Azerbaijan Journal of Mathematics
V. 12, No 1, 2022, January
ISSN 2218-6816

Three-dimensional Initial-boundary Value Problem for a Parabolic-hyperbolic Equation With a Degenerate Parabolic Part

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Abstract. Initial-boundary value problem for a non-homogeneous equation of mixed parabolic-hyperbolic type in three variables with a degenerate parabolic part in a rectangular parallelepiped is studied. A criterion for the uniqueness of a solution is established. The solution is constructed as the sum of an orthogonal series. When justifying the convergence of the series, the problem of small denominators of two natural arguments arises. Estimates on the separation of small denominators from zero with the corresponding asymptotics are established. These estimates made it possible to substantiate the convergence of the constructed series in the class of regular solutions of this equation. The stability of the solution with respect to the boundary function and the right-hand side of the equation is established.

Key Words and Phrases: equation of mixed parabolic-hyperbolic type, three-dimensional initial-boundary value problem, uniqueness, series, small denominators, existence, stability.

2010 Mathematics Subject Classifications: 35M10

1. Formulation of the problem

Consider an equation of mixed parabolic-hyperbolic type

$$Lu = F(x, y, t), \tag{1}$$

where

$$Lu = \begin{cases} u_t - t^l(u_{xx} + u_{yy}) + bt^l u, & t > 0, \\ u_{tt} - u_{xx} - u_{yy} + bu, & t < 0, \end{cases}$$

$$F(x,y,t) = \begin{cases} F_1(x,y,t), & t > 0, \\ F_2(x,y,t), & t < 0, \end{cases}$$

is given in the three-dimensional domain

$$Q = \{(x,y,t) |\, (x,y) \in D,\, t \in (-\alpha, \beta)\},$$

where

$$D = \{(x,y) |\, 0 < x < p,\, 0 < y < q\},$$

l, α, β, p, q are positive real numbers, b is any real number, and $F_i(x,y,t)$ $(i = 1,2)$ are the given functions. We consider the following problem.

Initial boundary value problem. *Find a function $u(x,y,t)$ defined in the domain Q and satisfying the following conditions:*

$$u(x,y,t) \in C(\overline{Q}) \cap C_t^1(Q) \cap C_{x,y}^1(\overline{Q}) \cap C_{x,y}^2(Q_+) \cap C^2(Q_-); \tag{2}$$

$$Lu(x,y,t) \equiv F(x,y,t), \quad (x,y,t) \in Q_+ \cup Q_-; \tag{3}$$

$$u(x,y,t)\big|_{x=0} = u(x,y,t)\big|_{x=p} = 0, \quad -\alpha \le t \le \beta; \tag{4}$$

$$u(x,y,t)\big|_{y=0} = u(x,y,t)\big|_{y=q} = 0, \quad -\alpha \le t \le \beta; \tag{5}$$

$$u(x,y,t)\big|_{t=-\alpha} = \psi(x,y), \quad (x,y) \in \overline{D}, \tag{6}$$

where $F(x,y,t)$ and $\psi(x,y)$ are the given sufficiently smooth functions, $Q_- = Q \cap \{t < 0\}$, $Q_+ = Q \cap \{t > 0\}$.

O.A. Ladyzhenskaya and L. Stupyalis [1, 2] considered initial-boundary boundary value problems of conjugation for parabolic-hyperbolic equations in a multidimensional space that arise in the study of the problem of motion of conducting fluid in an electromagnetic field. One of these problems is stated as follows. Let the bounded domain $\Omega \subset \mathbb{R}^n$ be the union of two domains Ω_1 and Ω_2 and the $(n-1)$-dimensional surface Γ separating them, so that $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. Let us denote by S the boundary of Ω, and by S_1 and S_2 the boundaries of Ω_1 and Ω_2 respectively, $Q_T^{(k)} = \Omega_k \times [0,T]$, $k = 1,2$. The problem is: find a function $u(x,t)$ that satisfies the parabolic equation

$$u_t + L_1 u = f_1(x,t)$$

in the domain $Q_T^{(1)}$, the hyperbolic equation

$$u_{tt} + L_2 u = f_2(x,t)$$

in the domain $Q_T^{(2)}$, the initial conditions

$$u\Big|_{t=0} = \varphi_1(x), \quad x \in \Omega_1;$$

$$u\Big|_{t=0} = \varphi_2(x), \quad u_t\Big|_{t=0} = \varphi_3(x), \quad x \in \Omega_2,$$

the boundary condition

$$u\Big|_S = \psi(x,t), \quad t \in [0,T],$$

on S and the conjugation conditions

$$u^{(1)}\Big|_\Gamma - u^{(2)}\Big|_\Gamma = \psi_1(x,t),$$

$$b^{(1)}(x,t)\frac{\partial u^{(1)}}{\partial N}\bigg|_\Gamma - b^{(2)}(x,t)\frac{\partial u^{(2)}}{\partial N}\bigg|_\Gamma = \psi_2(x,t), \quad t \in [0,T],$$

on Γ, where $u(x,t) = u^{(1)}(x,t)$ for $x \in \Omega_1$, $t \geq 0$, and $u(x,t) = u^{(2)}(x,t)$ for $x \in \Omega_2$, $t \geq 0$, $b^{(k)}(x,t) \geq \beta > 0$, $k = 1,2$, are the well-known functions,

$$L_k u \equiv -\frac{\partial}{\partial x_i}\left[a_{ij}^{(k)}(x,t)\frac{\partial u}{\partial x_j}\right] + a_i^{(k)}(x,t)\frac{\partial u}{\partial x_i} + a^{(k)}(x,t), \quad k = 1,2,$$

$a_{ij}^{(k)}(x,t)$, $a_i^{(k)}(x,t)$, $a^{(k)}(x,t)$, $f_k(x,t)$, $\varphi_j(x)$, $j = 1,2,3$, $\psi(x,t)$, $\psi_k(x,t)$ are the given functions.

Unlike the above articles, in this work, the conjugation conditions are specified not with respect to a spatial variable, but with respect to a time variable, i.e. on surface $t = 0$.

N.Yu. Kapustin [3], using the methods of functional analysis, proved the unique solvability of an analogue of the Tricomi problem in the space L_2 for an equation of parabolic-hyperbolic type in a plane mixed domain, where the parabolic part is a rectangle, and the hyperbolic part is a characteristic triangle with a base on the degeneration line. E.I. Moiseev [4] by the method of spectral analysis problems, studied the parabolic-hyperbolic equations and the corresponding one-dimensional spectral problems.

Initial boundary value problems (local and nonlocal) for homogeneous and non-homogeneous equations of mixed parabolic-hyperbolic type in a rectangular domain were first studied in the works of K.B. Sabitov and S.N. Sidorov [5], [6, p. 56–94], [7-10].

We also note our papers [11, 12], where, for three classes of one-dimensional parabolic-hyperbolic equations with a right-hand side $F(x,t)$: equations of mixed type with a degenerate hyperbolic part, equations of mixed type with a degenerate parabolic part and mixed-type equations with power-law degeneracy, an initial-boundary value problem in a rectangular domain $D = \{(x,t)|\, 0 < x < l,\, -\alpha < t < \beta\}$ with nonzero conditions on the boundary was studied $u(0,t) = h_1(t)$, $u(l,t) = h_2(t)$, $u(x,-\alpha) = \varphi(x)$.

The problem (2) – (6) for the equation (1) without a degenerate parabolic part, that is, for $l = 0$, was studied in [13].

We also note the works of S.A. Aldashev [14 – 17], in which boundary value problems were studied for degenerate multidimensional hyperbolic-parabolic equations in a mixed cylindrical domain by the method of separation of variables, which had been split into two independent boundary value problems for equations of parabolic and hyperbolic types, respectively. In [14], the solution to the Dirichlet problem for the hyperbolic equation was constructed as a sum of a series, the convergence of which is not substantiated due to the presence of small denominators. In [16], a boundary value problem was studied for a hyperbolic-parabolic equation, which decomposes into two initial-boundary value problems for equations of parabolic and hyperbolic types, respectively. The main Theorem 1 includes the condition $\cos \mu_{s,n}\alpha' \neq 0$, $s = 1, 2, ...$, $n = 0, 1, ...$, where $\mu_{s,n}$ are positive zeros of the Bessel function of the first kind $J_n(z)$, α' is a given number, but this condition is not used anywhere in the proof of this theorem.

The need to study the problem (2) – (6) for the non-homogeneous equation (1) arises in connection with the study of inverse problems for the equation (1) to find the factors of the right-hand side $F_i(x,y,t) = f_i(x,y)g_i(t)$, $i = 1, 2$, depending either on (x,y) or on t. In the one-dimensional case, inverse problems of finding unknown right-hand sides were investigated in [18 – 22].

In this work, using the idea of [13], a criterion for the uniqueness of the solution to the problem (2) – (6) is established. The solution is explicitly constructed as the sum of an orthogonal two-dimensional series. When justifying the convergence of the series, the problem of small denominators of two natural arguments arose for the first time, which complicates the convergence of the constructed series. In this connection, in order to prove the uniform convergence of the series, estimates were established on the separation of small denominators from zero, which made it possible to prove the existence of a regular solution. The stability of the solution with respect to the boundary function $\psi(x,y)$ and the right-hand side of the equation (1) is established.

2. A criterion for the uniqueness of the solution to the problem

Let there exist a solution $u(x,y,t)$ to problem (2) – (6), $F_1(x,y,t) \in C(D) \cap L(D)$ for $0 < t < \beta$, and $F_2(x,y,t) \in C(D) \cap L(D)$ for $-\alpha < t < 0$. Following

[13], consider the functions

$$u_{mn}(t) = \iint\limits_{D} u(x,y,t) v_{mn}(x,y)\,dxdy, \quad m,\, n \in \mathbb{N}, \tag{7}$$

where

$$v_{mn}(x,y) = \frac{2}{\sqrt{pq}} \sin\frac{m\pi x}{p} \sin\frac{n\pi y}{q} \tag{8}$$

is the complete orthonormal system of eigenfunctions of the Laplace operator in the rectangle D with zero Dirichlet boundary conditions. Note also that the system of functions (8) is complete, orthonormal, and forms a basis for the space $L_2(D)$.

Differentiating the equality (7) with respect to t for $t > 0$ once, and for $t < 0$ twice, and taking into account the equation (1), we obtain

$$u'_{mn}(t) = \iint\limits_{D} u_t(x,y,t) v_{mn}(x,y)\,dxdy =$$

$$= \iint\limits_{D_\varepsilon} \Big[t^l (u_{xx} + u_{yy} - bu) + F_1(x,y,t)\Big] v_{mn}(x,y)\,dxdy =$$

$$= t^l \iint\limits_{D} u_{xx} v_{mn}(x,y)\,dxdy + t^l \iint\limits_{D} u_{yy} v_{mn}(x,y)\,dxdy - bt^n u_{mn}(t) +$$

$$+ \iint\limits_{D} F_1(x,y,t) v_{mn}(x,y)\,dxdy,$$

$$u''_{mn}(t) = \iint\limits_{D} u_{tt}(x,y,t) v_{mn}(x,y)\,dxdy =$$

$$= \iint\limits_{D} \Big[u_{xx} + u_{yy} - bu + F_2(x,y,t)\Big] v_{mn}(x,y)\,dxdy =$$

$$= \iint\limits_{D} u_{xx} v_{mn}(x,y)\,dxdy + \iint\limits_{D} u_{yy} v_{mn}(x,y)\,dxdy - bu_{mn}(t) +$$

$$+ \iint\limits_{D} F_2(x,y,t) v_{mn}(x,y)\,dxdy.$$

In the integrals containing the derivatives u_{xx} and u_{yy}, integrating by parts twice and taking into account the boundary conditions (4) and (5), we obtain the differential equations

$$u'_{mn}(t) + \lambda^2_{mn} t^l u_{mn}(t) = \Phi_{1mn}(t), \quad t > 0, \tag{9}$$

$$u''_{mn}(t) + \lambda^2_{mn} u_{mn}(t) = \Phi_{2mn}(t), \quad t < 0, \tag{10}$$

where

$$\lambda^2_{mn} = b + \pi^2 \left[\left(\frac{m}{p} \right)^2 + \left(\frac{n}{q} \right)^2 \right], \tag{11}$$

$$\Phi_{imn}(t) = \iint\limits_D F_i(x,y,t) v_{mn}(x,y)\, dxdy, \quad i = 1, 2. \tag{12}$$

Note that in what follows, we will assume in (11), that $b = \mu^2 \geq 0$ ($\mu \geq 0$). Since if $b < 0$, then starting from some numbers $n > n_0$ or $m > m_0$ the right-hand side of (11) takes only positive values, i.e. the sign of the coefficient b does not affect the results obtained.

Differential equations (9) and (10) have common solutions

$$u_{mn}(t) = \begin{cases} a_{mn} e^{-\lambda^2_{mn} t^{l+1}/(l+1)} + \\ \qquad + \int\limits_0^t \Phi_{1mn}(s) e^{-\lambda^2_{mn}\left(t^{l+1}/(l+1) - s^{l+1}/(l+1)\right)}\, ds, \quad t > 0, \\ c_{mn} \cos \lambda_{mn} t + d_{mn} \sin \lambda_{mn} t - \\ \qquad - \dfrac{1}{\lambda_{mn}} \int\limits_t^0 \Phi_{2mn}(s) \sin\left[\lambda_{mn}(t-s)\right]\, ds, \quad t < 0. \end{cases} \tag{13}$$

The functions (13) satisfy conjugation conditions

$$u_{mn}(0+0) = u_{mn}(0-0), \quad u'_{mn}(0+0) = u'_{mn}(0-0), \quad m, n \in \mathbb{N}.$$

only if

$$a_{mn} = c_{mn}, \quad d_{mn} = \frac{1}{\lambda_{mn}} \Phi_{1mn}(0+0).$$

By virtue of the last equalities, the functions $u_{mn}(t)$ take the form

$$u_{mn}(t)=\begin{cases} a_{mn}e^{-\lambda_{mn}^2 t^{l+1}/(l+1)}+ \\ \qquad +\int\limits_0^t \Phi_{1mn}(s)e^{-\lambda_{mn}^2\left(t^{l+1}/(l+1)-s^{l+1}/(l+1)\right)}\,ds, \quad t>0, \\ a_{mn}\cos\lambda_{mn}t+\dfrac{\Phi_{1mn}(0+0)}{\lambda_{mn}}\sin\lambda_{mn}t- \\ \qquad -\dfrac{1}{\lambda_{mn}}\int\limits_t^0 \Phi_{2mn}(s)\sin\left[\lambda_{mn}(t-s)\right]\,ds, \quad t<0. \end{cases}$$

To find the constants a_{mn} in this formula, we use the boundary condition (6) and the formula (7):

$$u_{mn}(-\alpha)=\iint\limits_D u(x,y,-\alpha)v_{mn}(x,y)\,dxdy=\iint\limits_D \psi(x,y)v_{mn}(x,y)\,dxdy=\psi_{mn}. \tag{14}$$

Then we have

$$a_{mn}\cos\lambda_{mn}\alpha=\psi_{mn}+\frac{\Phi_{1mn}(0+0)}{\lambda_{mn}}\sin\lambda_{mn}\alpha-$$
$$-\frac{1}{\lambda_{mn}}\int\limits_{-\alpha}^0 \Phi_{2mn}(s)\sin\left[\lambda_{mn}(s+\alpha)\right]\,ds=w_{mn}. \tag{15}$$

From the equality (15) provided that for all m, $n\in\mathbb{N}$

$$\Delta_{mn}(\alpha)=\cos\lambda_{mn}\alpha\neq 0, \tag{16}$$

we find

$$a_{mn}=\frac{w_{mn}}{\Delta_{mn}(\alpha)}.$$

Hence, for the functions $u_{mn}(t)$ we obtain

$$u_{mn}(t)=\begin{cases} \dfrac{w_{mn}}{\cos\lambda_{mn}\alpha}e^{-\lambda_{mn}^2 t^{l+1}/(l+1)}+ \\ \qquad +\int\limits_0^t \Phi_{1mn}(s)e^{-\lambda_{mn}^2\left(t^{l+1}/(l+1)-s^{l+1}/(l+1)\right)}\,ds, \quad t>0, \\ \dfrac{w_{mn}}{\cos\lambda_{mn}\alpha}\cos\lambda_{mn}t+\dfrac{\Phi_{1mn}(0+0)}{\lambda_{mn}}\sin\lambda_{mn}t- \\ \qquad -\dfrac{1}{\lambda_{mn}}\int\limits_t^0 \Phi_{2mn}(s)\sin\left[\lambda_{mn}(t-s)\right]\,ds, \quad t<0. \end{cases} \tag{17}$$

Let $F_i(x,y,t) \equiv 0$, $i = 1,2$, $\psi(x,y) \equiv 0$ and the conditions (16), that is, $\Delta_{mn}(\alpha) \neq 0$ hold for all m, $n \in \mathbb{N}$. Then all $\psi_{mn} \equiv 0$, $\Phi_{imn}(t) \equiv 0$ and by virtue of (17) and (7) it follows that for all m, $n \in \mathbb{N}$ and any $t \in [-\alpha, \beta]$

$$\iint\limits_D u(x,y,t) v_{mn}(x,y)\, dxdy = 0.$$

Hence, since the system of functions (8) in $L_2(D)$ is complete, it follows that $u(x,y,t) = 0$ almost everywhere in $\overline{D}$ for any $t \in [-\alpha, \beta]$. By the condition (2), the function $u(x,y,t)$ is continuous on $\overline{D}$, therefore $u(x,y,t) \equiv 0$ in $\overline{D}$.

Let for some $m = m_0$ or $n = n_0$ the expression $\Delta_{m_0 n}(\alpha) = 0$ or $\Delta_{mn_0}(\alpha) = 0$. Let's assume that $\Delta_{mn_0}(\alpha) = 0$. Then the homogeneous problem (2) – (6) (where $\psi(x,y) \equiv 0$, $F_i(x,y,t) \equiv 0$, $i = 1,2$) has a nonzero solution

$$u_{mn_0}(x,y,t) = u_{mn_0}(t) v_{mn_0}(x,y),$$

where

$$u_{mn_0}(t) = \begin{cases} C_{mn_0} e^{-\lambda_{mn_0}^2 t^{l+1}/(l+1)}, & t \geq 0, \\ C_{mn_0} \cos \lambda_{mn_0} t, & t \leq 0, \end{cases}$$

and $C_{mn_0} \neq 0$ is an arbitrary constant.

From the formula (16) it is seen that $\Delta_{mn}(\alpha) = 0$ with respect to α only if

$$\alpha = \frac{\pi(1+2k)}{2\lambda_{mn}}, \quad k,\ m,\ n \in \mathbb{N}.$$

Thus, we have established the following criterion for the uniqueness of the solution to the problem (2) – (6).

Theorem 1. *If there is a solution to the problem* (2) – (6)*, then it is unique only if for all m and n the relation $\Delta_{mn}(\alpha) \neq 0$ holds.*

3. Existence of a solution to the problem

Under the conditions $\Delta_{mn}(\alpha) \neq 0$, the solution to the problem (2) – (6) is formally defined by the Fourier series with respect to the system of functions (8):

$$u(x,y,t) = \sum_{m,n=1}^{\infty} u_{mn}(t) v_{mn}(x,y), \tag{18}$$

where the coefficients are found from the formula (17). Since $\Delta_{mn}(\alpha)$ is the denominator of the coefficients of the series (18) and, as shown above, the equation

$\Delta_{mn}(\alpha) = 0$ has a countable set of zeros, the problem of small denominators is of more complex structure than in the flat case [5], [6, p. 61–66]. Therefore, to justify the convergence of the series (18) in the class of functions (2), it is necessary to establish an estimate of separation from zero for the expression $\Delta_{mn}(\alpha)$.

In what follows, for simplicity of research, we assume that $b = 0$. With this in mind, we represent the expression $\Delta_{mn}(\alpha)$ in the form

$$\Delta_{mn}(\nu) = \cos(\pi N \nu), \quad N = \max\{m, n\}, \tag{19}$$

where

$$\nu = \frac{\alpha}{pq}\sqrt{\left(\frac{q\,m}{N}\right)^2 + \left(\frac{p\,n}{N}\right)^2}.$$

Since ν depends on n and m, the question arises: what should the data of the problem α, p and q be in order for ν to take only rational or only irrational algebraic values? Let the ratio q/p be rational. In this case, without loss of generality, we can assume that p and q are integers and they are coprime. Then the formula for ν can be rewritten as

$$\nu = \begin{cases} \dfrac{\alpha}{q}\sqrt{1 + \left(\dfrac{q\,m}{p\,n}\right)^2} = \dfrac{\alpha}{q}\nu_+ & \text{for} \quad n > m, \\ \dfrac{\alpha}{q}\sqrt{1 + \left(\dfrac{q}{p}\right)^2} = \dfrac{\alpha}{q}\nu_0 & \text{for} \quad n = m, \\ \dfrac{\alpha}{q}\sqrt{\left(\dfrac{q}{p}\right)^2 + \left(\dfrac{n}{m}\right)^2} = \dfrac{\alpha}{q}\nu_- & \text{for} \quad n < m. \end{cases}$$

By virtue of Theorem 310 [23, p. 308] the square roots of ν_+ and ν_- take rational values only if

$$m = (a^2 - b^2)p, \quad n = 2abq \quad \text{for} \quad \nu_+,$$
$$n = (a^2 - b^2)q, \quad m = 2abp \quad \text{for} \quad \nu_-,$$

where $a > b > 0$, $(a, b) = 1$, and of the natural numbers a and b one of them is even, and the other is odd. The root ν_0 takes rational values only if

$$q = (a^2 - b^2), \quad p = 2ab.$$

It follows from these statements that there are countable sets of values n, m, p and q for which the roots ν_+, ν_- and ν_0 take only rational values. If these conditions are violated, then the roots ν_+, ν_- and ν_0 are algebraic numbers of degree 2. Therefore, if the ratio α/q is a rational number, then ν takes rational values under conditions for n, m, p and q, and when these conditions are violated it takes irrational algebraic values.

Lemma 1. *If $\nu = s/t$ is an arbitrary rational number, where s, $t \in \mathbb{N}$, $(s,t) = 1$, and t is an odd number, then there exists a positive constant C_0 such that for all $N \in \mathbb{N}$ the estimate*

$$|\Delta_{mn}(\nu)| \geq C_0 > 0 \tag{20}$$

holds.

Proof. Let $\nu = s \in \mathbb{N}$. Then from the equality (19) we have

$$\Delta_{mn}(\nu) = \cos(\pi N p) = (-1)^{pN}.$$

Hence it follows that

$$|\Delta_{mn}(\nu)| = C_0 = 1 > 0. \tag{21}$$

Let now $\nu = s/t$, $(s,t) = 1$. Dividing Ns by t, we have $Ns = kt + r$, k, $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leqslant r < t$. Then

$$\Delta(k) = \cos\frac{\pi N s}{t} = \cos\left(\pi k + \frac{r\pi}{t}\right) = (-1)^k \cos\frac{\pi r}{t}. \tag{22}$$

If $r = 0$, then we get the estimate (21). Let $r > 0$, then $1 \leqslant r \leqslant t-1$, $q \geqslant 2$, and the inequality

$$\frac{\pi}{t} \leqslant \frac{\pi r}{t} \leqslant \frac{(t-1)\pi}{t} = \pi - \frac{\pi}{t}$$

holds. If $r/t \neq 1/2$ for $r \in [1, t-1] \cap \mathbb{N}$ and $t \geqslant 2$, then

$$\cos\frac{\pi r}{t} \neq 0. \tag{23}$$

Indeed, if for some $r = r_0 \in [1, t-1] \cap \mathbb{N}$ the inequality (23) is violated, that is

$$\cos\frac{\pi r_0}{t} = 0 \Leftrightarrow \frac{\pi r_0}{t} = \frac{\pi}{2} + \pi m \Leftrightarrow \frac{r_0}{t} = \frac{1}{2} + m,\ m \in \mathbb{N}_0.$$

Since $r_0 < t$, the last equality is possible only for $m = 0$, therefore, we obtain $t = 2r_0$, which contradicts the condition, so the inequality (23) is true. Then from (22) and (23) the validity of the estimate (20) follows. ◄

Lemma 2. *Let ν be an irrational algebraic number of degree 2. Then there exists a positive constant C_0 such that for all $N \in \mathbb{N}$ the estimate*

$$|\Delta_{mn}(\nu)| > \frac{C_0}{N} \tag{24}$$

holds.

Proof. We represent the relation (19) in the form

$$|\Delta_{mn}(\nu)| = |\cos(\pi N\nu)| = \left|\sin\left(\pi N\nu + \frac{\pi}{2}\right)\right| = \left|\sin\left(\pi N\nu + \frac{\pi}{2} - \pi m\right)\right| =$$
$$= \left|\sin\left[\pi N\left(\nu - \frac{2m-1}{2N}\right)\right]\right| = \left|\sin\left[\frac{\pi N}{2}\left(2\nu - \frac{l}{N}\right)\right]\right|,$$

where $l = 2m - 1$, $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$.

Note that for every $N \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that

$$\left|2\nu - \frac{l}{N}\right| < \frac{1}{N}. \tag{25}$$

For this, it is enough to take

$$l = \begin{cases} [2\nu N], & \text{if } [2\nu N] \text{ is an odd number,} \\ [\nu N] + 1, & \text{if } [2\nu N] \text{ is an even number,} \end{cases}$$

where $[\nu N]$ is the integer part of the irrational number $2\nu N$.

We take the number l such that, by virtue of the inequality (25), the inequality

$$\left|\frac{\pi N}{2}\left(2\nu - \frac{l}{N}\right)\right| < \frac{\pi}{2} \tag{26}$$

holds. If 2ν is an irrational algebraic number of degree two, i.e. is a quadratic irrational number, then, by Liouville's theorem [24, p. 60], there exists a positive number δ depending on ν such that for any integer l and k, $k > 0$, inequality

$$\left|2\nu - \frac{l}{N}\right| > \frac{\delta}{N^2} \tag{27}$$

holds. The set of values of ν for any values of n and m is bounded, i.e.

$$\frac{\alpha}{T} < \nu < \frac{\alpha}{pq}\sqrt{p^2 + q^2}, \quad T = \max\{p, q\}.$$

Then, based on [6, p. 66], it can be shown that the set of values of δ is bounded, while $\delta > \delta_0 > 0$, where δ_0 is independent of N, that is, n and m.

Then the expression for $\Delta_{mn}(\nu)$ takes the form

$$|\Delta_{mn}(\nu)| = |\cos\pi N\nu| = \left|\sin\frac{\pi N}{2}\left(2\nu - \frac{l}{N}\right)\right|. \tag{28}$$

Based on (28), in view of (27), we have an estimate

$$|\Delta_{mn}(\nu)| = |\cos\pi N\nu| = \left|\sin\frac{\pi N}{2}\left(2\nu - \frac{l}{N}\right)\right| > N\left|2\nu - \frac{l}{N}\right| > \frac{\delta_0}{N},$$

which implies the validity of the lemma. ◄

Lemma 3. *If the number ν is an algebraic number of degree 1 or 2, then for all $N \in \mathbb{N}$ the following estimates hold:*

$$|u_{mn}(t)| \le M_1 \Big[N|\psi_{mn}| + \|\Phi_{1mn}\|_C + \|\Phi_{2mn}\|_C \Big], \;\; t \in [-\alpha, \beta],$$

$$|u'_{mn}(t)| \le M_2 \Big[N^2|\psi_{mn}| + N\|\Phi_{1mn}\|_C + N\|\Phi_{2mn}\|_C \Big], \;\; t \in [-\alpha, \beta],$$

$$|u''_{mn}(t)| \le M_3 \Big[N^3|\psi_{mn}| + N^2\|\Phi_{1mn}\|_C + N^2\|\Phi_{2mn}\|_C \Big], \;\; t \in [-\alpha, 0],$$

where

$$\|\Phi_{1mn}\|_C = \max_{0 \le t \le \beta} |\Phi_{1mn}(t)|, \quad \|\Phi_{2mn}\|_C = \max_{-\alpha \le t \le 0} |\Phi_{2mn}(t)|,$$

M_i are hereinafter positive constants.

Proof. Let's first estimate the expression w_{mn} appearing in the formula (15):

$$|w_{mn}| \le |\psi_{mn}| + \frac{|\Phi_{1mn}(0+0)|}{\lambda_{mn}}| + \frac{1}{\lambda_{mn}} \left| \int_{-\alpha}^{0} \Phi_{2mn}(s) \sin\left[\lambda_{mn}(s+\alpha)\right] \, ds \right| \le$$

$$\le \widetilde{M_1} \Big(|\psi_{mn}| + N^{-1}\|\Phi_{1mn}\|_C + N^{-1}\|\Phi_{2mn}\|_C \Big). \tag{29}$$

Here and below $\widetilde{M_i}$ are positive constants depending on p, q, b, α and β.

Using the formula (17) for $t \ge 0$, taking into account Lemma 2 and (29), we estimate the expression $u_{mn}(t)$ for $t \in [0, \beta]$:

$$|u_{mn}(t)| \le \frac{|w_{mn}|}{|\Delta_{mn}(\alpha)|} e^{-\lambda_{mn}^2 t^{l+1}/(l+1)} + \left| \int_0^t \Phi_{1mn}(s) e^{-\lambda_{mn}^2 \left(t^{l+1}/(l+1) - s^{l+1}/(l+1)\right)} \, ds \right| \le$$

$$\le \widetilde{M_2} \Big(N|\psi_{mn}| + \|\Phi_{1mn}\|_C + \|\Phi_{2mn}\|_C \Big) + \frac{1}{\lambda_{mn}^2} \max_{0 \le t \le \beta} |\Phi_{1mn}(t)| \le$$

$$\le \widetilde{M_2} \Big(N|\psi_{mn}| + \|\Phi_{1mn}\|_C + \|\Phi_{2mn}\|_C \Big). \tag{30}$$

Similarly, we obtain an estimate for $u'_{mn}(t)$ when $t \in [0, \beta]$:

$$|u'_{mn}(t)| \le \frac{|w_{mn}|}{|\Delta_{mn}\alpha|} \lambda_{mn}^2 e^{-\lambda_{mn}^2 t^{n+1}/(n+1)} + |\Phi_{1mn}(t)| \le$$

$$\le \widetilde{M_3} \Big(N^2|\psi_{mn}| + N\|\Phi_{1mn}\|_C + N\|\Phi_{2mn}\|_C \Big). \tag{31}$$

Now, based on the formula (17), evaluate the expression $u_{mn}(t)$ for $t \le 0$. Using Lemma 2 and the estimate (29), we obtain

$$|u_{mn}(t)| \le \widetilde{M}_4 \left(\frac{|w_{mn}|}{|\Delta_{mn}(\alpha)|} + \frac{|\Phi_{1mn}(0+0)|}{\lambda_{mn}} \right) +$$

$$+\frac{1}{\lambda_{mn}} \left| \int\limits_t^0 \Phi_{2mn}(s) \sin\left[\lambda_{mn}(t-s)\right] \, ds \right| \le$$

$$\le \widetilde{M}_5 \Big(N|\psi_{mn}| + \|\Phi_{1mn}\|_C + \|\Phi_{2mn}\|_C \Big); \tag{32}$$

$$|u'_{mn}(t)| \le \widetilde{M}_6 \left(\frac{|w_{mn}|}{|\Delta_{mn}(\alpha)|} \lambda_{mn} + |\Phi_{1mn}(0+0)| + \frac{1}{\lambda_{mn}} |\Phi_{1mn}(t)| \right) \le$$

$$\le \widetilde{M}_7 \Big(N^2|\psi_{mn}| + N\|\Phi_{1mn}\|_C + N\|\Phi_{2mn}\|_C \Big). \tag{33}$$

From the inequalities (30) and (32) we obtain the first estimate of the lemma, and from (31) and (33) the second one.

Since $u''_{mn}(t) = -\lambda^2_{mn} u_{mn}(t) + \Phi_{2mn}(t)$, based on the estimate (32) we obtain the third estimate of the lemma. ◀

Let the conditions of Lemma 3 be satisfied. The series (18) and the series consisting of the derivatives of this series found above, by virtue of Lemma 3, for any points $(x, y, t) \in \overline{Q}$ are majorized by the numerical series

$$M_4 \sum_{N=1}^{+\infty} N^3|\psi_{mn}| + N^2\|\Phi_{1mn}\|_C + N^2\|\Phi_{2mn}\|_C. \tag{34}$$

Lemma 4. *Let* $\psi(x, y) \in C^{5+h_1}(\overline{D})$, $0 < h_1 < 1$,

$$\psi(0, y) = \psi_{xx}(0, y) = \psi_{xxxx}(0, y) =$$

$$= \psi(p, y) = \psi_{xx}(p, y) = \psi_{xxxx}(p, y) = 0, \quad 0 \le y \le q,$$

$$\psi(x, 0) = \psi_{yy}(x, 0) = \psi_{yyyy}(x, 0) =$$

$$= \psi(x, q) = \psi_{yy}(x, q) = \psi_{yyyy}(x, q) = 0 \quad 0 \le x \le p,$$

$F_1(x, y, t) \in C(\overline{Q}_+) \cap C^{4+h_2}_{xy}(\overline{D}_+)$, $0 < h_2 < 1$,

$$F_1(0, y, t) = F_{1xx}(0, y, t) = F_{1xxxx}(0, y, t) = F_1(p, y, t) =$$

$$= F_{1xx}(p, y, t) = F_{1xxxx}(p, y, t) = 0, \quad 0 \le y \le q, \quad 0 \le t \le \beta,$$

$$F_1(x,0,t) = F_{1yy}(x,0,t) = F_{1yyyy}(x,0,t) = F_1(x,q,t) =$$

$$= F_{1yy}(x,q,t) = F_{1yyyy}(x,q,t) = 0, \quad 0 \le x \le q, \quad -\alpha \le t \le 0,$$

$F_2(x,y,t) \in C(\overline{Q}_-) \cap C_{xy}^{4+h_3}(\overline{D}_-)$, $\varepsilon < h_3 < 1$,

$$F_2(0,y,t) = F_{2xx}(0,y,t) = F_{2xxxx}(0,y,t) = F_2(p,y,t) =$$

$$= F_{2xx}(p,y,t) = F_{2xxxx}(p,y,t) = 0, \quad 0 \le y \le q, \quad 0 \le t \le \beta,$$

$$F_2(x,0,t) = F_{2yy}(x,0,t) = F_{2yyyy}(x,0,t) = F_2(x,q,t) =$$

$$= F_{2yy}(x,q,t) = F_{2yyyy}(x,q,t) = 0, \quad 0 \le x \le q, \quad -\alpha \le t \le 0.$$

Then the following estimates are valid:

$$|\psi_{mn}| \le \frac{M_5}{N^{5+h_1}}, \quad |\Phi_{1mn}(t)| < \frac{M_6}{N^{4+h_2}}, \quad |\Phi_{2mn}(t)| < \frac{M_7}{N^{4+h_3}}.$$

Proof. To prove the estimates of the lemma, it suffices to carry out integration by parts in the formulas (14) and (12), taking into account the conditions of the lemma four and three times, respectively, and then use the Holder property of the fourth and third order derivatives [25, p. 81]. ◄

In the series (34), the number of terms with a given N is of order N. Then, by Lemma 4, the series (34) is estimated by the convergent series

$$M_8 \sum_{N=1}^{\infty} \left(\frac{1}{N^{1+h_1}} + \frac{1}{N^{1+h_2}} + \frac{1}{N^{1+h_3}} \right).$$

Theorem 2. *Let the conditions of Lemma 1 or Lemma 2 be satisfied and the functions $\psi(x,y)$, $F_1(x,y,t)$ and $F_2(x,y,t)$ satisfy the conditions of Lemma 4. Then there is a unique solution to the problem* (2)-(6) *and it is defined by* (18).

4. Stability of solution

Consider the following norms:

$$\|u(x,y,t)\|_{L_2(D)} = \left(\iint_D u^2(x,y,t)\, dxdy \right)^{1/2},$$

$$\|u(x,y,t)\|_{C(\overline{Q})} = \max_{\overline{Q}} |u(x,y,t)|,$$

$$\|f(x,y)\|_{W_2^l(D)} = \left(\iint\limits_D \sum_{i+j=0}^{k} \left| \frac{\partial^{(i+j)} f(x,y)}{\partial x^i \partial y^j} \right| \right)^{1/2},$$

$$\|f(x,y)\|_{C^l(\overline{D})} = \sum_{i+j=0}^{k} \max_{\overline{D}} \left| \frac{\partial^{(i+j)} f(x,y)}{\partial x^i \partial y^j} \right|, \quad k \geq 1.$$

Theorem 3. *Let the conditions of Theorem 2 be satisfied. Then the solution* (18) *of the problem* (2) – (6) *satisfies the estimates*

$$\|u(x,y,t)\|_{L_2(D)} \leq M_9 \Big(\|\psi(x,y)\|_{W_2^1(D)} + \|F_1(x,y,t)\|_{L_2(D)} + \|F_2(x,y,t)\|_{L_2(D)} \Big),$$

$$\|u(x,y,t)\|_{C(\overline{Q})} \leq M_{10} \Big(\|\psi(x,y)\|_{C^3(D)} + \|F_1(x,y,t)\|_{C^2(D)} + \|F_2(x,y,t)\|_{C^2(D)} \Big).$$

Proof. Since the system of functions (8) is orthonormal in $L_2(D)$, from the formula (18), by Lemma 3, we have

$$\|u(x,y,t)\|^2_{L_2(D)} = \sum_{m,n=1}^{\infty} u^2_{mn}(t) = \sum_{n \geq m} u^2_{mn}(t) + \sum_{m > n} u^2_{mn}(t) \leq$$

$$\leq 2M_1^2 \sum_{n \geq m} n^2 \psi^2_{mn} + 2M_1^2 \sum_{m > n} m^2 \psi^2_{mn} + 2M_1^2 \sum_{m,n=1}^{\infty} \Big(|\Phi_{1mn}(t)|^2 + |\Phi_{2mn}(t)|^2 \Big). \tag{35}$$

Using

$$\psi_{mn} = \left(\frac{q}{n\pi}\right)^2 \psi^{(0,2)}_{mn} = \left(\frac{p}{m\pi}\right)^2 \psi^{(2,0)}_{mn},$$

where

$$\psi^{(0,1)}_{mn} = \frac{2}{\sqrt{p\,q}} \iint\limits_D \varphi_y(x,y) \sin\frac{m\pi x}{p} \cos\frac{n\pi y}{q}\, dxdy,$$

$$\psi^{(1,0)}_{mn} = \frac{2}{\sqrt{p\,q}} \iint\limits_D \varphi_x(x,y) \cos\frac{m\pi x}{p} \sin\frac{n\pi y}{q}\, dxdy,$$

from the inequality (35), we obtain

$$\|u(x,y,t)\|^2_{L_2(D)} \leq 2M_1^2 \Bigg[\left(\frac{q}{\pi}\right)^2 \sum_{n \geq m} |\psi^{(0,1)}_{mn}|^2 + \left(\frac{p}{\pi}\right)^2 \sum_{m > n} |\psi^{(1,0)}_{mn}|^2 +$$

$$+ \|F_1(x,y,t)\|^2_{L_2(D)} + \|F_2(x,y,t)\|^2_{L_2(D)} \Bigg] \leq$$

$$\leq M_9^2 \left[\sum_{m,n=1}^{\infty} \left(|\psi_{mn}^{(0,1)}|^2 + |\psi_{mn}^{(1,0)}|^2 \right) + \|F_1(x,y,t)\|^2_{L_2(D)} + \|F_2(x,y,t)\|^2_{L_2(D)} \right] \leq$$

$$\leq M_9^2 \left(\|\psi(x,y)\|^2_{W_2^1(D)} + \|F_1(x,y,t)\|^2_{L_2(D)} + \|F_2(x,y,t)\|^2_{L_2(D)} \right).$$

This implies the validity of the first estimate of the theorem.

Let (x,y,t) be an arbitrary point in $\overline{Q}$. Then from the formula (18) by Lemma 3 we have

$$|u(x,y,t)| \leq \sum_{m,n=1}^{\infty} |u_{mn}(t)| + |\Phi_{1mn}(t)| + |\Phi_{2mn}(t)| \leq$$

$$\leq \sum_{n \geq m} |u_{mn}(t)| + \sum_{m>n} |u_{mn}(t)| + \sum_{m,n=1}^{\infty} \Big(|\Phi_{1mn}(t)| + |\Phi_{2mn}(t)| \Big) \leq$$

$$\leq M_1 \sum_{n \geq m} n|\psi_{mn}| + M_1 \sum_{m>n} m\psi_{mn} + M_1 \sum_{m,n=1}^{\infty} \Big(|\Phi_{1mn}(t)| + |\Phi_{2mn}(t)| \Big), \quad (36)$$

Next, we use the equalities

$$\psi_{mn} = -\frac{q}{n\pi} \left(\frac{p}{m\pi} \right)^2 \psi_{mn}^{(2,1)} = -\frac{p}{m\pi} \left(\frac{q}{n\pi} \right)^2 \psi_{mn}^{(1,2)},$$

$$\Phi_{imn} = \frac{qp}{mn\pi^2} \Phi_{imn}^{(1,1)}, \quad i = 1, 2,$$

where

$$\psi_{mn}^{(2,1)} = \frac{2}{\sqrt{p\,q}} \iint_D \varphi_{xxy}(x,y) \sin \frac{m\pi x}{p} \cos \frac{n\pi y}{q} \, dxdy,$$

$$\psi_{mn}^{(1,2)} = \frac{2}{\sqrt{p\,q}} \iint_D \varphi_{xyy}(x,y) \cos \frac{m\pi x}{p} \sin \frac{n\pi y}{q} \, dxdy,$$

$$\Phi_{imn}^{(1,1)} = \frac{2}{\sqrt{p\,q}} \iint_D F_{ixy}(x,y,t) \cos \frac{m\pi x}{p} \cos \frac{n\pi y}{q} \, dxdy,$$

Then, continuing the estimate (36), we have

$$|u(x,y,t)| \leq M_1 \sum_{n \geq m} \frac{q^2 p}{\pi^3} \frac{1}{mn} |\psi_{mn}^{(1,2)}| + M_1 \sum_{m>n} \frac{p^2 q}{\pi^3} \frac{1}{mn} |\psi_{mn}^{(2,1)}| +$$

$$+ M_1 \sum_{m,n=1}^{\infty} \frac{qp}{\pi^2} \frac{1}{mn} \Big(|\Phi_{1mn}^{(1,1)}| + |\Phi_{2mn}^{(1,1)}| \Big).$$

Hence, by virtue of Bunyakovsky inequality, we obtain

$$|u(x,y,t)| \le \widetilde{M}_8 \sum_{m,n=1}^{\infty} \frac{1}{mn}\left(|\psi_{mn}^{(1,2)}| + |\psi_{mn}^{(2,1)}|\right) +$$

$$+\widetilde{M}_9 \sum_{m,n=1}^{\infty} \frac{1}{mn}|\Phi_{1mn}^{(1,1)}| + \widetilde{M}_9 \sum_{m,n=1}^{\infty} \frac{1}{mn}|\Phi_{2mn}^{(1,1)}| \le$$

$$\le \widetilde{M}_8 \left(\sum_{m,n=1}^{\infty}\left(\frac{1}{mn}\right)^2\right)^{1/2} \left[\left(\sum_{m,n=1}^{\infty}|\psi_{mn}^{(1,2)}|^2\right)^{1/2} + \left(\sum_{m,n=1}^{\infty}|\psi_{mn}^{(2,1)}|^2\right)^{1/2}\right] +$$

$$+\widetilde{M}_9 \left(\sum_{m,n=1}^{\infty}\left(\frac{1}{mn}\right)^2\right)^{1/2} \left(\sum_{m,n=1}^{\infty}|\Phi_{1mn}^{(1,1)}|^2\right)^{1/2} +$$

$$+\widetilde{M}_9 \left(\sum_{m,n=1}^{\infty}\left(\frac{1}{mn}\right)^2\right)^{1/2} \left(\sum_{m,n=1}^{\infty}|\Phi_{2mn}^{(1,1)}|^2\right)^{1/2} \le$$

$$\le \widetilde{M}_{10}\left(\|\psi(x,y)\|_{W_2^3(D)} + \|F_1(x,y,t)\|_{W_2^2(D)} + \|F_2(x,y,t)\|_{W_2^2(D)}\right) \le$$

$$\le M_{10}\left(\|\psi(x,y)\|_{C^3(D)} + \|F_1(x,y,t)\|_{C^2(D)} + \|F_2(x,y,t)\|_{C^2(D)}\right).$$

From the last inequality we have the second estimate of the theorem. ◄

5. Funding

This work was funded by Russian Foundation for Basic Research, project No. 19-31-60016.

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Received 15 April 2021
Accepted 14 May 2021