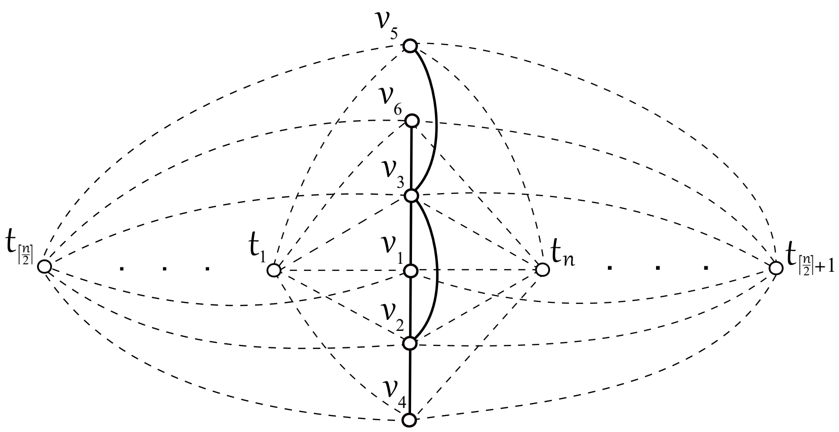

Figure 3: The good drawing of $G^* + D_n$ with $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ crossings.

Theorem 1. *If* $n \geq 1$, *then* $\mathrm{cr}(G^* + D_n) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor$.

Proof. In Figure 3, the edges of $K_{6,n}$ cross each other $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ times, each subgraph T^i, $i = 1, \dots, \lceil\frac{n}{2}\rceil$ on the left side does not cross the edges of G^* and each subgraph T^i, $i = \lceil\frac{n}{2}\rceil + 1, \dots, n$ on the right side crosses the edges of G^* exactly twice. Thus, $\mathrm{cr}(G^* + D_n) \leq 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$. We prove the reverse inequality by induction on n. The graph $G^* + D_1$ is planar; hence, $\mathrm{cr}(G^* + D_1) = 0$. It is clear from the possibility of adding a subgraph $T^k \in S_D$ with two crossings into the subdrawing of $\mathcal{A}_1$ in Figure 2 that $\mathrm{cr}(G^* + D_2) \leq 2$. The graph $G^* + D_2$ contains a subdivision of $K_{3,4}$, and therefore $\mathrm{cr}(G^*+D_2) \geq 2$. So, $\mathrm{cr}(G^*+D_2) = 2$ and the result is true for $n = 1$ and $n = 2$. Suppose now that, for some $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G^* + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor, \tag{5}$$

and that

$$\mathrm{cr}(G^* + D_m) = 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor \qquad \text{for any integer } m < n. \tag{6}$$

We claim that the considered drawing D must be antipode-free. For a contradiction, suppose that $\mathrm{cr}_D(T^k, T^l) = 0$ for two different subgraphs T^k and T^l. If at least one of T^k and T^l, say T^k, does not cross G^*, it is not difficult to verify in Figure 1 that T^l must cross $G^* \cup T^k$ at least twice, that is, $\mathrm{cr}_D(G^*, T^k \cup T^l) \geq 2$. By [11], we already know that $\mathrm{cr}(K_{6,3}) = 6$, which yields that any T^m, $m \neq k, l$, crosses the edges of the subgraph $T^k \cup T^l$ at least six times. So, the number of crossings of $G^* + D_n$ in D is given by

$$\mathrm{cr}_D\,(G^* + D_{n-2}) + \mathrm{cr}_D(K_{6,n-2}, T^k \cup T^l) + \mathrm{cr}_D(G^*, T^k \cup T^l) + \mathrm{cr}_D(T^k \cup T^l)$$

$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 6(n-2) + 2 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

This contradicts the assumption (5) and confirms that D is antipode-free. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (5) together with the well-known fact $\mathrm{cr}(K_{6,n}) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ imply that in D:

$$\mathrm{cr}_D(G^*) + \sum_{T^i \in R_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \in S_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \notin R_D \cup S_D} \mathrm{cr}_D(G^*, T^i) < 2\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$\mathrm{cr}_D(G^*) + 0r + s + 2(n - r - s) < 2\left\lfloor \frac{n}{2} \right\rfloor. \tag{7}$$

This readily enforces that $2r + s > 2n - 2\left\lfloor \frac{n}{2} \right\rfloor$ and $r > n - r - s$, that is, $r \geq 1$, and so there is at least one subgraph T^i whose edges do not cross the edges of G^*. Now, for $T^i \in R_D$, we will discuss the existence of possible configurations of subgraph $F^i = G^* \cup T^i$ in the drawing D and we show that in all cases the contradiction with the assumption (5) is obtained.

Case 1: $\mathrm{cr}_D(G^*) = 0$. Without loss of generality, we can choose the vertex notation of the graph G^* in a way shown in Figure 1(a). We deal with the following possibilities for the nonempty set of configurations $\mathcal{M}_D$.

Let us first consider that $\mathcal{A}_1 \in \mathcal{M}_D$, and let also for some $T^i \in R_D$ with the configuration $\mathcal{A}_1$ of F^i there be a subgraph $T^k \in S_D$ such that $\mathrm{cr}_D(T^i, T^k) = 1$. By Lemma 2, we already know that the subgraph T^k is uniquely represented by $\mathrm{rot}_D(t_k) = (124653)$. Let us denote $S_D(T^i) = \{T^k \in S_D : \mathrm{cr}_D(T^i, T^k) = 1\}$, and $s_1 = |S_D(T^i)|$. Note that $S_D(T^i)$ is a subset of S_D and $1 \leq s_1 \leq s$. Then, by fixing the graph $G^* \cup T^i \cup T^k$ and using the lower bounds in Lemma 2, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &= \mathrm{cr}_D(K_{6,n-2}) + \mathrm{cr}_D(K_{6,n-2}, G^* \cup T^i \cup T^k) + \mathrm{cr}_D(G^* \cup T^i \cup T^k) \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7(r-1) + 8(s_1-1) + 7(s-s_1) + 7(n-r-s) + 2 \\
&= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n + s_1 - 13 \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n + 1 - 13 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This contradicts the assumption of D. Now, suppose that for any $T^k \in S_D$, $\mathrm{cr}_D(T^i, T^k) \geq 2$ holds for each $T^i \in R_D$ with the configuration $\mathcal{A}_1 \in \mathcal{M}_D$ of F^i.

a) $\mathcal{A}_p \in \mathcal{M}_D$ for some $p \in \{4, 6\}$. Without loss of generality, let us assume that $T^n \in R_D$ with the configuration either $\mathcal{A}_4$ or $\mathcal{A}_6$ of F^n. Only for these two subcases, the reader can easily verify in seven possible regions of $D(F^n)$ that $\mathrm{cr}_D(G^* \cup T^n, T^k) \geq 4$ is true for any subgraph T^k, $k \neq n$. Thus, by fixing the subgraph $G^* \cup T^n$, we have

$$\mathrm{cr}_D(G^* + D_n) \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 0 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

This confirms a contradiction with the assumption in D, and therefore, we can suppose that $\mathcal{A}_p \notin \mathcal{M}_D$ for any $p = 4, 6$ in all following cases.

b) $\{\mathcal{A}_1, \mathcal{A}_2\} \subseteq \mathcal{M}_D$. Without loss of generality, let us consider two subgraphs $T^{n-1}, T^n \in R_D$ such that F^{n-1} and F^n have different configurations $\mathcal{A}_1, \mathcal{A}_2$, respectively. Then, $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^i) \geq 9$ is true for any $T^i \in R_D$ with $i \neq n-1, n$ by summing the values in the first two rows for each column of Table 1, and $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^k) \geq 7$ holds for any subgraph $T^k \in S_D$ by Corollary 1. Moreover, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^i) \geq 4$ is fulfilled for any subgraph T^i with $i \neq n-1, n$ according to the properties of the cyclic permutations. Since $\mathrm{cr}_D(T^{n-1} \cup T^n) \geq 3$, then by fixing the graph $G^* \cup T^{n-1} \cup T^n$, we have

$$\mathrm{cr}_D(G^* + D_n) \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 9(r-2) + 7s + 6(n-r-s) + 3$$

$$= 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 6n + r + (2r+s) - 15$$
$$\geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 6n + 1 + 2n - 2\left\lfloor\frac{n}{2}\right\rfloor + 1 - 15$$
$$\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.$$

This also contradicts the assumption of D.

c) $\{\mathcal{A}_1,\mathcal{A}_2\} \not\subseteq \mathcal{M}_D$. Here, two subcases may occur:

1. $\{\mathcal{A}_2,\mathcal{A}_3\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_2,\mathcal{A}_5\} \subseteq \mathcal{M}_D$. Without loss of generality, let us consider two subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have different configurations $\mathcal{A}_2,\mathcal{A}_3$, respectively. The condition (3) is true by summing the values in all remaining columns in the corresponding two rows of Table 1, and the condition (4) holds by Corollary 1. Thus, all assumptions of Lemma 1 are fulfilled. Due to the symmetry of the configurations, we are able to use the same arguments for the case of $\mathcal{A}_2$ of F^{n-1} and $\mathcal{A}_5$ of F^n.
2. $\{\mathcal{A}_2,\mathcal{A}_3\} \not\subseteq \mathcal{M}_D$ and $\{\mathcal{A}_2,\mathcal{A}_5\} \not\subseteq \mathcal{M}_D$. In this case, we discuss simultaneously either $\mathcal{M}_D = \{\mathcal{A}_q\}$ for only one $q \in \{1,2,3,5\}$ or $\{\mathcal{A}_p,\mathcal{A}_q\} \subseteq \mathcal{M}_D$ for some different $p,q \in \{1,3,5\}$. Without loss of generality, we can assume the configuration $\mathcal{A}_q \in \mathcal{M}_D$ of F^n. Let us denote $S_D(T^n) = \{T^k \in S_D : \mathrm{cr}_D(T^n,T^k) = 2\}$, and $s_2 = |S_D(T^n)|$. Remark that $S_D(T^n)$ is a subset of S_D and $s_2 \leq s$. If the set $S_D(T^n)$ is empty, then by fixing the graph $G^* \cup T^n$, we have

$$\mathrm{cr}_D(G^*+D_n) \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 5(r-1) + 4s + 3(n-r-s) + 0$$
$$\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + (2r+s) - 5$$
$$\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + 2n - 2\left\lfloor\frac{n}{2}\right\rfloor + 1 - 5$$
$$\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.$$

In addition, let T^k be a subgraph from the nonempty set $S_D(T^n)$. Since all rotations $\mathrm{rot}_D(t_k)$ for $\mathcal{A}_1$ are different from the intended $\mathrm{rot}_D(t_k)$ for $\mathcal{A}_3$ and $\mathcal{A}_5$, we have $\mathrm{cr}_D(G^* \cup T^n \cup T^k, T^i) \geq 6+2 = 8$ or $\mathrm{cr}_D(G^* \cup T^n \cup T^k, T^i) \geq 5+3 = 8$ for any $T^i \in R_D$, $i \neq n$. By Lemma 3, $\mathrm{cr}_D(G^* \cup T^n \cup T^k, T^i) \geq 7$ is fulfilled for any $T^i \in S_D$,

$i \neq k$. Moreover, $\mathrm{cr}_D(G^* \cup T^n \cup T^k, T^i) \geq 2 + 4 = 6$ holds for any $T^i \notin R_D \cup S_D$. Since $n-r-s \leq r-1$, by fixing the graph $G^* \cup T^n \cup T^k$, we have

$$6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-1) + 7(s_2-1) + 7(s-s_2) + 6(n-r-s) + 3$$

$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7(r-1) + 7(s_2-1) + 7(s-s_2) + 7(n-r-s) + 3$$

$$= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n - 11 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

All these subcases confirm a contradiction with the assumption in D.

Case 2: $\mathrm{cr}_D(G^*) \geq 1$. In all considered subdrawings of the graph G^*, without loss of generality, we can choose the vertex notation of the graph G^* in a way shown in Figure 1(b) – (m). In all cases without the subdrawing of G^* as in Figure 1(h), there are one or two or three or four configurations, and for $T^i \in R_D$, one can easily verify in all possible regions of $D(F^i)$ that $\mathrm{cr}_D(G^* \cup T^i, T^k) \geq 4$ holds for any subgraph T^k, $k \neq i$. Hence, by fixing the subgraph $G^* \cup T^i$, we have

$$\mathrm{cr}_D(G^* + D_n) \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 1 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Finally, for the subdrawing of G^* as in Figure 1(h), there are three configurations denoted by $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, and they are represented by the cyclic permutations (146532), (146352) and (143652), respectively. Of course again, in a fixed drawing of the graph $G^* + D_n$, some configurations from $\mathcal{N} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ need not appear. So, we denote by $\mathcal{N}_D$ the subset of $\mathcal{N}$ consisting of all configurations that exist in the drawing D. If $\mathcal{H}_p \in \mathcal{N}_D$ for some $p \in \{2, 3\}$, then we are able to apply the same idea as in the previous subdrawings provided that, for $T^i \in R_D$ with the configuration $\mathcal{H}_p$ of F^i, the edges of $G^* \cup T^i$ are crossed at least four times by edges of each subgraph T^k, $k \neq i$. In case of $\mathcal{N}_D = \{\mathcal{H}_1\}$, we have to consider the same two possibilities as in the last subcase of Case 1 according to the possibility of an existence of a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$ for the configuration $\mathcal{H}_1$ of F^i.

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $G^* + D_n$ with fewer than $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings, and the proof of Theorem 1 is done. ◀

4. Three Other Graphs

Finally, in the drawing in Figure 3, we are able to add the edges v_4v_5 and v_5v_6 to the graph G^* without additional crossings, and we obtain three new graphs G_i for $i = 1, 2, 3$ in Figure 4. Therefore, the drawings of the graphs G_1+D_n, G_2+D_n, and $G_3 + D_n$ with $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$ crossings are obtained. On the other hand, $G^* + D_n$ is a subgraph of each $G_i + D_n$, and therefore, $\mathrm{cr}(G_i + D_n) \geq \mathrm{cr}(G^* + D_n)$ for any $i = 1, 2, 3$. Thus, the following result is obvious.

Corollary 2. *If $n \geq 1$, then* $\mathrm{cr}(G_i + D_n) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor$*, where $i = 1, 2, 3$.*

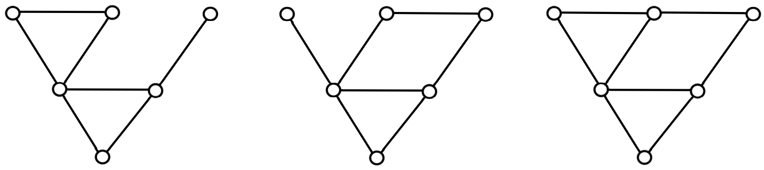

Figure 4: Three graphs G_1, G_2, and G_3 by adding new edges to the graph G^*.

Remark that the crossing numbers of the graphs $G_2 + D_n$ and $G_3 + D_n$ were already obtained in [20] also using the vertex rotation. Furthermore, in the drawing in Figure 3, it is possible to add $n - 1$ edges, which form the path P_n, $n \geq 2$ on the vertices of D_n without another crossing. Thus, the following result is also obvious.

Theorem 2. *If $n \geq 2$, then* $\mathrm{cr}(G^* + P_n) = \mathrm{cr}(G_1 + P_n) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor$.

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