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One Remark on the Transformation Operator for Perturbed Hill Operators

Agil Kh. Khanmamedov*, A.F. Mamedova

Abstract. The transformation operator for the perturbed Hill operators is considered. An error is indicated in the proof of the previously proved theorem on the triangular representation of the solution of the perturbed Hill equation. A method for eliminating this disadvantage for a special periodic potential is proposed.

Key Words and Phrases: Hill operators, transformation operators, integral equation, Riemann function.

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In [1], the transformation operator method is used to study the inverse scattering problem for the perturbed Hill operator

$$-y'' + p(x)\, y + q(x)\, y = \lambda y, -\infty < x < +\infty, \tag{1}$$

where $p(x)$ is a piecewise continuous periodic function, $\mathrm{Im}\, \mathrm{p}(\mathrm{x}) = \mathrm{Im}\, \mathrm{q}(\mathrm{x}) = 0$, and

$$\int_{-\infty}^{+\infty} (1 + |x|)\, |q(x)|\, dx < \infty.$$

One of the main goals in [1] was to obtain the analogues of the triangular representation of B.Ya. Levin [2] (these representations were also used in [3,4]). We introduce solutions $\varphi(x, \lambda)$ and $\theta(x, \lambda)$ of the unperturbed equation $-y'' + p(x)\, y = \lambda y, -\infty < x < +\infty$, satisfying the conditions $\varphi(0, \lambda) = 0$, $\varphi'(0, \lambda) = 1$ and $\theta(0, \lambda) = 1$, $\theta'(0, \lambda) = 0$. We put $F(\lambda) = \frac{\varphi'(1,\lambda)+\theta(1,\lambda)}{2}$. Let $k(\lambda) = \arcsin\left(i\sqrt{F^2(\lambda) - 1}\right)$ be the function introduced in [1], and $\Psi_{1,2}(x, k) = e^{\pm ikx}\chi(x, k)$, where $\chi(x + 1, k) = \chi(x, k)$ are the normalized Floquet solutions.

*Corresponding author.

It was proved in [1, 3] that the equation (1) has solutions $F_j(x,k), j = 1,2$, admitting the representations

$$F_j(x,k) = \Psi_j(x,k) \pm \int_x^{\pm\infty} A_j(x,t)\,\Psi_j(x,k)\,dx, j = 1,2. \tag{2}$$

Upon reviewing the proof of the latter fact, it became obvious that the given proof contains significant errors. Namely, in the proof in [1], the function $A(x,t,k)$ is defined ([1], pp. 835) by the formula

$$A(x,t,k) = W^{-1}(k)\left[\Psi_1(x,k)\,\Psi_2(t,k) - \Psi_1(t,k)\,\Psi_2(x,k)\right],$$

where $W(k)$ is the Wronskian of solutions $\Psi_1(x,k)$,$\Psi_2(x,k)$. In particular, this implies that $A(x,x,k) \equiv 0$. Let us introduce the function $\Gamma(x,t,y,z)$ defined by the formula ([1], pp. 835, formula (9))

$$\Gamma(x,t,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(x,t,k)\,\Psi_1(y,k)\,\Psi_2(z,k)\,dk,\ y \geq x.$$

We have $\Gamma(x,x,y,z) \equiv 0$. On the other hand, the kernel $A_1(x,t)$ of the triangular representation (2) (see [1], relations (7), (12)) satisfies the integral equation

$$A_1(x,y) = \int_{\frac{x+y}{2}}^{\infty} q(t)\,\Gamma(x,t,t,y)\,dt +$$

$$+\int_{\frac{x+y}{2}}^{\infty} dt \int_0^{\frac{y-x}{2}} q(t-z)\,A_1(t-z,t+z)\,\Gamma(x,t-z,t+z,y)\,dz.$$

If we put $y = x$ in the last equation, then we get the equality

$$A_1(x,x) = \int_x^{\infty} q(t)\,\Gamma(x,t,t,x)\,dt.$$

It is interesting that after solving the inverse problem the author concludes that the equality ([1], see pp. 841, formulas (23), (24))

$$A_1(x,x) = \frac{1}{2}\int_x^{\infty} q(t)\,dt$$

holds. These relations are valid for any $x \in (-\infty,\infty)$ and any real function $q(x)$ satisfying $\int_{-\infty}^{\infty}(1+|x|)\,|q(x)|\,dx < \infty$. This implies that, for any $t \geq x$, the identity

$$\Gamma(x,t,t,x) = \frac{1}{2}$$

is true, which contradicts the above equality $\Gamma(x,x,y,z) \equiv 0$. In this case, the formula (27) obtained in [1] is nothing but $\Gamma(x,x,x,x) = \frac{1}{2}$.

In this remark we show that the errors found in [1, 3] can be eliminated in the case of the potential $p(x) = \sin x$. Indeed, as follows from the general theory (see [5, 6]), representation (2) is valid if only the kernel $A_1(x,t)$ satisfies the second order hyperbolic equation

$$\frac{\partial A_1(x,t)}{\partial x^2} - \frac{\partial A_1(x,t)}{\partial t^2} - (p(x) - p(t) - q(x)) \ A_1(x,t) = 0, \ t > x,$$

and the boundary conditions $A_1(x,x) = \frac{1}{2}\int_x^\infty q\ (t)\ dt \ \lim\limits_{x+t\to\infty} A_1(x,t) = 0$. Setting $\frac{t+x}{2} = \xi$, $\frac{t-x}{2} = \eta$, $U(\xi,\eta) = U\left(\frac{t+x}{2}, \frac{t-x}{2}\right) = A_1(x,t)$, we obtain for the function $U(\xi,\eta)$ the following equation:

$$\frac{\partial^2 U(\xi,\eta)}{\partial\xi\partial\eta} + 2\cos\xi\sin\eta\cdot U(\xi,\eta) = U(\xi,\eta)\, q(\xi-\eta) \tag{3}$$

with the boundary conditions

$$U(\xi,0) = \frac{1}{2}\int_\xi^\infty q(\alpha)\, d\alpha, \tag{4}$$

$$\lim_{\xi\to\infty} U(\xi,\eta) = 0,\ \eta > 0. \tag{5}$$

Applying the Riemann method to equation (3), as in [1, 3], we reduce the problem (3)-(5) to the integral equation

$$U(\xi_0,\eta_0) = \frac{1}{2}\int_{\xi_0}^\infty R(\xi,0;\xi_0,\eta_0) q(\xi) d\xi + \int_{\xi_0}^\infty d\xi \int_0^{\eta_0} U(\xi,\eta) q(\xi-\eta) R(\xi,\eta;\xi_0,\eta_0) d\eta, \tag{6}$$

where

$$U(\xi,\eta) = K(\xi-\eta,\xi+\eta)\,, R(\xi,\eta,\xi_0,\eta_0) =$$

$$= J_0\left(2\sqrt{(\sin\xi - \sin\xi_0)(\cos\eta_0 - \cos\eta)}\right).$$

$J_0(z)$ is the Bessel function of the first kind. Integral equation (6) is a Volterra-type equation and can be solved by the method of successive approximations ([7, 8]). The validity of representation (2) for $j = 2$ is established in a similar way.

Remark 1. *The proposed method can be extended to the general case where $p(x)$ is any periodic function. This requires a detailed study of the corresponding Riemann function.*

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Agil Kh. Khanmamedov
Azerbaijan University, AZ 1007, Baku, Azerbaijan,
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
Baku Engineering University, AZ 0101, Baku, Azerbaijan
*E-mail:*agil_khanmamedov@yahoo.com

Afag F. Mamedova
Azerbaijan State University of Economics (UNEC), AZ1001, Baku,
E-mail: mammadova_afaq@unec.edu.az

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