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On a New Method for Investigation of Multiperiodic Solutions of Quasilinear Strictly Hyperbolic System

A.Kh. Zhumagaziyev*, Zh.A. Sartabanov, Ya.T. Sultanaev

Abstract. A method for investigating the problem of the existence and construction of a multiperiodic solution of the quasilinear strictly hyperbolic systems is proposed.

Key Words and Phrases: strictly hyperbolic system, multiperiodic solution, method of characteristics, projection operator, differentiation operators by vector fields.

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1. Introduction

We consider the quasilinear system of equations

$$Dx = Bx + f(\tau, t, x), \tag{1}$$

for $x = (x_1, ..., x_n)$ with a vector and matrix differentiation operator D

$$Dx \equiv \frac{\partial x}{\partial \tau} + \sum_{k=1}^{m} A_k \frac{\partial x}{\partial t_k}, \tag{2}$$

where A_k and B are constant $n \times n$-matrices; $f(\tau, t, x) = (f_1(\tau, t, x), ..., f_n(\tau, t, x))$ is a vector-function of independent variables $\tau \in \mathbb{R}$, $t = (t_1, ...t_m) \in \mathbb{R}^m$; $x \in \mathbb{R}^n_\Delta = \{x \in \mathbb{R}^n : |x| = \max\limits_{j=\overline{1,n}} |x_j| \leq \Delta\}$ is a required vector-function.

Assume that each of the matrices A_k has different real nonzero eigenvalues

$$\lambda_{kj} = \lambda_j(A_k) \neq 0, \quad \lambda_{ki} \neq \lambda_{kj}, \quad i \neq j, \quad \lambda_{kj} \in \mathbb{R}, \quad i, j = \overline{1, n}. \tag{3}$$

*Corresponding author.

Under condition (3), equation (1) is of hyperbolic type [1, p. 59], [2, p. 9].

The vector-function $f(\tau, t, x)$ has the properties of (θ, ω)-periodicity in (τ, t), continuity and smoothness on $(\tau, t, x) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n_\Delta$ of order $(0, e, \hat{e})$:

$$f(\tau + \theta, t + \omega, x) = f(\tau, t, x) \in C^{(0,e,\hat{e})}_{\tau,t,x}(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n_\Delta),$$

with rationally incommensurable periods θ, $\omega_1, ..., \omega_m$, $\omega = (\omega_1, ..., \omega_m)$, where e and $\hat{e}$ are vectors with unit components of dimensions m and n, respectively.

Our problem is stated as follows: establish the existence conditions for the solutions of initial value problem and the multiperiodic solutions of equation (1)-(2) which satisfy the conditions

$$x(\tau, t)|_{\tau=\tau_0} = x^0(t + \omega) = x^0(t) \in C^{(e)}_t(\mathbb{R}^m), \tag{4}$$

$$x(\tau + \theta, t + \omega) = x(\tau, t) \in C^{(1,e)}_{\tau,t}(\mathbb{R} \times \mathbb{R}^m),\ |x| \le \Delta, \tag{5}$$

respectively.

Nonlinear Euler equations from hydromechanics, which are called equations of fluid dynamics [3, p. 214], can be reduced to the form (1)-(2).

Quasilinear analogs of these equations describe various wave processes related to environments of noninteracting particles that belong to different types of systems of equations, in particular, to the strictly hyperbolic one.

When studying systems of such types, passing from the field of a continuous medium to particles, in many cases it is possible to find their close connection with systems of ordinary equations. The method of reduction of this nature is the art of solving problems for systems of partial differential equations. This principle is followed by the majority of researchers in this field. In this regard, as an example, we can note the approach of the study in the book [4], which is based on the hodograph method.

Studies of strongly hyperbolic systems, as noted above, often find application in applied problems, which is of some interest. Systems of this kind can also be encountered in theoretical developments. For the sake of fairness, we note that in [5] these systems appear in the study of problems about manifolds.

The problem under consideration is related to wave processes, therefore, they are described by globally defined solutions [6], which have oscillatory properties, in particular, periodic and multiperiodic [7, 8].

Also note that the study of waves that are almost periodic with respect to time leads to the study of multiperiodic solutions of such systems.

The research of a problem of this nature, as a rule [1, 2, 3], begins with the reduction to the canonical form of the differential operators of systems.

For this purpose, we reduce the equation (1)-(2) to the form

$$D^* y = Cy + \varphi(\tau, t, y) \tag{6}$$

with differentiation operator

$$D^* \equiv \frac{\partial}{\partial \tau} + \sum_{k=1}^{m} J_k \frac{\partial}{\partial t_k} \tag{7}$$

on the basis of condition (3) and the linear nonsingular replacement

$$x = Ky, \tag{8}$$

assuming

$$J_k = K^{-1} A_k K, \;\; C = K^{-1} B K, \;\; \varphi(\tau, t, y) = K^{-1} f(\tau, t, Ky).$$

Then the initial condition (4) and the multiperiodicity condition (5), by virtue of the substitution (8), become

$$y(\tau, t)|_{\tau=\tau_0} = K^{-1} x^0(t) = y^0(t) \in C_t^{(e)}(\mathbb{R}^m), \tag{9}$$

$$y(\tau + \theta, t + q\omega) = y(\tau, t) \in C_{\tau,t}^{(1,e)}(\mathbb{R} \times \mathbb{R}^m), \; |y| \le \Delta^*, \tag{10}$$

where $\Delta^* = \Delta / |K|$, $|K| = \max\limits_{j=\overline{1,n}} \sum\limits_{j=1}^{n} |k_{ij}|$.

Thus, the main problem of this paper is to develop the methods for establishing the existence of and constructing the solutions for the initial value problem and the problem (6)-(7), with to the conditions (9) and (10).

The methods of the theory of multiperiodic solutions of systems of partial differential equations are of fundamental importance for our problem. This theory that dates back to [9, 10, 11] and was further developed in [12, 13, 14, 15] allows to make special generalizations for the systems of hyperbolic type like in [11, 16, 17] related to our work.

Note that in the field of problems of this nature, there are two types of problems:

1) The problem of reducing the system (1) with operator (2) to the system (6) with operator (7) when the number m of variables t_k is more than one (i.e. when there are many of them). In this case, this barrier is overcome by condition (3).

2) The problem of controlling the characteristics corresponding to the variables t_k and the operator D_j^*. In this regard, it should be noted that the condition

(3) simplifies, but does not fully solve the problem. In this simplified version, different differentiation operators D_j^* correspond to different equations. Therefore, when solving the problem, different coordinates y_j of the required vector solution y are integrated along different characteristics $t_{kj} = h_{kj}$, corresponding to the variables t_k and operators D_j^*. In other words, the same variable t_k corresponds to the n characteristics h_{kj}, and it becomes difficult to determine where and by what characteristic y_j should be integrated, when the latter is determined by others y_i's, containing the same variables t_k.

In general, this issue has not yet been studied and remained open. We have proposed a new method that allows us to completely solve this problem.

2. Research method

From the form (9) of the operator $D^* = (D_1^*, ..., D_n^*)$, it can be seen that each coordinate y_j of the required solution y is determined by $D_j^* y_j$, which has its own characteristics $t_{kj} = h_{kj}$ corresponding to the variable t_k along which solution y is considered. In order to ensure the consideration of y_j on the characteristics h_{kj} by variables t_k, it becomes necessary to 1) assign t_k an identical designation t_{kj} with respect to j, and 2) ensure the transition from t_{kj} to t_{ki} in the case where y_i is defined by y_j.

Thus, we introduce an operator $\Pi = diag\,[\Pi_1, ...\Pi_n]$ that acts on a vector-function $y(\tau, t) = [y_j(\tau, t_1, ...t_m)]$ as follows:

$$\Pi y(\tau, t_1, ...t_m) = [\Pi_j y_j(\tau, t_1, ...t_m)] = [y_j(\tau, t_{1j}, ...t_{mj})] = \\ = \left[y_j(\tau, \bar{t}_j)\right] = y(\tau, \bar{t}),$$

where $\bar{t} = (\bar{t}_1, ..., \bar{t}_n)$, $\bar{t}_j = (t_{1j}, ..., t_{mj})$. Obviously, the operator Π is invertible.

In the sequel, we consider a space $\mathbb{S}$ of vector and matrix functions of variables $(\tau, \bar{t})$, which in the case of a vector-function have the form $x(\tau, \bar{t}) = \left[x_i(\tau, \bar{t}_i)\right]$ and in the case of a matrix are written in the form $X(\tau, t) = \left[x_{ij}(\tau, \bar{t}_i)\right]$, where the i-th row of matrices is expressed by $(\tau, \bar{t}_i) \in \mathbb{R} \times \mathbb{R}^{mn}$.

Now we introduce operators $P_i = diag\,[p_{i1}, ..., p_{in}]$ with projectors p_{ij} for the transition from the j-th coordinate of variables $\bar{t}_j$ to the i-th coordinate of variables $\bar{t}_i$ of the functions from the space $\mathbb{S} : p_{ij}\bar{t}_j = \bar{t}_i$.

Further, we give the necessary rules for the action of operators P_i which are used in the product and composition of vector and matrix functions from $\mathbb{S}$.

(a) The projectors p_{ij} act on the scalar coordinate function $x_j(\tau, \bar{t}_j)$ as follows: $p_{ij} x_j(\tau, \bar{t}_j) = x_j(\tau, p_{ij}\bar{t}_j) = x_j(\tau, \bar{t}_i)$. The matrix projector P_i acts on the vector-function $x(\tau, \bar{t}) = \left[x_j(\tau, \bar{t}_j)\right]$ as follows:

$$P_i x(\tau, \bar{t}) = \left[p_{ij} x_j(\tau, \bar{t}_j)\right] = \left[x_j(\tau, p_{ij}\bar{t}_j)\right] = \left[x_j(\tau, \bar{t}_i)\right].$$

(b) We define the operation of the product of a matrix $X(\tau,\bar{t})$ and a vector-function $x_j(\tau,\bar{t}_j)$ by means of the projector P_i as follows:

$$y(\tau,\bar{t}) = X(\tau,\bar{t}) \times x(\tau,\bar{t}) = P_i\left[x_{ij}(\tau,\bar{t}_i)\right]\left[x_i(\tau,\bar{t}_i)\right] = \left[x_{ij}(\tau,\bar{t}_i)p_{ij}\right]\left[x_i(\tau,\bar{t}_i)\right] =$$
$$= \left[\sum_j x_{ij}(\tau,\bar{t}_i)p_{ij}x_j(\tau,\bar{t}_j)\right] = \left[\sum_j x_{ij}(\tau,\bar{t}_i)x_j(\tau,\bar{t}_i)\right] = \left[y_{ij}(\tau,\bar{t}_i)\right] \in \mathbb{S},$$

where the sign of the product "$\times$" means the preliminary supply of the elements of the matrix X by the projector P_i when it is produced with the vector-function x.

(c) The product $Z(\tau,\bar{t})$ of matrices $X(\tau,\bar{t})$ and $Y(\tau,\bar{t})$ is defined by the relation

$$Z(\tau,\bar{t}) = X(\tau,\bar{t}) \times Y(\tau,\bar{t}) = P_i\left[x_{ij}(\tau,\bar{t}_i)\right]\left[y_{ij}(\tau,\bar{t}_i)\right] =$$
$$= \left[\sum_{r=1}^{n} x_{ir}(\tau,\bar{t}_i)p_{ir}y_{rj}(\tau,\bar{t}_r)\right] = \left[\sum_{r=1}^{n} x_{ir}(\tau,\bar{t}_i)y_{rj}(\tau,\bar{t}_i)\right] = \left[z_{ij}(\tau,\bar{t}_i)\right] \in \mathbb{S}.$$

(d) A composition $\varphi = f \otimes x$ with projector P_i of vector-functions $f(\tau,\bar{t},x) = \left[f_i(\tau,\bar{t}_i,x)\right]$ and $x(\tau,\bar{t}) = \left[x_i(\tau,\bar{t}_i)\right]$ is defined by the relation

$$\varphi(\tau,\bar{t}) = (f \otimes x)(\tau,\bar{t}) = (P_i f \circ x)(\tau,\bar{t}) = \left[f_i(\tau,\bar{t}_i,P_i x(\tau,\bar{t}))\right] =$$
$$= \left[f_i(\tau,\bar{t}_i,P_i\left[x_j(\tau,\bar{t}_j)\right])\right] = \left[f_i(\tau,\bar{t}_i,\left[p_{ij}x_j(\tau,\bar{t}_j)\right])\right] = \left[f_i(\tau,\bar{t}_i,\left[x_j(\tau,p_{ij}\bar{t}_j)\right])\right] =$$
$$= \left[f_i(\tau,\bar{t}_i,\left[x_j(\tau,\bar{t}_i)\right])\right] = \left[\varphi_i(\tau,\bar{t}_i)\right] \in \mathbb{S},$$

where the sign of the composition "$\otimes$" means the preliminary supply of the elements of the matrix f by the projector P_i when composing it with the vector-function x.

Now we define vector and matrix functions from the space $\mathbb{S}$ along the characteristics $\bar{h}_i(\tau^0,\tau,\bar{t}_i))$ of differentiation operators D_i.

(e) The vector-function $x(\tau,\bar{t}) = \left[x_i(\tau,\bar{t}_i)\right]$ on the characteristic is defined by assignment of the form $(\tau,\bar{t}_i) := (\tau^0,\bar{h}_i(\tau^0,\tau,\bar{t}_i))$ and we have the relation

$$x(\tau^0,\bar{h}(\tau^0,\tau,\bar{t})) = \left[x_i(\tau^0,\bar{h}_i(\tau^0,\tau,\bar{t}_i))\right] \in \mathbb{S},$$

where $\bar{h}_i(\tau^0,\tau,\bar{t}_i) = (\bar{h}_{1i}(\tau^0,\tau,\bar{t}_{1i}),...,\bar{h}_{ni}(\tau^0,\tau,\bar{t}_{ni}))$.

(f) Similarly, along the characteristic, the matrix $X(\tau,\bar{t}) = \left[x_{ij}(\tau,\bar{t}_i)\right]$ is defined by the relation of the form

$$X(\tau^0,\bar{h}(\tau^0,\tau,\bar{t})) = \left[x_{ij}(\tau^0,\bar{h}_i(\tau^0,\tau,\bar{t}_i))\right] \in \mathbb{S}.$$

3. Main results

Now, based on the above concepts, from the problem with operator D in variables (τ, t), we pass to the study of the problem with operator D^* in variables $(\tau, \overline{t})$, in the process of which the essence of the proposed method of projectors is revealed. For this purpose, taking into account the commutativity of the operators D^* and Π, and using the operator Π from (6)-(7), we consider the problem for the equation

$$D^* y(\tau, \overline{t}) = C y(\tau, \overline{t}) + \varphi(\tau, \overline{t}, y(\tau, \overline{t}))$$

with the initial condition

$$y(\tau, \overline{t})|_{\tau=\tau^0} = y^0(\overline{t}).$$

We reduce this problem by means of the operator $P = diag\,[P_1, ..., P_n]$, to the problem

$$D^* y(\tau, \overline{t}) = C \times y(\tau, \overline{t}) + (\varphi \otimes y)(\tau, \overline{t}) \tag{11}$$

with the conditions

$$y(\tau, \overline{t})|_{\tau=\tau^0} = y^0(\overline{t}) \in C_{\overline{t}}^{(\overline{e})}(\mathbb{R}^{mn}), \tag{12}$$

$$y(\tau + \theta, \overline{t} + \overline{\omega}) = y(\tau, \overline{t}) \in C_{\tau, \overline{t}}^{(1, \overline{e})}(\mathbb{R} \times \mathbb{R}^{mn}), \ |y| \leq \Delta^*, \tag{13}$$

where $\overline{\omega} = \overline{e}\omega$, $\overline{e} = (1, ..., 1)$ is an mn-vector.

Linear vector and matrix equations with differentiation operator on the directions of characteristics. We consider the vector equation

$$D^* y(\tau, \overline{t}) = 0, \tag{14}$$

which in the coordinate form breaks down into independent equations

$$D_j^* y_j(\tau, \overline{t}_j) = 0, \tag{15}$$

with characteristic equations

$$\frac{dt_{kj}}{d\tau} = \lambda_{kj},$$

from which the characteristics are defined as follows:

$$t_{kj} = t_{kj}^0 + \lambda_{kj}(\tau - \tau^0) \equiv h_{kj}(\tau, \tau^0, t_{kj}^0).$$

If $y_j^0(\overline{t}_j) \in C_{\overline{t}_j}^{(e)}(\mathbb{R}^m)$ is an arbitrary differentiable function of the vector variable $\overline{t}_j$, then, by virtue of $\overline{t}_j = (\overline{t}_{1j}, ..., \overline{t}_{nj}) = (h_{1j}(\tau, \tau^0, t_{1j}^0), ..., h_{nj}(\tau, \tau^0, t_{nj}^0)) = \overline{h}_j(\tau, \tau^0, \overline{t}_j^0)$, according to [1], the function

$$y_j(\tau^0, \tau, \overline{t}_j) = (y_j^0 \circ \overline{h}_j)(\tau^0, \tau, \overline{t}_j) = y_j^0(\overline{h}_j(\tau, \tau^0, \overline{t}_j)) \tag{16}$$

is a solution of the equation (15) satisfying the condition (12), where $\overline{h}_j(\tau^0,\tau,\overline{t}_j) = (\overline{h}_1(\tau,\tau^0,\overline{t}_1^0),...,\overline{h}_n(\tau,\tau^0,\overline{t}_n^0))$.

Lemma 1. *The problem (14), (12) is uniquely solvable and its solution can be represented by the relation (16).*

Now let us investigate the question of $(\theta,\overline{\omega})$-periodicity of solutions to this problem.

Lemma 2. *The vector and matrix equation (14) has a ω-periodic solution in $\overline{t}_j$ if and only if the initial function is ω-periodic in $\overline{t}_j$.*

The proof of Lemma 2 follows from (16).

Since the initial function $\tilde{y}_j^0(\overline{t}_j) \in C_{\overline{t}_j}^{(e)}(\mathbb{R}^m)$ is ω-periodic in $\overline{t}_j$, then, in the case $\tilde{y}_j^0 \neq const$, for the θ-periodicity of solution (16) in τ, it is necessary and sufficient that the condition

$$\lambda_{kj}\theta = l_{kj}\omega_k,\ l_{kj} \in \mathbb{Z}, \tag{17}$$

is satisfied.

(θ,ω)-periodic in $(\tau,\overline{t}_j)$ solutions, subject to condition (17), are defined by the formula (16) with a given initial function $\tilde{y}_j^0(\overline{t}_j)$ as follows:

$$y^*(\tau,\overline{h}(\tau^0,\tau,\overline{t})) = (\tilde{y}_1^0(\tau,\overline{h}_1(\tau^0,\tau,\overline{t}_1),...,\tilde{y}_n^0(\tau,\overline{h}_n(\tau^0,\tau,\overline{t}_n)). \tag{18}$$

Thus, the following lemma is proved.

Lemma 3. *Let the conditions of Lemma 2 be satisfied. Then the $(\theta,\overline{\omega})$-periodic solutions of the vector and matrix equation (14) are 1) only constants in the absence of condition (17), and 2) along with constant solutions, the equation (14) has solutions of the form (18), if condition (17) is satisfied.*

Now we consider the homogeneous linear vector and matrix equation

$$D^*y(\tau,\overline{t}) = C \times y(\tau,\overline{t}). \tag{19}$$

Obviously, the matrix $Y(\tau) = exp\,[C\tau]$ satisfies the equation (19):

$$D^*Y(\tau) = CY(\tau),\ Y(0) = E \tag{20}$$

and is called its matricant.

It is easy to check that, if $y^0(\overline{t}) \in C_{\overline{t}}^{(\overline{e})}(\mathbb{R}^{mn})$, then

$$y^0(\overline{h}(\tau^0,\tau,\overline{t})) = \left[y_j^0(\overline{h}_j(\tau^0,\tau,\overline{t}_j)\right] \in \mathbb{S}$$

is the zero of the operator D^*:

$$D^* y^0(\overline{h}(\tau^0, \tau, \overline{t})) = 0. \tag{21}$$

Then, by virtue of (20) and (21)

$$y(\tau^0, \tau, \overline{t}) = Y(\tau - \tau^0) \times y^0(\overline{h}(\tau^0, \tau, \overline{t})) \tag{22}$$

is a solution of the initial value problem for (19) with the condition (12).

Lemma 4. *The problem (19), (12) has a unique solution of the form (22), defined on the basis of the product with the projector of matricant and the zero of operator D^*.*

Next, we investigate $(\theta, \overline{\omega})$-periodic solutions of the equation (19) that correspond to the multiperiodic zeros of the operator D^*.

In this regard, according to Lemma 3, we consider the solution (22) of problem (19), (12) with the initial functions $y^0(\overline{t} + \overline{\omega}) = y^0(\overline{t}) \in C_{\overline{t}}^{(\overline{e})}(\mathbb{R}^{mn})$.

Let us prove the following theorem.

Theorem 1. *Solution (22) is $(\theta, \overline{\omega})$-periodic if and only if the condition*

$$[Y(\theta) - E] \times y^0(\overline{t}) = 0 \tag{23}$$

is satisfied.

Proof. Necessity. Let the solution (22) be $(\theta, \overline{\omega})$-periodic, that is

$$y(\tau^0, \tau + \theta, \overline{t} + \overline{\omega}) = y(\tau^0, \tau, \overline{t}). \tag{24}$$

Then we have condition (23) from (24) for $\tau = \tau^0$.
Sufficiency. Suppose condition (23) is satisfied. Let us prove the validity of (24). Along with solution (22), consider the solution

$$\tilde{y}(\tau^0, \tau, \overline{t}) = y(\tau^0, \tau + \theta, \overline{t} + \overline{\omega}) = Y(\tau + \theta - \tau^0) \times y^0(\overline{h}(\tau^0, \tau + \theta, \overline{t} + \overline{\omega})) =$$
$$= Y(\tau + \theta - \tau^0) \times y^0(\overline{h}(\tau^0, \tau, \overline{t})).$$

Hence, for $\tau = \tau^0$, by virtue of condition (23), we have

$$\tilde{y}(\tau^0, \tau, \overline{t})|_{\tau=\tau^0} = Y(\theta) \times y^0(\overline{t})) = [(Y(\theta) - E) + E] \times y^0(\overline{t}) =$$
$$= [(Y(\theta) - E)] \times y^0(\overline{t}) + y^0(\overline{t}) = y^0(\overline{t}).$$

Consequently, the solutions $y(\tau^0, \tau, \overline{t})$ and $\tilde{y}(\tau^0, \tau, \overline{t})$ satisfy the same initial condition. Then, by virtue of the uniqueness property of the solution, we have $\tilde{y}(\tau^0, \tau, \overline{t}) \equiv y(\tau^0, \tau, \overline{t})$ or $y(\tau^0, \tau + \theta, \overline{t} + \overline{\omega}) \equiv y(\tau^0, \tau, \overline{t})$. ◄

Corollary 1. *Under the conditions of Theorem 1, vector and matrix equation (19) has only a zero $(\theta, \overline{\omega})$-periodic solution if and only if the following condition is satisfied*

$$det\,[Y(\theta) - E] \neq 0. \tag{25}$$

The proof of the corollary follows from the fact that the condition (25) is equivalent to the existence of only a trivial solution to equation (23).

Next, consider the non-homogeneous linear vector and matrix equation

$$D^* y(\tau, \bar{t}) = C \times y(\tau, \bar{t}) + \varphi(\tau, \bar{t}) \tag{26}$$

with constant matrix C and free term

$$\varphi(\tau + \theta, \bar{t} + \overline{\omega}) = \varphi(\tau, \bar{t}) \in C^{(0,\bar{e})}_{\tau,\bar{t}}(\mathbb{R} \times \mathbb{R}^{mn}). \tag{27}$$

It is clear that the solution $y(\tau^0, \tau, \bar{t})$ of equation (19) with the initial condition (12) consists of the sum of the solution (22) of the corresponding homogeneous equation and the solution of equation (19) with the zero initial condition

$$\tilde{y}(\tau^0, \tau, \bar{t}) = \int\limits_{\tau^0}^{\tau} Y(\tau - s) \times \varphi(s, \overline{h}(s, \tau, \bar{t}))\, ds. \tag{28}$$

The fact that the vector-function (28) satisfies the equation (26) can be verified directly.

Consequently,

$$y(\tau^0, \tau, \bar{t}) = Y(\tau - \tau^0) \times y^0(\overline{h}(\tau^0, \tau, \bar{t})) + \int\limits_{\tau^0}^{\tau} Y(\tau - s) \times \varphi(s, \overline{h}(s, \tau, \bar{t}))\, ds \tag{29}$$

is a solution to equation (26) with the initial condition (12).

Thus, we have proved the following lemma.

Lemma 5. *Problem (26), (12) has a unique solution of the form (29), defined on the basis of products with a projector of 1) zero of operator D^* and 2) a free term defined along a characteristic with a matricant $Y(\tau - s)$.*

In applied problems, the definition of a multiperiodic solution of non-homogeneous equation corresponding to a unique trivial solution of homogeneous equation is of special interest.

Therefore, in the sequel, we assume that the condition (25) is fulfilled.

According to the constructive method, assuming the existence of a unique multiperiodic solution $y^*(\tau^0, \tau, \overline{t})$ to non-homogeneous equation (26), from the relation (29) we define the initial data $y_*^0(\overline{t})$ corresponding to this solution, for which $y_*^0(\overline{h}(\tau^0, \tau, \overline{t}))$ is a $(\theta, \overline{\omega})$-periodic zero of the operator D^*. In this regard, we have a solution

$$y^*(\tau, \overline{t}) = Y(\tau - \tau^0) \times y_*^0(\overline{h}(\tau^0, \tau, \overline{t})) + \int\limits_{\tau^0}^{\tau} Y(\tau - s) \times \varphi(s, \overline{h}(s, \tau, \overline{t}))\, ds. \quad (30)$$

In order to construct this solution, shifting τ to θ in (30), we have

$$y^*(\tau, \overline{t}) = y^*(\tau + \theta, \overline{t}) = Y(\tau + \theta - \tau^0) \times y_*^0(\overline{h}(\tau^0, \tau + \theta, \overline{t})) + $$
$$+ \int\limits_{\tau^0}^{\tau+\theta} Y(\tau + \theta - s) \times \varphi(s, \overline{h}(s, \tau + \theta, \overline{t}))\, ds =$$
$$= Y(\tau + \theta - \tau^0) \times y_*^0(\overline{h}(\tau^0, \tau, \overline{t})) + \int\limits_{\tau^0}^{\tau+\theta} Y(\tau + \theta - s) \times \varphi(s, \overline{h}(s, \tau + \theta, \overline{t}))\, ds.$$

Further, by replacing s with $s + \theta$, we obtain

$$y^*(\tau, \overline{t}) = Y(\tau + \theta - \tau^0) \times y_*^0(\overline{h}(\tau^0, \tau, \overline{t})) + \int\limits_{\tau^0-\theta}^{\tau} Y(\tau - s) \times \varphi(s, \overline{h}(s, \tau, \overline{t}))\, ds. \quad (31)$$

Multiplying (31) and (30) on the left by matrices $Y^{-1}(\tau + \theta - \tau^0)$ and $Y^{-1}(\tau - \tau^0)$, respectevely, we obtain

$$Y^{-1}(\tau + \theta - \tau^0) \times y^*(\tau, \overline{t}) = y_*^0(\overline{h}(\tau^0, \tau, \overline{t})) +$$
$$+ Y^{-1}(\tau + \theta - \tau^0) \int\limits_{\tau^0-\theta}^{\tau} Y(\tau - s) \times \varphi(s, \overline{h}(s, \tau, \overline{t}))\, ds, \quad (32)$$
$$Y^{-1}(\tau - \tau^0) \times y^*(\tau, \overline{t}) = y_*^0(\overline{h}(\tau^0, \tau, \overline{t})) +$$
$$+ Y^{-1}(\tau - \tau^0) \int\limits_{\tau^0}^{\tau} Y(\tau - s) \times \varphi(s, \overline{h}(s, \tau, \overline{t}))\, ds. \quad (33)$$

Further, excluding the initial function $y_*^0(\overline{h}(\tau^0, \tau, \overline{t}))$ from the system of equations (36)-(37), we obtain

$$\left[Y^{-1}(\tau + \theta - \tau^0) - Y^{-1}(\tau - \tau^0)\right] \times y^*(\tau, \overline{t}) =$$

$$= \int\limits_{\tau^0-\theta}^{\tau} Y(\tau^0-\theta-s)\times\varphi(s,\overline{h}(s,\tau,\overline{t}))ds + \int\limits_{\tau^0}^{\tau} Y(\tau^0-s)\times\varphi(s,\overline{h}(s,\tau,\overline{t}))ds. \quad (34)$$

Since, by virtue of condition (25), we have

$$det\left[Y^{-1}(\tau+\theta-\tau^0)-Y^{-1}(\tau-\tau^0)\right] = det\left[Y^{-1}(\tau-\tau^0)(E-Y(\theta))Y^{-1}(\theta)\right] \neq 0,$$

the required solution (34) can be represented as

$$y^*(\tau,\overline{t}) = \left[Y^{-1}(\tau+\theta-\tau^0)-Y^{-1}(\tau-\tau^0)\right]^{-1}$$

$$\Big[\int\limits_{\tau^0-\theta}^{\tau} Y(\tau^0-\theta-s)\times\varphi(s,\overline{h}(s,\tau,\overline{t}))\,ds + \int\limits_{\tau}^{\tau^0} Y(\tau^0-s)\times\varphi(s,\overline{h}(s,\tau,\overline{t}))\,ds\Big].$$

Since

$$\left[Y^{-1}(\tau+\theta-\tau^0)-Y^{-1}(\tau-\tau^0)\right]^{-1} = \left[Y^{-1}(\tau+\theta)-Y^{-1}(\tau)\right]^{-1} - Y^{-1}(\tau^0),$$
$$Y(\tau^0-\theta-s) = Y^{-1}(\tau^0)Y(s+\theta),$$

we finally have

$$y^*(\tau,\overline{t}) = \left[Y^{-1}(\tau+\theta)-Y^{-1}(\tau)\right]^{-1}\Big[\int\limits_{\tau^0-\theta}^{\tau} Y^{-1}(s+\theta)\times\varphi(s,\overline{h}(s,\tau,\overline{t}))\,ds+$$

$$+\int\limits_{\tau}^{\tau^0} Y^{-1}(s)\times\varphi(s,\overline{h}(s,\tau,\overline{t}))\,ds\Big]. \quad (35)$$

It is clear that $y^*(\tau,\overline{t})$ is a solution to equation (26), defined by relation (35).

Let us introduce the function $s^*(\tau) = [\tau/\theta]\,\theta$, where $[\cdot]$ is a sign of integer part. Obviously, $s^*(\tau)$ is differentiable when $\tau \neq \nu\theta$, $\nu \in \mathbb{Z}$, moreover, $\dot{s}^*(\tau) = 0$. Points $\tau = \nu\theta$ are removable singular points of the derivative $\dot{s}^*(\tau)$.

If we put $\dot{s}^*(\tau)|_{\tau=\nu\theta} = 0$, then the singularity of the derivative is eliminated and we have $\dot{s}^*(\tau) = 0$ with respect to $\tau \in \mathbb{R}$. Obviously, $s^*(\tau)$ has the property $s^*(\tau+\nu\theta) = s^*(\tau)+\nu\theta$.

Now, if we take into account $\dot{s}^*(\tau) = 0$, then for $\tau^0 = s^*(\tau+\theta-0)$ relation (35) remains a solution of equation (26) of the form

$$y^*(\tau,\overline{t}) = \left[Y^{-1}(\tau+\theta)-Y^{-1}(\tau)\right]^{-1}\Big[\int\limits_{s^*(\tau)}^{\tau} Y^{-1}(s+\theta)\times\varphi(s,\overline{h}(s,\tau,\overline{t}))\,ds+$$

$$+ \int\limits_{\tau}^{s^*(\tau+\theta-0)} Y^{-1}(s) \times \varphi(s, \overline{h}(s,\tau,\overline{t}))\, ds\Big]. \tag{36}$$

Moreover, it is θ-periodic with respect to τ. This can be verified directly.

Further, we introduce the matrix function

$$G(s,\tau)) = \begin{cases} \left[Y^{-1}(\tau+\theta) - Y^{-1}(\tau)\right]^{-1} Y^{-1}(s+\theta), & s^*(\tau) \le s \le \tau, \\ \left[Y^{-1}(\tau+\theta) - Y^{-1}(\tau)\right]^{-1} Y^{-1}(s), & \tau < s \le s^*(\tau+\theta-0) \end{cases} \tag{37}$$

for a compact representation of the solution (36) in the form

$$y^*(\tau, \overline{t}) = \int\limits_{s^*(\tau)}^{s^*(\tau+\theta-0)} G(s,\tau) \times \varphi(s, \overline{h}(s,\tau,\overline{t}))\, ds. \tag{38}$$

The function (37) can be called a Green's type function, which has the following easily verifiable properties:

$$\begin{aligned} &1^0.\ D^*G(s,\tau) = CG(s,\tau), \quad s \ne \tau; \\ &2^0.\ G(\tau-0,\tau) - G(\tau+0,\tau) = G(\tau,\tau+0) - G(\tau,\tau-0) = E; \\ &3^0.\ G(s^*(\tau+\theta-0),\tau) - G(s^*(\tau),\tau) = 0; \\ &4^0.\ G(s+\theta,\tau+\theta) = G(s,\tau); \\ &5^0.\ \int\limits_{s^*(\tau)}^{s^*(\tau+\theta-0)} G(s,\tau) \times \varphi(s, \overline{h}(s,\tau,\overline{t}))\, ds \le \gamma, \end{aligned} \tag{39}$$

where E is an identity matrix, and γ is a positive constant.

Consequently, by (39), the solution (38) satisfies the estimate

$$\|y^*\| \le \gamma \, \|\varphi\|, \tag{40}$$

where $\|y^*\| = \max\limits_{j=\overline{1,n}} \sup\limits_{\mathbb{R}\times\mathbb{R}^m} \left| y_j^*(\tau, \overline{t}_j) \right|$.

By virtue of condition (25), equation (26) has a unique $(\theta, \overline{\omega})$-periodic solution. Thus, the following theorem is proved.

Theorem 2. *Under conditions (25) and (27), equation (26) admits a unique $(\theta, \overline{\omega})$-periodic solution of the form (38), for which the estimate (40) is valid.*

Quasilinear vector and matrix equations with differentiation operator in the direction of characteristics. Now we proceed to study the quasilinear vector and matrix equation

$$D^*y = Cy + \varphi(\tau, \bar{t}, y), \tag{41}$$

with the initial condition

$$y(\tau, \bar{t})|_{\tau=\tau^0} = y^0(\bar{t} + \bar{\omega}) = y^0(\bar{t}) \in C_{\bar{t}}^{(\bar{e})}(\mathbb{R}^{mn}), \quad \left|y^0\right| \leq \Delta^0, \tag{42}$$

where the vector-function $\varphi(\tau, \bar{t}, y) = \left[\varphi_i(\tau, \bar{t}_i, y)\right] \in \mathbb{S}$ has the property

$$\varphi(\tau + \theta, \bar{t} + \bar{\omega}, y) = \varphi(\tau, \bar{t}, y) \in C_{\tau,\bar{t},y}^{(0,\bar{e},\hat{e})}(\mathbb{R} \times \mathbb{R}^{mn} \times \mathbb{R}_{\Delta^*}^n) \tag{43}$$

and $\mathbb{R}_{\Delta^*}^n = \{y \in \mathbb{R}^n : |y| \leq \Delta^*\}$, $|\cdot|$ is one of the three known types of vector norms.

Let us state the consequences of condition (43) with respect to $(\tau, \bar{t}) \in (\mathbb{R} \times \mathbb{R}^{mn})$:

$$\left|\varphi(\tau, \bar{t}, \tilde{y}) - \varphi(\tau, \bar{t}, y)\right| \leq \ell\,|\tilde{y} - y|\,, \quad \tilde{y}, y \in \mathbb{R}_{\Delta^*}^n, \tag{44}$$

$$\left|\varphi(\tau, \bar{t}, 0)\right| \leq \kappa, \quad \left|\varphi(\tau, \bar{t}, y)\right| \leq \left|\varphi(\tau, \bar{t}, 0) + \ell\,|y|\right| \leq \kappa + \ell\,|y|\,, \quad y \in \mathbb{R}_{\Delta^*}^n. \tag{45}$$

Under the condition (43), based on Lemma 5, the problem (41)-(42) is presented in the form of an equivalent integral equation

$$y(\tau, \bar{t}) = Y(\tau - \tau^0) \times y^0(\bar{h}(\tau^0, \tau, \bar{t})) + \int_{\tau^0}^{\tau} Y(\tau - s) \times (\varphi \otimes y)(s, \bar{h}(s, \tau, \bar{t}))\, ds. \tag{46}$$

The solution $y^*(\tau, \bar{t})$ of equation (46) is determined under the condition

$$\delta|C| < 1, \quad \delta(2\Delta^0 + \alpha(\kappa + \ell\Delta^*)) < \Delta^*,$$

by the method of successive approximations in the space $\mathbb{S}_{\delta,\Delta^*}^{\omega}(\mathbb{R}_\delta \times \mathbb{R}^{mn})$ of n-vector-functions $y(\tau, \bar{t})$ continuous in $(\tau, \bar{t}) \in \mathbb{R}_\delta \times \mathbb{R}^{mn}$, $\bar{\omega}$-periodic in $\bar{t} \in \mathbb{R}^{mn}$ and bounded in the norm $\left|\, y - y^0\right| \leq \Delta^*$ in $\tau \in \mathbb{R}_\delta = \left\{\tau \in \mathbb{R} : |\tau - \tau^0| \leq \delta\right\}$, $\delta = const > 0$, $\tau^0 \in \mathbb{R}$.

It is clear that $\bar{t}$ behaves like an mn-dimensional parameter. Moreover, the right-hand side of equation (46) has continuous partial derivatives both with respect to the required vector-function and with respect to the parameter $\bar{t}$. Here the theorem on the existence of continuous derivatives of the solution of the integral equation with respect to the parameter is applicable. Therefore, for sufficiently small values of $\delta > 0$, the limit point $y^*(\tau, \bar{t})$ of successive approximations is continuously differentiable with respect to the coordinates $\bar{t}$.

The solution $y^*(\tau, \overline{t})$ is also continuously differentiable with respect to τ, since it is a transition variable from the initial value problem for the differential equation to the equivalent integral equation for sufficiently small values of $\delta > 0$.

Thus, the smoothness of the solution of integral equation is ensured for a sufficiently small value of $\delta = \delta_*$, and the following theorem on the solvability of the initial value problem is proved.

Theorem 3. *Under condition (43), initial value problem (41)-(42) is uniquely solvable in the space* $\mathbb{S}^{\overline{\omega}}_{\delta_*,\Delta^*}(\mathbb{R}_{\delta_*} \times \mathbb{R}^{mn})$.

Since Theorem 3 is valid for all $\tau^0 \in \mathbb{R}$, our problem has a unique solution for all $\tau \in \mathbb{R}$.

Corollary 2. *Under the conditions of Theorem 3, problem (41)-(42) is solvable for all* $(\tau, \overline{t}) \in \mathbb{R} \times \mathbb{R}^{mn}$*. Moreover,* $y^*(\tau, \overline{t}) \in \mathbb{S}^{\overline{\omega}}_{\Delta^*}(\mathbb{R} \times \mathbb{R}^{mn})$.

Theorem 3 allows us to extend Theorem 2 to the quasilinear case.

In this regard, we still assume that the condition of Corollary 1 is satisfied and, in accordance with the integral representation (38), we introduce the operator

$$Q^* y^*(\tau, \overline{t}) = \int\limits_{s^*(\tau)}^{s^*(\tau+\theta-0)} G(s, \tau) \times \varphi(s, \overline{h}(s, \tau, \overline{t}))\, ds \tag{47}$$

in the space $\mathbb{S}^{\theta,\overline{\omega}}_{\Delta^*}(\mathbb{R} \times \mathbb{R}^{mn})$ of n-vector-functions $y^*(\tau, \overline{t})$ that are continuous and $(\theta, \overline{\omega})$-periodic with respect to $(\tau, \overline{t})$ and bounded in the norm $|y| \leq \Delta^*$.

Obviously, if $y^*(\tau, \overline{t}) \in \mathbb{S}^{\theta,\overline{\omega}}_{\Delta^*}(\mathbb{R} \times \mathbb{R}^{mn})$, then, by virtue of properties (39) and inequalities (45), estimating (47), we have

$$\big|Q^* y^*(\tau, \overline{t})\big| = \Bigg| \int\limits_{s^*(\tau)}^{s^*(\tau+\theta-0)} G(s, \tau) \times \varphi(s, \overline{h}(s, \tau, \overline{t}))\, ds \Bigg| \leq \gamma(\kappa + \ell\Delta^*). \tag{48}$$

Therefore, for

$$\gamma(\kappa + \ell\Delta^*) < \Delta^*, \tag{49}$$

from the estimate (48) we obtain $\|Q^* y\| \leq \Delta^*$. Then, according to Theorem 2, we have $Q^* y(\tau, \overline{t}) \in \mathbb{S}^{\theta,\overline{\omega}}_{\Delta^*}(\mathbb{R} \times \mathbb{R}^{mn})$ for $y(\tau, \overline{t}) \in \mathbb{S}^{\theta,\overline{\omega}}_{\Delta^*}(\mathbb{R} \times \mathbb{R}^{mn})$.

Thus, we have proved that the operator Q^* maps the space $\mathbb{S}^{\theta,\overline{\omega}}_{\Delta^*}(\mathbb{R} \times \mathbb{R}^{mn})$ into itself.

By virtue of properties (39) and condition (44) for $\tilde{y} = \tilde{y}(\tau, \bar{t})$ and $y = y(\tau, \bar{t})$ from the space $\mathbb{S}_{\Delta^*}^{\theta,\overline{\omega}}(\mathbb{R} \times \mathbb{R}^{mn})$, we have

$$\left|Q^*\tilde{y}(\tau, \bar{t}) - Q^*y(\tau, \bar{t})\right| \leq q^* \left|\tilde{y} - y\right|, \qquad (50)$$

where $q^* = \gamma\ell$.

Then from condition (49) we have $q^* < 1$. Hence, by virtue of (50), Q^* is a contraction operator in $\mathbb{S}_{\Delta^*}^{\theta,\overline{\omega}}(\mathbb{R} \times \mathbb{R}^{mn})$ and has a unique fixed point

$$y^*(\tau, \bar{t}) = Q^*y^*(\tau, \bar{t}).$$

The existence of a unique solution of the equation (47) in the space $\mathbb{S}_{\Delta^*}^{\theta,\overline{\omega}}(\mathbb{R} \times \mathbb{R}^{mn})$ follows from the last identity. Moreover, by virtue of Theorem 3, the solution has a smoothness property with respect to $(\tau, \bar{t})$.

Thus, the following theorem on the existence of a $(\theta, \overline{\omega})$-periodic solution of the equation under consideration is proved.

Theorem 4. *Under conditions (25), (43), and (49), quasilinear equation (41) has a unique solution* $y^*(\tau, \bar{t})$ *in the space* $\mathbb{S}_{\Delta^*}^{\theta,\overline{\omega}}(\mathbb{R} \times \mathbb{R}^{mn})$*.*

In conclusion, we note that the method has been developed for studying the initial value problem (11)-(12) and the problem (11), (13) for quasilinear vector and matrix equations with differentiation operators $D^* = (D_1^*, ..., D_n^*)$ with different components D_j^*, based on the introduction of the operator Π and the projector P. Upon completion of the integration of systems with canonical differentiation operators D^* , the reverse transition to the previous variables is carried out, based on the operator $\Pi^{-1}\bar{t} = t$. Otherwise we have $t_{i1} = t_{i2} = ... = t_{in} = t_i$. Thus, we get the results expressed by the previous variables $(t_1, ..., t_n) = t$. The final results are formulated after the replacement $y = K^{-1}x$ in terms of variables (τ, t) for the problem with operator D.

According to this method, the existence of solutions of the problems is proved and the integral representations of these solutions in the linear case, which allowed to extend the idea of method to the quasilinear case, are obtained.

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Amire Kh. Zhumagaziyev
K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan
E-mail: charmeda@mail.ru

Zhaishylyk A. Sartabanov
K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan
E-mail: sartabanov42@mail.ru

Yaudat T. Sultanaev
Lomonosov Moscow State University, Center of Fundamental and Applied Mathematics, Moscow, Russia
E-mail: sultanaevyt@gmail.com

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