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On the Existence and Uniqueness of Generalized Solutions of Second Order Partial Operator-Differential Equations

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Abstract. In this paper, we consider the existence and uniqueness of the generalized solution of a second order partial operator-differential equation in Hilbert space. To this end, at first we prove lemmas on possibility of continuation of some functional generated by the minor terms of the equation on all the Hilbert space. Then we prove a theorem on the existence of a unique generalized solution of a binomial non-homogeneous equation. Using this theorem we prove the existence and uniqueness of the generalized solution of the given operator-differential equation.

Key Words and Phrases: Hilbert space, operator, operator-differential equation, bilinear functional, generalized solution, positive-definite operator, self-adjoint operator.

2010 Mathematics Subject Classifications: 35J25, 47A10, 58J40

1. Introduction

This paper is dedicated to the existence and uniqueness of the generalized solution of second order partial linear operator-differential equations in Hilbert space. This theory dates back to the works by E.Hille, K.Iosida, S.Agmon, R.Lax and others, where theorems on the existence of the solutions of equations with unbounded operator coefficients in Hilbert and Banach spaces have been obtained.

The foundation for theory of solvability of linear operator-differential equations in the case where the number of independent variables $n = 1$ was laid in the books of S.G.Krein [9], S.Ya.Yakubov [21], in the papers of B.A.Plamenevsky [20], V.G.Mazya and B.A.Plamenevsky [19], Yu.A.Dubinsky [8], M.G.Gasymov [7], V.K.Romanko [10], S.S.Mirzoyev [15], A.A.Shkalikov [22], A.R.Aliyev [1] and others.

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Compared to ordinary operator-differential equations, there are not enough papers on the investigation of solvability of partial operator-differential equations in Hilbert spaces.

The solvability of boundary value problems for some classes of degenerate partial operator-differential equations was considered by V.B.Shakhmurov [23,24], V.B.Shakhmurov and Azad A.Babayev [25], where theorems on continuity, on compactness and quasi-nuclearity of the imbedding operator in abstract anisotropic spaces have been proved.

The obtained imbedding theorems allow to consider a coercive solvability of a new class of strongly degenerate partial differential-operator equations.

Unique, normal and Fredholm solvability and asymptotic behavior of solutions of higher order partial operator-differential equations in Hilbert space are studied by G.I.Aslanov in [3-6]. Theorems on multiple completeness of the system of eigen and associated elements of such operators are proved.

The obtained abstract theorems on the existence of solutions of operator-differential equations are applied to the existence of solutions of the Dirichlet problem and the Neumann problem for second order elliptic equations in infinite multidimensional layer type domains.

Regular solutions of second order operator-differential equations were considered in the papers of S.S.Mirzoyev and N.F.Ismailov [17], S.S. Mirzoyev and N.M. Suleymanov [18]. Generalized solutions of fourth order operator-differential equations were studied in [16]. Well-defined solvability of a class of second order partial operator-differential equations were considered in [11,12].

Let H be a Hilbert space, C be a positive-definite self-adjoint operator with domain of definition $D(C)$. Then for $\gamma \geq 0$ the domain of definition H_γ of the operator C^γ becomes a Hilbert space with respect to the scalar product $(x,y)_\gamma = (C^\gamma x, C^\gamma y)$, $x, y \in D(C^\gamma)$. For $\gamma = 0$ we assume $H_0 = H$.

Let $D(R^n, H_2)$ be a linear set of infinitely differentiable functions with compact supports with the values in H_2 [14]. Suppose that $W_2^1(R^n, H)$ is a completion of the set $D(R^n, H_2)$ with the norm

$$\|u\|_{W_2^1(R^n;H)} = \left(\sum_{k=1}^{n} \left\| \frac{\partial u(x)}{\partial x_k} \right\|_{L_2(R^n;H)}^2 + \|Cu\|_{L_2(R^n;H)}^2 \right)^{1/2}.$$

Here $L_2(R^n; H)$ is a Hilbert space of vector-functions $f(x) = f(x_1, x_2, \ldots, x_n)$, defined almost everywhere in R^n with the values in H with the norm

$$\|f\|_{L_2(R^n;H)} = \left(\int_{R^n} \|f(x_1, x_2, \ldots, x_n)\|_H^2 \, dx_1 dx_2 \ldots dx_n \right)^{1/2}.$$

In space H we consider the equation

$$-\sum_{k=1}^{n} a_k \frac{\partial^2 u(x)}{\partial x_k^2} + C^2 u(x) + \sum_{k=1}^{n} R_k \frac{\partial u(x)}{\partial x_k} + Tu(x) = f(x), \tag{1}$$

where $R_k\,(k = 1, 2 \ldots n)$ and T are linear, in general, unbounded operators, $a_k > 0\ \ (k = 1, 2 \ldots n)$ ..

The goal of the present paper is to prove the existence and uniqueness of the generalized solution of equation (1) in the space $W_2^1\,(R^n, H)$.

2. Auxiliary lemmas on continuation of bilinear functional

As can be seen, equation (1) is a second order equation. The solution of this equation in the space $W_2^1\,(R^n, H)$, must be defined in a suitable way so that it would make sense. Therefore, below we define the generalized solution of equation (1).

To this end, we first introduce the following expressions in $D\,(R^n, H_2)$:

$$L_0 u = -\sum_{k=1}^{n} a_k \frac{\partial^2 u(x)}{\partial x_k^2} + C^2 u, u(x) \in D\,(R^n; H_2)\,,$$

$$L_1 u = \sum_{k=1}^{n} R_k \frac{\partial u(x)}{\partial x_k} + Tu(x), u(x) \in D\,(R^n; H_2)\,,$$

and

$$Lu = L_0 u + L_1 u, u(x) \in D\,(R^n; H_2)\,.$$

Assume that the vector-function $\psi(x) = \psi\,(x_1, x_2, \ldots x_n) \in D\,(R^n; H_2)$, and consider the bilinear functional $(L_1 u, \psi)_{L_2(R^n,H)}$, defined at first in the space $D\,(R^n; H) \oplus D\,(R^n; H)$, i.e. $u \in D\,(R^n; H)\,,\ \psi \in D\,(R^n; H)\,.$

Now we show the conditions on the coefficients $R_k\,(k = 1, 2, \ldots n)$ and T that provide possibility of continuous continuation of the functional $(L_1 u, \psi)_{L_2(R^n;H)}$ from $D\,(R^n; H) \oplus D\,(R^n; H)$ to the whole space $W_2^2\,(R^n; H) \oplus W_2^2\,(R^n; H)$.

We have the following lemma.

Lemma 1. *Let C be a positive-definite self-adjoint operator, the operators $Q_k = R_k C^{-1}\,(k = 1, 2, \ldots n)$ and $F = C^{-1} T C^{-1}$ be bounded in H.*

Then the bilinear functional $(L_1 u, \psi)_{L_2(R^n,H)}$, defined on the linear set $D\,(R^n; H) \oplus D\,(R^n; H)$ continues up to the bilinear functional $L_1\,(u, \psi)$, acting on the space $W_2^2\,(R^n; H) \oplus W_2^2\,(R^n; H)$, where

$$L_1\,(u, \psi) = \sum_{k=1}^{n} (R_k u, \psi)_{L_2(R^n;H)} + (Tu, \psi)_{L_2(R^n;H)} \tag{2}$$

Proof. For $u \in D\left(R^n; H\right), \, \psi \in D\left(R^n; H\right)$ we have:

$$(L_1 u, \psi)_{L_2(R^n;H)} =$$

$$= \left(\sum_{k=1}^{n} R_k \frac{\partial u}{\partial x_k} + Tu, \psi\right)_{L_2(R^n;H)} = -\sum_{k=1}^{n} \left(R_k u, \frac{\partial \psi}{\partial x_k}\right)_{L_2(R^n;H)} + (Tu, \psi)_{L_2(R^n;H)} =$$

$$= -\sum_{k=1}^{n} \left(R_k C^{-1} Cu, \frac{\partial \psi}{\partial x_k}\right)_{L_2(R^n;H)} + \left(C^{-1} T C^{-1} Cu, C\psi\right)_{L_2(R^n;H)} \qquad (3)$$

For $k = 1, 2, \ldots n$ we have:

$$\left|\left(-R_k C^{-1} Cu, \frac{\partial \psi}{\partial x_k}\right)_{L_2(R^n;H)}\right| \le \left\|R_k C^{-1}\right\| \cdot \|Cu\|_{L_2(R^n;H)} \cdot \left\|\frac{\partial \psi}{\partial x_k}\right\|_{L_2(R^n;H)}$$

and

$$\left|\left(C^{-1} T C^{-1} Cu, C\psi\right)_{L_2(R^n;H)}\right| \le \left\|C^{-1} T C^{-1}\right\| \cdot \|Cu\|_{L_2(R^n;H)} \cdot \|C\psi\|_{L_2(R^n;H)}$$

Hence

$$\left|\left(-R_k C^{-1} Cu, \frac{\partial \psi}{\partial x_k}\right)_{L_2(R^n;H)}\right| \le \frac{1}{2} \|Q_k\| \left(\|Cu\|^2_{L_2(R^n;H)} + \left\|\frac{\partial \psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}\right) \qquad (4)$$

and

$$\left|\left(C^{-1} T C^{-1} Cu, C\psi\right)_{L_2(R^n;H)}\right| \le \frac{1}{2} \|F\| \left(\|Cu\|^2_{L_2(R^n;H)} + \|C\psi\|^2_{L_2(R^n;H)}\right) \qquad (5)$$

From equality (3) with regard to inequalities (4) and (5) we get:

$$\left|-\sum_{k=1}^{n} \left(R_k u, \frac{\partial \psi}{\partial x_k}\right)_{L_2(R^n;H)} + (Tu, \psi)_{L_2(R^n;H)}\right| \le$$

$$\le \sum_{k=1}^{n} \frac{1}{2} \|Q_k\| \left(\|Cu\|^2_{L_2(R^n;H)} + \left\|\frac{\partial \psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}\right) +$$

$$+ \frac{1}{2} \|F\| \left(\|Cu\|^2_{L_2(R^n;H)} + \|C\psi\|^2_{L_2(R^n;H)}\right) \le$$

$$\le const \cdot \left(\|u\|^2_{W_2^1(R^n;H)} + \|\psi\|^2_{W_2^1(R^n;H)}\right). \qquad (6)$$

Hence it follows that the bilinear functional $(L_1u,\psi)_{L_2(R^n;H)}$ can be continuously continued from $D\left(R^n;H\right)\oplus D\left(R^n;H\right)$ to $W_2^1\left(R^n;H\right)\oplus W_2^1\left(R^n;H\right)$ since the linear set $D\left(R^n;H_2\right)$ is dense in the space $W_2^1\left(R^n;H\right)$ [see 14].

Lemma 1 is proved. ◄

A similar estimate of the form (6) may be obtained for the bilinear functional

$$(L_0u,\psi)=\left(-\sum_{k=1}^{n}a_k\frac{\partial^2u}{\partial x_k}+C^2u,\psi\right)=$$

$$=\sum_{k=1}^{n}a_k\left(\frac{\partial u}{\partial x_k},\frac{\partial\psi}{\partial x_k}\right)_{L_2(R^n;H)}+(Cu,C\psi)_{L_2(R^n;H)}$$

defined in $D\left(R^n;H\right)\oplus D\left(R^n;H\right)$:

$$\left|(L_0u,\psi)_{L_2(R^n;H)}\right|\leq\sum_{k=1}^{n}a_k\left\|\frac{\partial u}{\partial x_k}\right\|_{L_2(R^n;H)}\left\|\frac{\partial\psi}{\partial x_k}\right\|_{L_2(R^n;H)}+$$

$$+\|Cu\|_{L_2(R^n;H)}\cdot\|C\psi\|_{L_2(R^n;H)}\leq\sum_{k=1}^{n}\frac{1}{2}a_k\left\|\frac{\partial u}{\partial x_k}\right\|^2_{L_2(R^n;H)}+$$

$$+\sum_{k=1}^{n}\frac{1}{2}a_k\left\|\frac{\partial\psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}+\frac{1}{2}\|Cu\|^2_{L_2(R^n;H)}+\frac{1}{2}\|C\psi\|^2_{L_2(R^n;H)}\leq$$

$$\leq const\left(\|u\|^2_{W_2^1(R^n;H)}+\|\psi\|^2_{W_2^1(R^n;H)}\right)\tag{7}$$

Hence it follows that the bilinear functional $(L_0u,\psi)_{L_2(R^n;H)}$ can also be continuously continued from $D\left(R^n;H\right)\oplus D\left(R^n;H\right)$ to $W_2^1\left(R^n;H\right)\oplus W_2^1\left(R^n;H\right)$. As a result we get:

Lemma 2. *Subject to the condition of Lemma 1, the bilinear functional* $(Lu,\psi)_{L_2(R^n;H)}=(L_0u,\psi)_{L_2(R^n;H)}+(L_1u,\psi)_{L_2(R^n;H)}$ *may be continuously continued from the linear set* $D\left(R^n;H\right)\oplus D\left(R^n;H\right)$ *to* $W_2^1\left(R^n;H\right)\oplus W_2^1\left(R^n;H\right)$ *in the following way:*

$$L\left(u,\psi\right)=L_0\left(u,\psi\right)+L_1\left(u,\psi\right)$$

where

$$L_0\left(u,\psi\right)=\sum_{k=1}^{n}a_k\left(\frac{\partial u}{\partial x_k},\frac{\partial\psi}{\partial x_k}\right)_{L_2(R^n;H)}+(Cu,C\psi)_{L_2(R^n;H)}\tag{8}$$

$$L_1\left(u,\psi\right)=-\sum_{k=1}^{n}\left(R_ku,\frac{\partial\psi}{\partial x_k}\right)_{L_2(R^n;H)}+(Tu,\psi)_{L_2(R^n;H)}\tag{9}$$

We now can give the definition of the generalized solution of equation (1).

Definition 1. *If for any* $f(x) \in L_2(R^n; H)$ *there exists a vector-function* $u(x) \in W_2^1(R^n; H)$, *satisfying the relation*

$$L(u, \psi) = L_0(u, \psi) + L_1(u, \psi) = (f, \psi)_{L_2(R^n; H)}$$

for any $\psi \in W_2^1(R^n; H)$, *then* $u(x)$ *is called a generalized solution of equation* (1).

Now let's prove the following theorem.

Theorem 1. *Let* C *be a positive-definite self-adjoint operator. Then for any* $f(x) \in L_2(R^n; H)$ *the equation*

$$L_0 u = -\sum_{k=1}^{n} a_k \frac{\partial^2 u}{\partial x_k^2} + C^2 u = f(x) \tag{10}$$

has a unique generalized solution.

Proof. Let $f(x) = f(x_1, x_2, \ldots, x_n) \in L_2(R^n; H)$. Then it is easy to see that

$$u_0(x) =$$

$$\frac{1}{(2\pi)^{n/2}} \int_{R^n} \left(\sum_{k=1}^{n} a_k \xi_k^2 + C^2 \right)^{-1} \hat{f}(\xi_1, \xi_2, \ldots \xi_n) e^{i(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n)} d\xi_1 d\xi_2 \ldots d\xi_n$$

belongs to the space $W_2^1(R^n; H)$ and

$$L_0(u_0, \psi) = \sum_{k=1}^{n} a_k \left(\frac{\partial u_0}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n; H)} + (Cu, C\psi)_{L_2(R^n; H)} = (f, \psi)$$

is fulfilled. Indeed,

$$(L_0 u_0, \psi)_{L_2(R^n; H)} = \left(-\sum_{k=1}^{n} a_k \frac{\partial^2 u_0}{\partial x_k^2}, \psi \right)_{L_2(R^n; H)} + \left(C^2 u_0, \psi\right)_{L_2(R^n; H)} =$$

$$= \sum_{k=1}^{n} a_k \left(\frac{\partial u_0}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n; H)} + (Cu, C\psi)_{L_2(R^n; H)} .$$

Since $L_0 u_0 = f$, we have

$$\sum_{k=1}^{n} a_k \left(\frac{\partial u}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n; H)} + (Cu, C\psi)_{L_2(R^n; H)} = (f, \psi)_{L_2(R^n; H)} .$$

On the other hand, for $f(x)=0$ we get

$$(L_0u_0,\psi)_{L_2(R^n;H)}=\sum_{k=1}^{n}a_k\left(\frac{\partial u_0}{\partial x_k},\frac{\partial \psi}{\partial x_k}\right)_{L_2(R^n;H)}+(Cu_0,C\psi)_{L_2(R^n;H)}=0.$$

In particular, if we put $\psi=u_0$ we get

$$(L_0u_0,u_0)_{L_2(R^n;H)}=\sum_{k=1}^{n}a_k\left(\frac{\partial u_0}{\partial x_k},\frac{\partial u_0}{\partial x_k}\right)_{L_2(R^n;H)}+(Cu_0,Cu_0)=$$

$$=\sum_{k=1}^{n}a_k\left\|\frac{\partial u_0}{\partial x_k}\right\|^2_{L_2(R^n;H)}+\|Cu_0\|^2_{L_2(R^n;H)}=0$$

Hence it follows that $u_0=0$. Theorem 1 is proved. ◄

By definition, the norm of the element $u\in W_2^1\left(R^n;H\right)$ is defined as follows:

$$\|u\|^2_{W_2^1(R^n;H)}=\sum_{k=1}^{n}\left\|\frac{\partial u}{\partial x_k}\right\|^2_{L_2(R^n;H)}+\|Cu\|^2_{L_2(R^n;H)}$$

Then for any $\psi\in W_2^1\left(R^n;H\right)$ we have the inequalities

$$\|C\psi\|^2_{L_2(R^n;H)}\le\|\psi\|^2_{L_2(R^n;H)}$$

$$\left\|\frac{\partial \psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}\le\|\psi\|^2_{L_2(R^n;H)}$$

or

$$\|C\psi\|_{L_2(R^n;H)}\le\|\psi\|_{W_2(R^n;H)} \tag{11}$$

$$\left\|\frac{\partial \psi}{\partial x_k}\right\|_{L_2(R^n;H)}\le\|\psi\|_{W_2^1(R^n;H)}\,,\ k=1,2,\ldots n \tag{12}$$

The following theorem is true.

Theorem 2. *Let C be a positive-definite operator, the numbers $a_k>0\,(k=1,2,\ldots n)$, the operators $Q_k=R_kC^{-1}\,(1,2,\ldots n)$, $F=C^{-1}TC^{-1}$ be bounded in the space H, and the condition*

$$\frac{1}{2}\sum_{k=1}^{n}\|Q_k\|^2+\|F\|<\min\left(a_1,a_2,\ldots,a_n,1\right)$$

be fulfilled.

Then equation (1) for any $f(x)\in L_2\left(R^n;H\right)$ has a unique generalized solution

Proof. By Theorem 1, the equation

$$L_0 u = -\sum_{k=1}^{n} a_k \frac{\partial^2 u}{\partial x_k^2} + C^2 u = f(x),\ x \in R^n$$

has a unique generalized solution $u_0(x)$ for all $f(x) \in L_2(R^n; H)$.

We will look for the solution of equation (1) in the form $u(x) = u_0(x) + \omega(x)$, where $\omega(x) \in W_2^1(R^n; H)$.
Then we have:

$$L_0(u, \psi) = \sum_{k=1}^{n} a_k \left(\frac{\partial u}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n;H)} + (Cu, C\psi)_{L_2(R^n;H)} =$$

$$= \sum_{k=1}^{n} a_k \left(\frac{\partial (u_0 + \omega)}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n;H)} + (C(u_0 + \omega), C\psi)_{L_2(R^n;H)}.$$

Therefore we get:

$$L(u, \psi) = L_0(u, \psi) + L_1(u, \psi) = \sum_{k=1}^{n} a_k \left(\frac{\partial (u_0 + \omega)}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n;H)} +$$

$$+ (C(u_0 + \omega), C\psi)_{L_2(R^n;H)} + L_1(u_0, \psi) + L_1(\omega, \psi) = (f, \psi)_{L_2(R^n;H)}.$$

Since

$$\sum_{k=1}^{n} a_k \left(\frac{\partial u_0}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n;H)} + (Cu_0, C\psi)_{L_2(R^n;H)} = (f, \psi)_{L_2(R^n;H)},$$

for the function $\omega(x)$ we get the relation

$$\sum_{k=1}^{n} a_k \left(\frac{\partial \omega}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n;H)} + (C\omega, C\psi)_{L_2(R^n;H)} + L_1(\omega, \psi) = -L_1(u_0, \psi). \quad (13)$$

Since

$$|L_1(u_0, \psi)| \le const \left(\|u_0\|_{W_2^1(R^n:H)} + \|\psi\|_{W_2^1(R^n:H)} \right),$$

the right-hand side of equation (13) is a linear functional in the space $W_2^1(R^n; H)$ with respect to Ψ.

We denote the left hand side of relation (13) by $\langle \omega, \psi \rangle$, i.e.

$$\langle \omega, \psi \rangle = \sum_{k=1}^{n} a_k \left(\frac{\partial \omega}{\partial x_k}, \frac{\partial \psi}{\partial x_k} \right)_{L_2(R^n;H)} + (C\omega, C\psi)_{L_2(R^n;H)} + L_1(\omega, \psi).$$

Then for $\omega=\psi$ we get

$$\langle\psi,\psi\rangle=\sum_{k=1}^{n}a_k\left\|\frac{\partial\psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}+\|C\psi\|^2_{L_2(R^n;H)}+L_1(\psi,\psi).$$

Hence we have

$$\langle\psi,\psi\rangle\geq\sum_{k=1}^{n}a_k\left\|\frac{\partial\psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}+\|C\psi\|^2_{L_2(R^n;H)}-|L_1(\psi,\psi)|\geq$$

$$\geq\min(a_1,a_2,\dots a_n,1)\sum_{k=1}^{n}\left\|\frac{\partial\psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}+\|C\psi\|_{L_2(R^n;H)}+|L_1(\psi,\psi)|\geq$$

$$\geq\min(a_1,a_2,\dots a_{n-1})\|\psi\|^2_{L_2(R^n;H)}-|L_1(\psi,\psi)|. \tag{14}$$

On the other hand,

$$|L_1(\psi,\psi)|=\sum_{k=1}^{n}\left(R_k\psi,\frac{\partial\psi}{\partial x_k}\right)_{L_2(R^n;H)}+\left|(T\psi,\psi)_{L_2(R^n;H)}\right|\leq$$

$$\leq\sum_{k=1}^{n}\left|\left(R_kC^{-1}C\psi,\frac{\partial\psi}{\partial x_k}\right)_{L_2(R^n;H)}\right|+\left|\left(C^{-1}TC^{-1}C\psi,C\psi\right)_{L_2(R^n;H)}\right|\leq$$

$$\leq\sum_{k=1}^{n}\|Q_k\|\cdot\|C\psi\|_{L_2(R^n;H)}\cdot\left\|\frac{\partial\psi}{\partial x_k}\right\|_{L_2(R^n;H)}+\|F\|\cdot\|C\psi\|^2_{L_2(R^n;H)}\leq$$

$$\leq\sum_{k=1}^{n}\|Q_k\|\left\{\frac{1}{2}\|C\psi\|^2_{L_2(R^n;H)}+\frac{1}{2}\left\|\frac{\partial\psi}{\partial x_k}\right\|^2_{L_2(R^n;H)}\right\}+\|F\|^2\cdot\|C\psi\|_{L_2(R^n;H)}\leq$$

$$\leq\frac{1}{2}\sum_{k=1}^{n}\|Q_k\|\cdot\|\psi\|^2_{W_2^1(R^n;H)}+\|F\|\cdot\|\psi\|^2_{W_2^1(R^n;H)}=$$

$$=\left(\frac{1}{2}\sum_{k=1}^{n}\|Q_k\|+\|F\|\,\|\Psi\|\right)_{W_2^1(R^n;H)}.$$

By the condition of theorem,

$$\gamma=\min(a_1,a_2,\dots a_n,1)-\left(\frac{1}{2}\sum_{k=1}^{n}\|Q_k\|+\|F\|\right)>0.$$

Then we get

$$\langle\psi,\psi\rangle\geq\gamma\|\psi\|^2_{W_2^1(R^n;H)}. \tag{15}$$

The left-hand side of the expression $\langle \omega, \psi \rangle$ is a linear functional and from the inequality (15) we get that all conditions of the Lax-Milgram theorem [13] are fulfilled. Then it follows from this theorem that there exists a unique function $\omega \in W_2^1 (R^n; H)$ satisfying the relation

$$\langle \omega, \psi \rangle = -L_1 (u_0, \psi)$$

for all $\psi \in W_2^1 (R^n; H)$ i.e. the function $\omega(x)$ is a desired function. Hence we get that the function $u = u_0 + \omega$ is a generalized solution of equation (1). Theorem 2 is proved. ◄

Corollary 1. *If C is a positive-definite self-adjoint operator in H, the operators $Q_k = R_k C^{-1} \, (k = 1, 2, \dots n)$ and $F = C^{-1} T C^{-1}$ are bounded in H and the estimate*

$$\frac{1}{2} \sum_{k=1}^{n} \|Q_k\| + \|F\| < 1$$

holds, then the equation

$$-\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} + C^2 u + \sum_{k=1}^{n} R_k \frac{\partial u}{\partial x_k} + Tu = f(x), \;\; x \in R^n$$

has a unique generalized solution for any $f(x) \in L_2 (R^n; H)$.

Indeed, in this case, it suffices to assume $a_1 = a_2 = \ldots = a_n = 1$.

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References

[1] A.R. Aliyev, *On generalized solution of boundary value problems for a class of operator-differential equations with a discontinuous coefficient*, Proc. of the scientific conference "The problems of functional analysis and mathematical physics" devoted to 80 years of BSU: Baku, 1999, 169-176.

[2] B.A. Aliyev, *Solvability of a boundary value problem with an operator in boundary conditions for a second order elliptic differential equation*, DAN Azerb. SSR, **37(6)**, 1981, 17-21.

[3] G.I. Aslanov, *On differential equations with operator coefficients in Hilbert spaces*, Matem. zametki, **53(3)**, 1993, 153-155.

[4] G.I. Aslanov, *On differential equations with unbounded operator coefficients in Hilbert spaces*, DAN SSSR, **337(1)**, 1994, 329-331.

[5] G.I. Aslanov, *On solvability and asymptotic behavior of solutions of differential equations in Hilbert space*, Doklady AN Azerb., **46(7-8)**, 1990, 17-23.

[6] G.I. Aslanov, *On solvability and asymptotic behavior of solutions of differential equations with operator coefficients in Hilbert space*, Uspekhi mat. nauk., **4**, 1993, 172-173.

[7] M.G. Gasymov, *On solvability of boundary value problems for a class of operator-differential equations*, DAN SSSR, **235(3)**, 1977, 505-508.

[8] Yu.A. Dubinsky, *Boundary value problem for some classes of arbitrary order differential operator equations*, DAN SSSR, **196(1)**, 1971, 32-35.

[9] S.G. Krein, *Linear equations in Banach space*, Moscow, Nauka, 1967.

[10] V.K. Romanko, *To theory of operator $u^{(m)} - A$*, Diff. Uravn., **3(11)**, 1967, 1957-1970.

[11] R.F. Gatamova, *Well-defined solvability of a class of second order partial operator-differential equations in Hilbert space*, Journal of Contem. Appl. Math., **9(2)**, 2019, 25-31.

[12] R.F. Gatamova, *On solvability of second order partial operator-differential equations in all space*, Journal of Baku Engineering University, Mathematics and Computer Sc., **2(2)**, 2018, 85-91.

[13] K. Iosida, *Functional Analysis*, Moscow, Mir, 1967.

[14] J.L. Lions, E. Magenes, *Inhomogeneous boundary value problems and their applications*, Moscow, Mir, 1971.

[15] S.S. Mirzoev, *On a boundary value problem for second order operator-differential equations*, Trudy IMM AN Azerb., **VIII(XVI)**, 1998, 154-161.

[16] S.S. Mirzoev, M.F. Ismayilova, *On solvability of fourth order partial operator-differential equations in Hilbert space*, Vestnik Bakinskogo Univ., Issue Phys. Math. Sc., **4**, 2006, 5-11.

[17] S.S. Mirzoev, I.J. Jafarov, *On solvability of a boundary value problem for second order partial operator-differential equations*, Mathematicheskiye zametki, **91(3)**, 2012, 470-472.

[18] S.S. Mirzoev, N.M. Suleymanov, *On solvability of a class of partial operator-differential equations*, Vestnik BSU, Issue Phys. Math. Sc., **2**, 2015, 5-12.

[19] V.G. Mazya, B.A. Plamenevsky, *On asymptotics of the solution of differential equations with operator coefficients*, DAN SSSR, **196(3)**, 1971, 512-515.

[20] B.A. Plamenevsky, *On the existence and asymptotics of solutions of differential equations with unbounded operator coefficients in Banach space*, Izv. AN SSSR, Ser. Math., **36(6)**, 1972, 1348-1401.

[21] S.Ya. Yakubov, *Linear differential equations and their applications*, Baku, Elm, 1985.

[22] A.A. Shkalikov, *Elliptic equations in Hilbert space and related problems*, Proc. of I.G. Petrovsky, **14**, 1989, 140-224.

[23] V.B. Shakhmurov, *Coercive solvability of general boundary value problems for partial differential operator equations*, Dokl. An Azerb. SSR, 1978, **6**, 3-7.

[24] V.B. Shakhmurov, *Coercive boundary value problems for strongly degenerate differential equations*, DAN SSSR, **290(3)**, 1986, 553-556.

[25] V.B. Shakhmurov, A.A. Babayev, *Coercive problems for a equation with parameters*, DAN SSSR, 315(1), 1992, 37-40.

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