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On a New Generalization of Jacobsthal Hybrid Numbers

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Abstract. We define a two-parameter generalization of Jacobsthal hybrid numbers. We give Binet formula, the generating functions and some identities for these numbers.

Key Words and Phrases: Jacobsthal numbers, hybrid numbers, Jacobsthal hybrid numbers, recurrence relations, generating functions.

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1. Introduction

In 2018, ([8]) Özdemir introduced a new generalization of complex, hyperbolic and dual numbers – hybrid numbers.

Let $\mathbb{K}$ denote the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}, \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ε, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1 \tag{2}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{3}$$

Let $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ be any two hybrid numbers. We define equality, addition, subtraction and multiplication by scalar $s \in \mathbb{R}$ in the following way:

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$\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
$\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}$,
$\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h}$,
$s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h}$.

The hybrid numbers multiplication is defined using (2) and (3). Note that using the formulas (2) and (3) we can find the product of any two hybrid units. The Table 1 presents products of $\mathbf{i}$, ε, and $\mathbf{h}$.

Table 1: The hybrid number multiplication.

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Using the rules given in the Table 1 the multiplication of hybrid numbers can be carried out similar to the multiplication of algebraic expressions.

Addition operation on the hybrid numbers is both commutative and associative. Zero $\mathbf{0} = 0 + 0\mathbf{i} + 0\varepsilon + 0\mathbf{h}$ is the null element. With respect to the addition operation, the inverse element of $\mathbf{Z}$ is $-\mathbf{Z} = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. The multiplication is not commutative, but associative. Moreover, $(\mathbb{K}, +, \cdot)$ is a non-commutative ring (with identity element $\mathbf{1} = 1 + 0\mathbf{i} + 0\varepsilon + 0\mathbf{h}$), see [8].

The conjugate of a hybrid number $\mathbf{Z}$ is defined by

$$\overline{\mathbf{Z}} = \overline{a + b\mathbf{i} + c\varepsilon + d\mathbf{h}} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}. \tag{4}$$

The real number

$$C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2 \tag{5}$$

is called the character of the hybrid number $\mathbf{Z}$. Hybrid numbers are classified as spacelike, timelike and lightlike according to its character. We say that a hybrid number $\mathbf{Z}$ is spacelike, timelike or lightlike if $C(\mathbf{Z}) < 0$, $C(\mathbf{Z}) > 0$ or $C(\mathbf{Z}) = 0$, respectively. For the basics on hybrid numbers, see [8].

A special kind of hybrid numbers, namely Horadam hybrid numbers, were introduced in [9]. Interesting results of Jacobsthal hybrid numbers (which are a subset of Horadam hybrid numbers) obtained recently can be found in [9, 12]. In [7], Kızılateş defined the $q-$hybrid Fibonacci numbers and $q-$hybrid Lucas numbers. The Jacobsthal hybrid numbers are a special case of the $q-$hybrid Fibonacci numbers. Note that these numbers were defined using notations related to $q-$calculus. The theory of the quantum ($q-$) calculus has been studied in physics, electrochemistry, biology, mathematics etc. In this paper, we introduce and study a two-parameter generalization of the Jacobsthal hybrid numbers – the (s,p)-Jacobsthal hybrid numbers.

2. The (s,p)-Jacobsthal numbers

Let $n \geq 0$ be an integer. The Jacobsthal sequence $\{J_n\}$ is defined by the second order linear recurrence

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2 \tag{6}$$

with initial terms $J_0 = 0$, $J_1 = 1$. The direct formula for the nth Jacobsthal number has the form $J_n = \frac{2^n-(-1)^n}{3}$, named as the Binet formula for the Jacobsthal numbers. The first ten terms of the sequence are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171. There are many generalizations of the sequence in the literature. The second order recurrence (6) has been generalized in two ways: first by preserving the initial conditions and second by preserving the recurrence relation. We recall some of such generalizations:

1) k-Jacobsthal sequence $\{j_{k,n}\}$ [6], $j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $j_{k,0} = 0$, $j_{k,1} = 1$,

2) k-Jacobsthal sequence $\{J_{k,n}\}$ [4], $J_{k,n+1} = J_{k,n} + kJ_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $J_{k,0} = 0$, $J_{k,1} = 1$,

3) generalized Jacobsthal p-sequence $\{J_p\}$ [3], for any $p \in \mathbb{Z}^+$ and $n > p+1$ $J_p(n) = J_p(n-1) + 2J_p(n-p-1)$ with $J_p(1) = J_p(2) = \ldots = J_p(p+1) = 1$,

4) Jacobsthal r-sequence $\{J(r,n)\}$ [2], for $r \geq 0$ $J(r,n) = 2^r J(r,n-1) + (2^r + 4^r)J(r,n-2)$ for $n \geq 2$ with $J(r,0) = 1$, $J(r,1) = 1 + 2^{r+1}$,

5) (s,t)-Jacobsthal sequence $\{\hat{j}_n(s,t)\}$ [14], $\hat{j}_n(s,t) = s\hat{j}_{n-1}(s,t) + 2t\hat{j}_{n-2}(s,t)$ for $n \geq 2$ with $\hat{j}_0(s,t) = 0$ and $\hat{j}_1(s,t) = 1$, for real numbers s,t, $s > 0$, $t \neq 0$ and $s^2 + 8t > 0$,

6) Jacobsthal sequence $\{J(d,t,n)\}$ [10], $J(d,t,n) = J(d,t,n-1) + tJ(d,t,n-d)$ for $n \geq d$ with $J(d,t,0) = 1$, $J(d,t,n) = 1$ for $n = 1,\ldots,d$, $t \geq 1, d \geq 2$.

In [1], a two-parameter generalization of the Jacobsthal numbers was investigated. We recall this generalization and some properties of these numbers.

Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers, and the sequence $\{J_n(s,p)\}$ be defined by the following recurrence

$$J_n(s,p) = 2^{s+p}J_{n-1}(s,p) + (2^{2s+p} + 2^{s+2p})J_{n-2}(s,p) \text{ for } n \geq 2 \tag{7}$$

with initial conditions $J_0(s,p) = 1$, $J_1(s,p) = 2^s + 2^p + 2^{s+p}$.

It is easily seen that for $s = p = 0$ we have $J_n(0,0) = J_{n+2}$.

By (7) we obtain

$$\begin{aligned} J_0(s,p) &= 1 \\ J_1(s,p) &= 2^s + 2^p + 2^{s+p} \\ J_2(s,p) &= 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p} \\ J_3(s,p) &= 2^{3s+2p+1} + 2^{2s+3p+1} + 2^{3s+3p} + 2^{3s+p} \\ &\quad + 2^{2s+2p+1} + 2^{3s+2p} + 2^{2s+3p} + 2^{s+3p}. \end{aligned}$$

Theorem 1. [1] (*Binet formula*) *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then the nth (s,p)-Jacobsthal number is given by*

$$J_n(s,p) = c_1 r_1^n + c_2 r_2^n, \tag{8}$$

where

$$r_1 = 2^{s+p-1} + \tfrac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)},$$

$$r_2 = 2^{s+p-1} - \tfrac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)},$$

$$c_1 = \frac{2^s + 2^p + 2^{s+p} - 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}},$$

$$c_2 = \frac{-2^s - 2^p - 2^{s+p} + 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}.$$

Theorem 2. [1] *The generating function of the sequence $\{J_n(s,p)\}$ has the following form*

$$f(x) = \frac{1 + (2^s + 2^p)x}{1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2}.$$

Theorem 3. [1] *Let $n \geq 1$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\sum_{l=0}^{n-1} J_l(s,p) = \frac{J_n(s,p) + (2^{2s+p} + 2^{s+2p})J_{n-1}(s,p) - 1 - 2^s - 2^p}{2^{s+p}(1 + 2^s + 2^p) - 1}.$$

Application of the Jacobsthal sequence to curves was presented in [5]. Moreover, the Jacobsthal sequence has an application in the theory of graphs. Let G be a finite, undirected, simple graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subset V(G)$ is an independent set of G if for any two distinct vertices $x, y \in S$ the relation $xy \notin E(G)$ holds. Moreover, a subset of $V(G)$ containing only one vertex and the empty set are independent sets of G. The number of independent sets of a graph G is denoted by $NI(G)$.

Consider a graph G (Figure 1).

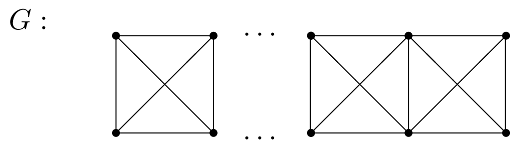

Figure 1.

In [11] it was proved that

$$NI(G) = J_{n+2}.$$

Other applications of generalizations of the Jacobsthal numbers in the graph theory can be found in [1, 2].

3. The (s, p)-Jacobsthal hybrid numbers

Let $n \geq 0$ be an integer. We define the nth (s, p)-Jacobsthal hybrid number $JH_n^{s,p}$ by the following relation:

$$JH_n^{s,p} = J_n(s,p) + \mathbf{i}J_{n+1}(s,p) + \varepsilon J_{n+2}(s,p) + \mathbf{h}J_{n+3}(s,p), \tag{9}$$

where $J_n(s, p)$ is given by (7).

Note that for $s = p = 0$ we obtain $JH_n^{0,0} = JH_{n+2}$, where JH_n denotes nth Jacobsthal hybrid number defined in [12].

By some elementary calculations we find the following recurrence relation for the (s, p)-Jacobsthal hybrid numbers.

Theorem 4. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$2^{s+p} JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p}) JH_n^{s,p} = JH_{n+2}^{s,p}.$$

Proof.

$$\begin{aligned}
&2^{s+p} JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p}) JH_n^{s,p} = \\
&= 2^{s+p}(J_{n+1}(s,p) + \mathbf{i}J_{n+2}(s,p) + \varepsilon J_{n+3}(s,p) + \mathbf{h}J_{n+4}(s,p)) \\
&\quad + (2^{2s+p} + 2^{s+2p})(J_n(s,p) + \mathbf{i}J_{n+1}(s,p) + \varepsilon J_{n+2}(s,p) + \mathbf{h}J_{n+3}(s,p)) \\
&= J_{n+2}(s,p) + \mathbf{i}J_{n+3}(s,p) + \varepsilon J_{n+4}(s,p) + \mathbf{h}J_{n+5}(s,p) = JH_{n+2}^{s,p}.
\end{aligned}$$

◀

Theorem 5. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\begin{aligned}&C(JH_n^{s,p}) = (1 - 2^{6s+4p} - 2^{4s+6p} - 2^{5s+5p+1})J_n^2(s,p)\\ &-\left(2^{2s+p+1} + 2^{s+2p+1}\right)(1 + 2^{3s+3p} + 2^{3s+2p} + 2^{2s+3p}\right) J_n(s,p)J_{n+1}(s,p)\\ &+(1 - 2^{s+p+1} - (2^{2s+2p} + 2^{2s+p} + 2^{s+2p})^2)J_{n+1}^2(s,p).\end{aligned}$$

Proof. By formula (7) we have

$$\begin{aligned}J_{n+3}(s,p) &= (2^{2s+2p} + 2^{s+2p} + 2^{2s+p})J_{n+1}(s,p)\\ &\quad + 2^{s+p}(2^{2s+p} + 2^{s+2p})J_n(s,p),\\ J_{n+2}(s,p) &= 2^{s+p}J_{n+1}(s,p) + (2^{s+2p} + 2^{2s+p})J_n(s,p).\end{aligned}$$

Hence

$$\begin{aligned}C(JH_n^{s,p}) = J_n^2(s,p) &+ J_{n+1}^2(s,p) - 2J_{n+1}(s,p)(2^{s+p}J_{n+1}(s,p)\\ &+ (2^{s+2p} + 2^{2s+p})J_n(s,p))\\ &- [(2^{2s+2p} + 2^{s+2p} + 2^{2s+p})J_{n+1}(s,p) + 2^{s+p}(2^{2s+p} + 2^{s+2p})J_n(s,p)]^2.\end{aligned}$$

After simple calculations we get

$$\begin{aligned}&C(JH_n^{s,p}) = (1 - 2^{2s+2p}(2^{2s+p} + 2^{s+2p})^2)J_n^2(s,p)\\ &-\left(2^{2s+p+1} + 2^{s+2p+1}\right)(1 + 2^{s+p}(2^{2s+2p} + 2^{2s+p} + 2^{s+2p})\right) J_n(s,p)J_{n+1}(s,p)\\ &+(1 - 2^{s+p+1} - (2^{2s+2p} + (2^{2s+p} + 2^{s+2p})^2)J_{n+1}^2(s,p)\\ &= (1 - 2^{6s+4p} - 2^{4s+6p} - 2^{5s+5p+1})J_n^2(s,p)\\ &-\left(2^{2s+p+1} + 2^{s+2p+1}\right)(1 + 2^{3s+3p} + 2^{3s+2p} + 2^{2s+3p}\right) J_n(s,p)J_{n+1}(s,p)\\ &+(1 - 2^{s+p+1} - (2^{2s+2p} + 2^{2s+p} + 2^{s+2p})^2)J_{n+1}^2(s,p).\end{aligned}$$

◀

Remark 1. *For $s = p = 0$ we obtain the result from* [12] *– the character of the Jacobsthal hybrid number JH_n:*

$$C(JH_n^{0,0}) = C(JH_{n+2}) = -3J_{n+2}^2 - 10J_{n+3}^2 - 16J_{n+2}J_{n+3}.$$

Theorem 6. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

(i) $JH_n^{s,p} + \overline{JH_n^{s,p}} = 2J_n(s,p)$,

(ii) $(JH_n^{s,p})^2 = 2J_n(s,p)JH_n^{s,p} - C(JH_n^{s,p})$.

Proof. (i) By formula (4) we get

$$JH_n^{s,p} + \overline{JH_n^{s,p}} = J_n(s,p) + \mathbf{i}J_{n+1}(s,p) + \varepsilon J_{n+2}(s,p) + \mathbf{h}J_{n+3}(s,p) + J_n(s,p) - \mathbf{i}J_{n+1}(s,p) - \varepsilon J_{n+2}(s,p) - \mathbf{h}J_{n+3}(s,p) = 2J_n(s,p).$$

(ii)

$$\begin{aligned}(JH_n^{s,p})^2 &= J_n^2(s,p) - J_{n+1}^2(s,p) + J_{n+3}^2(s,p)\\ &\quad + 2\mathbf{i}J_n(s,p)J_{n+1}(s,p) + 2\varepsilon J_n(s,p)J_{n+2}(s,p) + 2\mathbf{h}J_n(s,p)J_{n+3}(s,p)\\ &\quad + (\mathbf{i}\varepsilon + \varepsilon\mathbf{i})J_{n+1}(s,p)J_{n+2}(s,p) + (\mathbf{ih} + \mathbf{hi})J_{n+1}(s,p)J_{n+3}(s,p)\\ &\quad + (\varepsilon\mathbf{h} + \mathbf{h}\varepsilon)J_{n+2}(s,p)J_{n+3}(s,p)\\ &= J_n^2(s,p) - J_{n+1}^2(s,p) + J_{n+3}^2(s,p) + 2J_{n+1}(s,p)J_{n+2}(s,p)\\ &\quad + 2(\mathbf{i}J_n(s,p)J_{n+1}(s,p) + \varepsilon J_n(s,p)J_{n+2}(s,p) + \mathbf{h}J_n(s,p)J_{n+3}(s,p))\\ &= 2J_n(s,p)JH_n^{s,p} - (J_n^2(s,p) + J_{n+1}^2(s,p)\\ &\quad - 2J_{n+1}(s,p)J_{n+2}(s,p) + J_{n+3}^2(s,p))\\ &= 2J_n(s,p)JH_n^{s,p} - C(JH_n^{s,p}).\end{aligned}$$

◀

Theorem 7. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\begin{aligned}&JH_n^{s,p} - \mathbf{i}JH_{n+1}^{s,p} - \varepsilon JH_{n+2}^{s,p} - \mathbf{h}JH_{n+3}^{s,p} =\\ &= J_n(s,p) + J_{n+2}(s,p) - 2J_{n+3}(s,p) - J_{n+6}(s,p).\end{aligned}$$

Proof. By the definition of the (s,p)-Jacobsthal hybrid numbers we get

$$\begin{aligned}&JH_n^{s,p} - \mathbf{i}JH_{n+1}^{s,p} - \varepsilon JH_{n+2}^{s,p} - \mathbf{h}JH_{n+3}^{s,p} =\\ &= J_n(s,p) + \mathbf{i}J_{n+1}(s,p) + \varepsilon J_{n+2}(s,p) + \mathbf{h}J_{n+3}(s,p)\\ &\quad - \mathbf{i}(J_{n+1}(s,p) + \mathbf{i}J_{n+2}(s,p) + \varepsilon J_{n+3}(s,p) + \mathbf{h}J_{n+4}(s,p))\\ &\quad - \varepsilon(J_{n+2}(s,p) + \mathbf{i}J_{n+3}(s,p) + \varepsilon J_{n+4}(s,p) + \mathbf{h}J_{n+5}(s,p))\\ &\quad - \mathbf{h}(J_{n+3}(s,p) + \mathbf{i}J_{n+4}(s,p) + \varepsilon J_{n+5}(s,p) + \mathbf{h}J_{n+6}(s,p))\\ &= J_n(s,p) + J_{n+2}(s,p) - (1 - \mathbf{h})J_{n+3}(s,p)\\ &\quad + (\varepsilon + \mathbf{i})J_{n+4}(s,p) - (\mathbf{h} + 1)J_{n+3}(s,p) - (\varepsilon + \mathbf{i})J_{n+4}(s,p) - J_{n+6}(s,p)\\ &= J_n(s,p) + J_{n+2}(s,p) - 2J_{n+3}(s,p) - J_{n+6}(s,p).\end{aligned}$$

◀

Theorem 8. (*Binet formula*) *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$JH_n^{s,p} = c_1 \hat{r_1} r_1^n + c_2 \hat{r_2} r_2^n,$$

where

$$\begin{aligned}
r_1 &= 2^{s+p-1} + \tfrac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}, \\
r_2 &= 2^{s+p-1} - \tfrac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}, \\
c_1 &= \frac{2^s + 2^p + 2^{s+p} - 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}, \\
c_2 &= \frac{-2^s - 2^p - 2^{s+p} + 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}, \\
\hat{r_1} &= 1 + \mathbf{i} r_1 + \varepsilon r_1^2 + \mathbf{h} r_1^3, \\
\hat{r_2} &= 1 + \mathbf{i} r_2 + \varepsilon r_2^2 + \mathbf{h} r_2^3.
\end{aligned}$$

Proof. By Theorem 1 we get

$$\begin{aligned}
JH_n^{s,p} &= J_n(s,p) + \mathbf{i} J_{n+1}(s,p) + \varepsilon J_{n+2}(s,p) + \mathbf{h} J_{n+3}(s,p) \\
&= c_1 r_1^n + c_2 r_2^n + \mathbf{i}\left(c_1 r_1^{n+1} + c_2 r_2^{n+1}\right) \\
&\quad + \varepsilon\left(c_1 r_1^{n+2} + c_2 r_2^{n+2}\right) + \mathbf{h}\left(c_1 r_1^{n+3} + c_2 r_2^{n+3}\right) \\
&= c_1 r_1^n (1 + \mathbf{i} r_1 + \varepsilon r_1^2 + \mathbf{h} r_1^3) + c_2 r_2^n (1 + \mathbf{i} r_2 + \varepsilon r_2^2 + \mathbf{h} r_2^3) \\
&= c_1 \hat{r_1} r_1^n + c_2 \hat{r_2} r_2^n,
\end{aligned}$$

which ends the proof. ◀

The next theorem presents a summation formula for the (s,p)-Jacobsthal hybrid numbers.

Theorem 9. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\begin{aligned}
\sum_{l=0}^{n} JH_l^{s,p} \;=\; & \frac{JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p}) JH_n^{s,p} - (1 + 2^s + 2^p)(1 + \mathbf{i} + \varepsilon + \mathbf{h})}{2^{s+p}(1 + 2^s + 2^p) - 1} \\
& -\mathbf{i} - \varepsilon(1 + 2^s + 2^p + 2^{s+p}) \\
& -\mathbf{h}(1 + 2^s + 2^p + 2^{s+p} + 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p}).
\end{aligned}$$

Proof. By the definition of the (s,p)-Jacobsthal hybrid numbers we have

$$\begin{aligned}
\sum_{l=0}^{n} JH_l^{s,p} &= JH_0^{s,p} + JH_1^{s,p} + \ldots + JH_n^{s,p} \\
&= J_0(s,p) + \mathbf{i}J_1(s,p) + \varepsilon J_2(s,p) + \mathbf{h}J_3(s,p) \\
&\quad + J_1(s,p) + \mathbf{i}J_2(s,p) + \varepsilon J_3(s,p) + \mathbf{h}J_4(s,p) + \ldots \\
&\quad + J_n(s,p) + \mathbf{i}J_{n+1}(s,p) + \varepsilon J_{n+2}(s,p) + \mathbf{h}J_{n+3}(s,p) \\
&= J_0(s,p) + J_1(s,p) + \ldots + J_n(s,p) \\
&\quad + \mathbf{i}\left(J_1(s,p) + J_2(s,p) + \ldots + J_{n+1}(s,p) + J_0(s,p) - J_0(s,p)\right) \\
&\quad + \varepsilon(J_2(s,p) + J_3(s,p) + \ldots + J_{n+2}(s,p) + J_0(s,p) + J_1(s,p) \\
&\quad - J_0(s,p) - J_1(s,p)) \\
&\quad + \mathbf{h}(J_3(s,p) + J_4(s,p) + \ldots + J_{n+3}(s,p) + J_0(s,p) + J_1(s,p) \\
&\quad + J_2(s,p) - J_0(s,p) - J_1(s,p) - J_2(s,p)).
\end{aligned}$$

Using Theorem 3, we obtain

$$\begin{aligned}
\sum_{l=0}^{n} JH_l^{s,p} &= \tfrac{1}{2^{s+p}(1+2^s+2^p)-1}[J_{n+1}(s,p) + (2^{2s+p} + 2^{s+2p})J_n(s,p) - 1 - 2^s - 2^p \\
&\quad + \mathbf{i}(J_{n+2}(s,p) + (2^{2s+p} + 2^{s+2p})J_{n+1}(s,p) - 1 - 2^s - 2^p) \\
&\quad + \varepsilon(J_{n+3}(s,p) + (2^{2s+p} + 2^{s+2p})J_{n+2}(s,p) - 1 - 2^s - 2^p) \\
&\quad + \mathbf{h}(J_{n+4}(s,p) + (2^{2s+p} + 2^{s+2p})J_{n+3}(s,p) - 1 - 2^s - 2^p))] \\
&\quad - (\mathbf{i}J_0(s,p) + \varepsilon(J_0(s,p) + J_1(s,p)) + \mathbf{h}(J_0(s,p) + J_1(s,p) + J_2(s,p))) \\
&= \tfrac{1}{2^{s+p}(1+2^s+2^p)-1}[J_{n+1}(s,p) + \mathbf{i}J_{n+2}(s,p) + \varepsilon J_{n+3}(s,p) + \mathbf{h}J_{n+4}(s,p) \\
&\quad + (2^{2s+p} + 2^{s+2p})(J_n(s,p) + \mathbf{i}J_{n+1}(s,p) + \varepsilon J_{n+2}(s,p) + \mathbf{h}J_{n+3}(s,p)) \\
&\quad - (1 + 2^s + 2^p)(1 + \mathbf{i} + \varepsilon + \mathbf{h})] \\
&\quad - \mathbf{i} - \varepsilon(1 + 2^s + 2^p + 2^{s+p}) \\
&\quad - \mathbf{h}(1 + 2^s + 2^p + 2^{s+p} + 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p}) \\
&= \frac{JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p})JH_n^{s,p} - (1 + 2^s + 2^p)(1 + \mathbf{i} + \varepsilon + \mathbf{h})}{2^{s+p}(1 + 2^s + 2^p) - 1} \\
&\quad - \mathbf{i} - \varepsilon(1 + 2^s + 2^p + 2^{s+p}) \\
&\quad - \mathbf{h}(1 + 2^s + 2^p + 2^{s+p} + 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p}).
\end{aligned}$$

◀

In particular, we obtain the following formula for the Jacobsthal hybrid numbers.

Corollary 1. *Let $n \geq 1$ be an integer. Then*

$$\sum_{l=0}^{n} JH_l = \frac{JH_{n+2} - JH_1}{2}.$$

Proof. By Theorem 9, for $s = p = 0$ we have

$$\begin{aligned}\sum_{l=0}^{n} JH_l^{0,0} &= \frac{JH_{n+1}^{0,0} + 2JH_n^{0,0} - 3(1+\mathbf{i}+\varepsilon+\mathbf{h})}{2} - (\mathbf{i}+4\varepsilon+9\mathbf{h}) \\ &= \frac{JH_{n+2}^{0,0} - (3+5\mathbf{i}+11\varepsilon+21\mathbf{h})}{2}.\end{aligned}$$

Using the fact that $J_n(0,0) = J_{n+2}$ and $JH_0 = \mathbf{i}+\varepsilon+3\mathbf{h}$, $JH_1 = 1+\mathbf{i}+\varepsilon+5\mathbf{h}$, we get

$$\begin{aligned}\sum_{l=0}^{n} JH_l &= \frac{JH_{n+2} - (3+5\mathbf{i}+11\varepsilon+21\mathbf{h})}{2} + JH_0 + JH_1 \\ &= \frac{JH_{n+2} - (3+5\mathbf{i}+11\varepsilon+21\mathbf{h}) + 2(1+2\mathbf{i}+4\varepsilon+8\mathbf{h})}{2} \\ &= \frac{JH_{n+2} - (1+\mathbf{i}+3\varepsilon+5\mathbf{h})}{2} = \frac{JH_{n+2} - JH_1}{2},\end{aligned}$$

which ends the proof. ◀

Now, we state the followig theorem on the ordinary generating function for the (s,p)-Jacobsthal hybrid numbers.

Theorem 10. *The generating function for the (s,p)-Jacobsthal hybrid sequence $\{JH_n^{s,p}\}$ has the following form*

$$G(x) = \frac{JH_0^{s,p} + (JH_1^{s,p} - 2^{s+p}JH_0^{s,p})x}{1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2}.$$

Proof. Assuming that the generating function of the (s,p)-Jacobsthal hybrid sequence $\{JH_n^{s,p}\}$ has the form $G(x) = \sum_{n=0}^{\infty} JH_n^{s,p}x^n$, we obtain

$$\begin{aligned}&(1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2)G(x) = \\ &= (1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2) \cdot (JH_0^{s,p} + JH_1^{s,p}x + JH_2^{s,p}x^2 + \ldots) \\ &= JH_0^{s,p} + JH_1^{s,p}x + JH_2^{s,p}x^2 + \ldots \\ &\quad - 2^{s+p}JH_0^{s,p}x - 2^{s+p}JH_1^{s,p}x^2 - 2^{s+p}JH_2^{s,p}x^3 - \ldots \\ &\quad - (2^{2s+p} + 2^{s+2p})JH_0^{s,p}x^2 - (2^{2s+p} + 2^{s+2p})JH_1^{s,p}x^3 \\ &\quad - (2^{2s+p} + 2^{s+2p})JH_2^{s,p}x^4 - \ldots \\ &= JH_0^{s,p} + (JH_1^{s,p} - 2^{s+p}JH_0^{s,p})x,\end{aligned}$$

since $JH_n^{s,p} = 2^{s+p}JH_{n-1}^{s,p} + (2^{2s+p} + 2^{s+2p})JH_{n-2}^{s,p}$ and the coefficients of x^n for $n \geq 2$ are equal to zero. ◀

4. Concluding remarks

In [13], it was shown that if the corresponding sequences are increasing, then the hybrid numbers based on these sequences are spacelike. Hence the sequence $\{JH_n^{s,p}\}$ is spacelike. It seems interesting to study, which of the other hybrid numbers with generalized Jacobsthal coefficients are spacelike.

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References

[1] D. Bród, *On a two-parameter generalization of Jacobsthal numbers and its graph interpretation*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, 2018, **LXXII(2)**, 21–28.

[2] D. Bród, *On a new Jacobsthal-type sequence*, Ars Combin., 2020, **150**, 21–29.

[3] A. Dasdemir, *The Representation, Generalized Binet Formula and Sums of The Generalized Jacobsthal p-Sequence*, Hittite Journal of Sci. and Engineering, 2016, **3(2)**, 99–104.

[4] S. Falcon, *On the k-Jacobsthal Numbers*, American Review of Mathematics and Statistics, 2014, **2(1)**, 67–77.

[5] A.F. Horadam, *Jacobsthal and Pell Curves*, Fibonacci Quart., 1988, **26.1**, 79–83.

[6] D. Jhala, K. Sisodiya, G.P.S. Rathore, *On Some Identities for k-Jacobsthal Numbers*, Int. Journal of Math. Analysis, 2013, **7(12)**, 551–556.

[7] C. Kızılateş, *A new generalization of Fibonacci hybrid and Lucas hybrid numbers*, Chaos Solitons Fractals, 2020, **130**, 109449.

[8] M. Özdemir, *Introduction to Hybrid Numbers*, Adv. Appl. Clifford Algebr., 2018, **28** (2018), 1-32, https://doi.org/10.1007/s00006-018-0833-3.

[9] A. Szynal-Liana, *The Horadam hybrid numbers*, Discuss. Math. Gen. Algebra Appl., 2018, **38(1)**, 91–98.

[10] A. Szynal-Liana, A. Włoch, I. Włoch, *On generalized Pell numbers generated by Fibonacci and Lucas numbers*, Ars Combin., 2014, **115**, 411–423.

[11] A. Szynal-Liana, I. Włoch, *On distance Pell numbers and their connections with Fibonacci numbers*, Ars Combin., 2014, **113A**, 65–75.

[12] A. Szynal-Liana, I. Włoch, *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Ann. Math. Sil., 2019, **33**, 276–283.

[13] A. Szynal-Liana, I. Włoch, *On Special Spacelike Hybrid Numbers*, Mathematics, 2020, **8**, 1671, 1–10.

[14] S. Uygun, *The (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas Sequences*, Appl. Math. Sci., 2015, **9(70)**, 3467–3476.

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