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Applying the Fourier Method to Solve One Class of Third Order Differential Equations in Banach Spaces

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Abstract. Mixed problem for one class of third order differential equations with non-linear operator on the right-hand side is considered in the Banach space $B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$, $1 < p < +\infty$, $\alpha_p = \max\{p-2;0\}$. The concept of generalized solution in $B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$ is introduced, and the existence and uniqueness theorems for generalized solution of considered problem are proved. Note that for $p \geq 2$ this problem has been treated in ([1, 2]). The results obtained in this work are the generalizations of previously known corresponding results for $p \geq 2$.

Key Words and Phrases: mixed problem, generalized solution, Fourier method, Banach-valued spaces.

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1. Introduction

Many problems of physics, elasticity theory, mechanics and mathematical physics are reduced to various mixed problems with initial and boundary conditions. Fourier method, Riemann method, Laplace transform method and various iteration methods are used to solve such problems (see [3-17]). In oceanology, when treating the interaction between solitary waves in elastic rods, wave propagation in stratified liquids and many other problems, one has to consider the mixed problems of the form

$$u_{tt}(t,x) - \alpha u_{txx}(t,x) = F(u)(t,x) \tag{1}$$

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with the initial and boundary conditions

$$u(0,x)=\varphi(x), u_t(0,x)=\psi(x), \tag{2}$$

$$u(t,\pi)=u(t,0)=0, \tag{3}$$

where $0<\alpha$ is a fixed number, and F is in general nonlinear operator. Note that the mixed problems for various cases of equation (1) with the conditions close to (2) and (3) have been considered in [1, 2, 18-28]. In [1], the classical, almost everywhere and generalized solutions of the problem (1)-(3) have been studied in the space $B^{2,1}_{2,2,T}$ by using Fourier method. The authors in [1] reduce the problem to the countable system of nonlinear integro-differential equations, which, in turn, is reduced to finding the fixed point of some nonlinear operator in the corresponding space.

In this work, the generalized solution of the mixed problem (1)-(3) is studied in the Banach space $B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}$, $\alpha_p=\max\{p-2;0\}$, $1<p<+\infty$. The concept of generalized solution of (1)-(3) belonging to the Banach space $B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}$ is introduced and the conditions for its existence and uniqueness are found using the method of [1]. The obtained results imply in particular the validity of previously known corresponding facts for $p=2$ and $p>2$.

2. Preliminaries

Let $1<p<+\infty$ and q,p be conjugate numbers. Denote by $L_{p,|p-2|}(0,\pi)$ a Banach space of functions $f(x)\in L_p(0,\pi)$ with the norm

$$\|f\|_{L_{p,|p-2|}(0,\pi)}=\left(\sum_{n=1}^{\infty}n^{|p-2|}\left|f_n\right|^p\right)^{\frac{1}{p}},$$

where $f_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nxdx$. In case where $f_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx$, the corresponding space will be denoted by $L^c_{p,\alpha_p}(0,\pi)$. Consider the space $L_p([0,T],$ $L_{p,|p-2|}(0,\pi))$, a Banach space of functions $f(t,x)\in L_p(D)$, $D=[0,T]\times[0,\pi]$ with the norm

$$\|f\|_{L_p([0,T],L_{p,|p-2|}(0,\pi))}=\left(\sum_{n=1}^{\infty}n^{|p-2|}\int_0^T\left|f_n(t)\right|^p dt\right)^{\frac{1}{p}}.$$

Denote by l_{p,α_p} a Banach space of sequences $\{f_n\}_{n\in N}$ with the norm

$$\left\|\{f_n\}_{n\in N}\right\|_{l_{p,\alpha_p}}=\left(\sum_{n=1}^{\infty}n^{\alpha_p}\left|f_n\right|^p\right)^{\frac{1}{p}}.$$

Let $W^n_{p_1}((0,T), L_{p_2}(0,\pi))$ be a set of functions $f(t,x)$ such that $\left\|\frac{\partial^m f}{\partial t^m}\right\|_{L_{p_2}(0,\pi)} \in L_{p_1}(0,T)$ for every $0 \le m \le n$, and $B^{\alpha_0,\alpha_1,..,\alpha_k}_{\beta_0,\beta_1,..,\beta_k,T}$ be a Banach space of functions $u(t,x)$ of the form

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin nx,$$

considered on the rectangle D, such that $u_n(t) \in C^{(k)}([0,T])$, with the norm

$$\|u\|_{B^{\alpha_0,\alpha_1,..,\alpha_k}_{\beta_0,\beta_1,..,\beta_k T}} = \sum_{i=0}^{k}\left(\sum_{n=1}^{\infty}\left(n^{\alpha_i}\max_{0\le t\le T}\left|u_n^{(i)}(t)\right|\right)^{\beta_i}\right)^{\frac{1}{\beta_i}},$$

where $0 \le \alpha_i$, $1 \le \beta_i$, $i = 0,..,k$, $k \ge 0$ is an integer.

For more information on the properties of the spaces $B^{\alpha_0,\alpha_1,..,\alpha_k}_{\beta_0,\beta_1,..,\beta_k,T}$, we refer the readers to [1].

3. Solvability of the problem (1)-(3)

Let $\alpha_p = \max\{p-2;0\}$ and consider the space $B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}$.

The following remark is true on the relationship between the spaces $B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}$ and $B^{2,1}_{2,2,T}$:

Remark 1. *The space $B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}$ is continuously embedded in the space $B^{2,1}_{2,2,T}$ and*

$$\|u\|_{B^{2,1}_{2,2,T}} \le M_p\,\|u\|_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}},$$

where $M_p = \max\{1; \left(\frac{\pi^2}{6}\right)^{\frac{p-2}{2p}}\}$.

In fact, for $p \ge 2$ we have $B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T} = B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$, and consequently, $M_p = \left(\frac{\pi^2}{6}\right)^{\frac{p-2}{2p}}$ (see [2], Remark 2.1). Let $p < 2$. Then $B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T} = B^{2,1}_{p,p,T}$ and for $u(t,x) \in B^{2,1}_{p,p,T}$ we have

$$\|u\|_{B^{2,1}_{2,2,T}} = \left(\sum_{n=1}^{\infty}\left(n^2\max_{0\le t\le T}|u_n(t)|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty}\left(n\max_{0\le t\le T}\left|u_n^{/}(t)\right|\right)^2\right)^{\frac{1}{2}} \le$$

$$\le \left(\sum_{n=1}^{\infty}\left(n^2\max_{0\le t\le T}|u_n(t)|\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(n\max_{0\le t\le T}\left|u_n^{/}(t)\right|\right)^p\right)^{\frac{1}{p}} = \|u\|_{B^{2,1}_{p,p,T}}.$$

Let's state some properties of the functions belonging to $B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$.

Remark 2. *For $u(t,x) \in B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$, the following properties are true:*
1) $u(t,x), u_t(t,x), u_x(t,x) \in C(D)$;
2) $u_{xx}(t,x), u_{tx}(t,x) \in C([0,T]; L_p(0,\pi))$ and $u_{xx}(t,x) = -\sum_{n=1}^{\infty} n^2 u_n(t) \sin nx$, $u_{tx}(t,x) = \sum_{n=1}^{\infty} n u_n'(t) \cos nx$.

In fact, in case $p \geq 2$ the above statement follows from Remark 1 of ([2]). Let $p < 2$. Then $\alpha_p = 0$ and $u(t,x) \in B_{p,p,T}^{2,1}$. As $u(t,x) \in B_{2,2,T}^{2,1}$, from the results of ([1], Remark 1.1) it follows that $u(t,x), u_t(t,x), u_x(t,x) \in C(D)$ and there exist $u_{xx}(t,x), u_{tx}(t,x) \in C([0,T]; L_2(0,\pi))$ such that $u_{xx}(t,x) = -\sum_{n=1}^{\infty} n^2 u_n(t) \sin nx$, $u_{tx}(t,x) = \sum_{n=1}^{\infty} n u_n'(t) \cos nx$. Further, for $\forall t \in [0,T]$ the convergence of the series $\sum_{n=1}^{\infty} \left(n^2 |u_n(t)|\right)^p$ and $\sum_{n=1}^{\infty} (n |u_n'(t)|)^p$ implies by Riesz theorem ([29], XII. Theorem 2.8) that

$$\left(\int_0^{\pi} |u_{xx}(t,x)|^q \, dx\right)^{\frac{1}{q}} \leq \left(\sum_{n=1}^{\infty} (n^2 |u_n(t)|)^p\right)^{\frac{1}{p}},$$

$$\left(\int_0^{\pi} |u_{tx}(t,x)|^q \, dx\right)^{\frac{1}{q}} \leq \left(\sum_{n=1}^{\infty} (n \left|u_n'(t)\right|)^p\right)^{\frac{1}{p}}.$$

Hence, in view of $p < q$, we obtain

$$\int_0^{\pi} |u_{xx}(t,x)|^p \, dx \leq \pi^{\frac{p(q-p)}{p}} \left(\int_0^{\pi} |u_{xx}(t,x)|^q \, dx\right)^{\frac{p}{q}} \leq \pi^{\frac{p(q-p)}{p}} \sum_{n=1}^{\infty} (n^2 \|u_n\|_{C([0,T])})^p,$$

$$\int_0^{\pi} |u_{tx}(t,x)|^p \, dx \leq \pi^{\frac{p(q-p)}{p}} \left(\int_0^{\pi} |u_{tx}(t,x)|^q \, dx\right)^{\frac{p}{q}} \leq \pi^{\frac{p(q-p)}{p}} \sum_{n=1}^{\infty} (n \left\|u_n'\right\|_{C([0,T])})^p.$$

Thus, $u_{xx}(t,x), u_{tx}(t,x) \in C([0,T]; L_p(0,\pi))$.

Let's give a definition for the generalized solution of the problem (1)-(3) in $B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$.

Definition 1. *Generalized solution of the problem* (1)-(3) *is defined as a function $u(t,x) \in B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$, which satisfies the condition* (2) *and the integral identity*

$$\int_0^T \int_0^{\pi} \left\{u_t(t,x)v_t(t,x) - \alpha u_{xx}(t,x)v_t(t,x) + F(u)(t,x)v(t,x)\right\} dxdt-$$

$$-\alpha \int_0^\pi \varphi^{//}(x)v(0,x)dx + \int_0^\pi \psi(x)v(0,x)dx = 0 \tag{4}$$

for every function $v(t,x)$ *such that* $v(t,x) \in W_1^1([0,T];L_q(0,\pi))$, $v(T,x) = 0$ *for a.e.* $x \in [0,\pi]$.

Let

$$\varphi_n = \frac{2}{\pi}\int_0^\pi \varphi(x)\sin nxdx, \psi_n = \frac{2}{\pi}\int_0^\pi \psi(x)\sin nxdx.$$

Introduce the function

$$w(t,x) = \sum_{n=1}^{\infty} w_n(t)\sin nx,$$

where $w_n(t) = \varphi_n + \frac{1}{\alpha n^2}\left(1 - e^{-\alpha n^2 t}\right)\psi_n$.

The following lemma is true:

Lemma 1. *Assume that* $\varphi(x) \in C^{(1)}([0,\pi]) \bigcap W_p^2(0,\pi)$, $\left\{n^2\varphi_n\right\}_{n\in N} \in l_{p,\alpha_p}$, $\varphi(0) = \varphi(\pi) = 0$, $\psi(x) \in C([0,\pi]) \bigcap W_p^1(0,\pi)$, $\{n\psi_n\}_{n\in N} \in l_{p,\alpha_p}$, $\psi(0) = \psi(\pi) = 0$. *Then* $w(t,x) = \sum_{n=1}^\infty w_n(t)\sin nx \in B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$.

Proof. It is clear that $\|w_n\|_{C[0,T]} \leq |\varphi_n| + \frac{1}{\alpha n^2}|\psi_n|$ and $\left\|w_n^{/}\right\|_{C[0,T]} = |\psi_n|$. Consequently, as $\left\{n^2\varphi_n\right\}_{n\in N}, \{n\psi_n\}_{n\in N} \in l_{p,\alpha_p}$, we obtain

$$\|w\|_{B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}} =$$

$$= \left(\sum_{n=1}^\infty \left(n^{2+\frac{\alpha_p}{p}} \max_{0\leq t\leq T}|w_n(t)|\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^\infty \left(n^{1+\frac{\alpha_p}{p}} \max_{0\leq t\leq T}\left|w_n^{/}(t)\right|\right)^p\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\sum_{n=1}^\infty \left(n^{2+\frac{\alpha_p}{p}}\left(|\varphi_n| + \frac{1}{\alpha n^2}|\psi_n|\right)\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^\infty \left(n^{1+\frac{\alpha_p}{p}}|\psi_n|\right)^p\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\sum_{n=1}^\infty n^{2p+\alpha_p}|\varphi_n|^p\right)^{\frac{1}{p}} + \frac{1}{\alpha}\left(\sum_{n=1}^\infty n^{\alpha_p}|\psi_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^\infty n^{p+\alpha_p}|\psi_n|^p\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\sum_{n=1}^\infty n^{\alpha_p}\left(n^2|\varphi_n|\right)^p\right)^{\frac{1}{p}} + \frac{1+\alpha}{\alpha}\left(\sum_{n=1}^\infty n^{\alpha_p}(n|\psi_n|)^p\right)^{\frac{1}{p}} < +\infty.$$

The lemma is proved. ◄

The next lemma states that the coefficients of the Fourier series of generalized solution satisfy a system of integral equations.

Lemma 2. *If $u(t,x)$ is a generalized solution of the problem (1)-(3) and $F(u) \in L_p([0,T],\, L_{p,|p-2|}(0,\pi))$, then the coefficients $u_n(t)$ of the series $u(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin nx$ satisfy the following countable system of nonlinear integral equations:*

$$u_n(t) = \varphi_n + \frac{1}{\alpha n^2}\left(1 - e^{-\alpha n^2 t}\right)\psi_n + \frac{1}{\alpha n^2}\int_0^t F_n(u,\tau)\left(1 - e^{-\alpha n^2 (t-\tau)}\right)d\tau, \quad (5)$$

where $F_n(u,t)$'s are the Fourier coefficients of the function $u(t,x)$.

Proof. Similar to the case $p = 2$ (see [1], Lemma 1.1), we substitute in (4) arbitrary function $v(t,x)$ of the form

$$v_{\tau,n}(t,x) = \begin{cases} \frac{2}{\pi}(t-\tau)\sin nx, & 0 \le t \le \tau, 0 \le x \le \pi \\ 0, \tau < t \le T, 0 \le x \le \pi \end{cases}$$

$n \in N$, $\tau \in [0,T]$, and obtain the system (5). The lemma is proved. ◄

Consider in the space $L_p([0,T], L_{p,|p-2|}(0,\pi))$ the operator P defined by the formula

$$P(f)(t,x) =$$

$$= \sum_{n=1}^{\infty} \frac{1}{\alpha n^2}\int_0^t f_n(\tau)\left(1 - e^{-\alpha n^2 (t-\tau)}\right)d\tau \sin nx, f \in L_p([0,T], L_{p,|p-2|}(0,\pi)), \quad (6)$$

where $f_n(t) = \frac{2}{\pi}\int_0^{\pi} f(t,x)\sin nx dx$.

Theorem below establishes the boundedness of the operator P.

Theorem 1. *Let the operator P be defined in the space $L_p([0,T], L_{p,|p-2|}(0,\pi))$ by the formula (6). Then $P : L_p([0,T], L_{p,|p-2|}(0,\pi)) \to B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}$, and the relation*

$$\|P(f)\|_{B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} \le L\,\|f\|_{L_p([0,T],L_{p|,p-2|}(0,\pi))} \quad (7)$$

holds, where $L = \frac{q^{\frac{1}{q}}T^{\frac{1}{q}} + \alpha^{\frac{1}{p}}}{\alpha q^{\frac{1}{q}}}$.

Proof. For every $\forall u(t,x) \in B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}$ we have

$$\|P(f)\|_{B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} = \left(\sum_{n=1}^{\infty}\left\{n^{2+\frac{\alpha p}{p}}\max_{[0,T]}\left|\frac{1}{\alpha n^2}\int_0^t f_n(\tau)(1 - e^{-\alpha n^2(t-\tau)})d\tau\right|\right\}^p\right)^{\frac{1}{p}} +$$

$$+\left(\sum_{n=1}^{\infty}\left\{n^{1+\frac{\alpha_p}{p}}\max_{[0,T]}\left|\int_0^t f_n(\tau)e^{-\alpha n^2(t-\tau)}d\tau\right|\right\}^p\right)^{\frac{1}{p}}\leq$$

$$\leq\frac{1}{\alpha}\left(\sum_{n=1}^{\infty}n^{\alpha_p}\int_0^T|f_n(\tau)|^p\,d\tau\max_{[0,T]}\left(\int_0^t(1-e^{-\alpha n^2(t-\tau)})^q d\tau\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}+$$

$$+\left(\sum_{n=1}^{\infty}n^{p+\alpha_p}\int_0^T|f_n(\tau)|^p\,dt\max_{[0,T]}\left(\int_0^t e^{-\alpha n^2q(t-\tau)}d\tau\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\leq$$

$$\leq\frac{T^{\frac{1}{q}}}{\alpha}\left(\sum_{n=1}^{\infty}n^{\alpha_p}\int_0^T|f_n(\tau)|^p\,d\tau\right)^{\frac{1}{p}}+$$

$$+\left(\sum_{n=1}^{\infty}n^{p+\alpha_p}\int_0^T|f_n(\tau)|^p\,dt\left(\frac{1}{\alpha n^2q}\right)^{\frac{p}{q}}\max_{[0,T]}\left(1-e^{-\alpha n^2q\,t}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\leq$$

$$\leq\frac{T^{\frac{1}{q}}}{\alpha}\left(\sum_{n=1}^{\infty}n^{\alpha_p}\int_0^T|f_n(\tau)|^p\,d\tau\right)^{\frac{1}{p}}+\frac{1}{\alpha^{\frac{1}{q}}q^{\frac{1}{q}}}\left(\sum_{n=1}^{\infty}n^{\alpha_p+2-p}\int_0^T|f_n(\tau)|^p\,dt\right)^{\frac{1}{p}}\leq$$

$$\leq\frac{T^{\frac{1}{q}}q^{\frac{1}{q}}+\alpha^{\frac{1}{p}}}{\alpha q^{\frac{1}{q}}}\left(\sum_{n=1}^{\infty}n^{|p-2|}\int_0^T|f_n(\tau)|^p\,d\tau\right)^{\frac{1}{p}}=L\left(\sum_{n=1}^{\infty}n^{|p-2|}\int_0^T|f_n(\tau)|^p\,d\tau\right)^{\frac{1}{p}}.$$

The theorem is proved. ◀

Theorem below proves the uniqueness of generalized solution of the problem (1)- (3).

Theorem 2. *Let the following conditions be satisfied:*

1) $F(u)\in L_p([0,T],L_{p,|p-2|}(0,\pi)),\forall u(t,x)\in B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$;

2) $\forall u(t,x),v(t,x)\in B_{p,p,T}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}$ *and* $t\in[0,T]$*:*

$$\|F(u)(t,\cdot)-F(v)(t,\cdot)\|_{L_{p,|p-2|}(0,\pi)}\leq c(t)\,\|u-v\|_{B_{p,p,t}^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}},\tag{8}$$

where $c(t)\in L_p(0,T)$.

Then the problem (1)-(3) *cannot have more than one generalized solution.*

Proof. Assume the contrary, i.e. assume that the problem (1)-(3) has at least two different generalized solutions $u(t,x)$ and $v(t,x)$. Let $\{u_n(t)\}_{n\in N}$ and

$\{v_n(t)\}_{n\in N}$ be the sequences of coefficients of the functions $u(t,x)$ and $v(t,x)$, respectively. By Lemma 2 and (6) we obtain

$$u(t,x) - v(t,x) = P(F(u) - F(v))(t,x).$$

Then, due to (7) and (8), for $\forall t \in [0;T]$ we have

$$\|u-v\|^p_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,t}} = \|P(F(u)-F(v))\|^p_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,t}} \le$$

$$\le L^p \int_0^t \|F(u)(\tau,\cdot) - F(v)(\tau,\cdot)\|^p_{L_{p,|p-2|}(0,\pi)}\, dt \le$$

$$\le L^p \int_0^t c^p(\tau)\, \|u(\tau,\cdot) - v(\tau,\cdot)\|^p_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,\tau}}\, d\tau.$$

Applying Gronwall-Bellman inequality, we hence obtain $\|u-v\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} = 0$. From this equality it follows that $u(t,x) = v(t,x)$. The theorem is proved. ◄

Next theorem establishes the existence and uniqueness of generalized solution of the problem (1)-(3).

Theorem 3. *Let the following conditions be satisfied:*

1) $\varphi(x) \in C^{(1)}([0,\pi]) \bigcap W_p^2(0,\pi), \{n^2\varphi_n\}_{n\in N} \in l_{p,\alpha_p}, \varphi(0) = \varphi(\pi) = 0, \psi(x) \in C([0,\pi]) \bigcap W_p^1(0,\pi), \{n\psi_n\}_{n\in N} \in l_{p,\alpha_p}, \psi(0) = \psi(\pi) = 0.$

2) $F : B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T} \to L_p([0,T], L_{p,|p-2|}(0,\pi)),\ \forall u \in B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}$ *and* $t \in [0,T]$*:*

$$\|F(u(t,\cdot)\|_{L_{p,|p-2|}(0,\pi)} \le a(t) + b(t)\, \|u\|_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,t}}\,, \tag{9}$$

where $a(t), b(t) \in L_p(0,T)$*;*

3) $\forall u(t,x), v(t,x) \in K\left(\|u\|_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}} \le R\right)$ *and* $t \in [0,T]$*:*

$$\|F(u)(t,\cdot) - F(v)(t,\cdot)\|_{L_{p,|p-2|}(0,\pi)} \le c(t)\, \|u-v\|_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,t}}\,, \tag{10}$$

where

$c(t) \in L_p(0,T),\ R^p = A \exp \int_0^T B^p(t)dt,$

$A = 2^{p-1}\, \|w\|^p_{B^{2+\frac{\alpha_p}{p},1+\frac{\alpha_p}{p}}_{p,p,T}} + L_0^p\, \|a\|^p_{L_p(0,T)},\ \ B(t) = L_0 b(t),\ \ L_0 = 2^{\frac{2}{q}} L.$

Then the problem (1)-(3) *has a unique generalized solution.*

Proof. Define the operator Q in the space $B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}$ by the formula

$$Q(u)(t,x) = w(t,x) + P(F(u))(t,x).$$

From condition 2) and Theorem 1 it follows that $Q : B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}} \to B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}$. Using the inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, by (7) and (9) we obtain

$$\|Q(u)\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} \leq 2^{p-1}\left(\|w\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} + \|P(F(u))\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}}\right) \leq$$

$$\leq 2^{p-1}\left(\|w\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} + L^p \int_0^t \|F(u)(\tau,\cdot)\|^p_{L_{p,|p-2|}(0,\pi)}\, dt\right) \leq$$

$$\leq 2^{p-1}\left(\|w\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} + 2^{p-1}L^p \int_0^t (a^p(\tau) + b^p(\tau)\,\|u\|^p_{B_{p,p,\tau}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}})d\tau\right) =$$

$$= 2^{p-1}\,\|w\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} + L_0^p \int_0^t a^p(\tau)d\tau + L_0^p \int_0^t b^p(\tau)\,\|u\|^p_{B_{p,p,\tau}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}}\, d\tau \leq$$

$$\leq A + \int_0^t B^p(\tau)\,\|u\|^p_{B_{p,p,\tau}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}}\, d\tau. \tag{11}$$

Let's construct the sequence $\{u_k\}_{k=0}^{\infty} \subset B_{p,p,T}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}$ as follows:

$$u_0(t,x) = 0, u_k(t,x) = Q(u_{k-1})(t,x), k = 1,2,..,..$$

By (11), for every $t \in [0,T]$ we obtain

$$\|u_1\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} = \|Q(u_0)\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} \leq A \leq A + A\int_0^t B^p(\tau)d\tau$$

$$\|u_2\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} = \|Q(u_1)\|^p_{B_{p,p,t}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}} \leq A + \int_0^t B^p(\tau)\,\|u_1\|^p_{B_{p,p,\tau}^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}}\, d\tau \leq$$

$$\leq A + \int_0^t B^p(\tau)(A + A\int_0^\tau B^p(s)ds)d\tau =$$

$$= A + A\int_0^t B^p(\tau)d\tau + A\int_0^t B^p(\tau)\int_0^\tau B^p(s)dsd\tau =$$

$$= A(1 + \int_0^t B^p(\tau)d\tau + \int_0^t \frac{\frac{d}{dt}\left(\int_0^\tau B^p(s)ds\right)^2}{2} d\tau) =$$

$$= A(1 + \int_0^t B^p(\tau)d\tau + \frac{\left(\int_0^t B^p(\tau)d\tau\right)^2}{2}).$$

Repeating this reasoning, we obtain

$$\|u_k\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,t}} \leq A(1 + \int_0^t B^p(\tau)d\tau + ... + \frac{\left(\int_0^t B^p(\tau)d\tau\right)^k}{k!}), k \in N.$$

Hence,

$$\|u_k\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}} \leq A \exp \int_0^T B^p(t)dt = R^p,$$

i.e. $u_k(t,x) \in K$, $k \in N$

Now let's prove that the sequence $\{u_k\}_{k\in N} \subset B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}$ is convergent and its limit is the required generalized solution of the problem (1)-(3). For this, let's estimate $\|u_{n+k} - u_k\|_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}}$ for every $n, k = 1, 2, ...$ Taking into account (7) and (10), we have

$$\|u_{n+k} - u_k\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,t}} = \|P(F(u_{n+k-1}) - F(u_{k-1}))\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,t}} =$$

$$\leq L^p \int_0^t \|F(u_{n+k-1})(\tau,\cdot) - F(u_{k-1})(\tau,\cdot)\|^p_{L_{p,|p-2|}(0,\pi)} dt \leq$$

$$\leq L^p \int_0^t c^p(\tau) \|u_{n+k-1} - u_{k-1}\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,\tau}} d\tau.$$

Then

$$\|u_{n+k} - u_k\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,t}} \leq L^p \int_0^t c^p(\tau) \|u_{n+k-1} - u_{k-1}\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,\tau}} d\tau \leq$$

$$\leq L^p \int_0^t c^p(\tau) \left(L^p \int_0^\tau c^p(s) \|u_{n+k-2} - u_{k-2}\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,s}} ds \right) d\tau \leq$$

$$\leq L^{2p} \int_0^t \frac{\frac{d}{d\tau}\left(\int_0^\tau c^p(s)ds\right)^2}{2} \|u_{n+k-2} - u_{k-2}\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,\tau}} d\tau \leq$$

....

$$\le L^{pk}\int_0^t \frac{\frac{d}{d\tau}\left(\int_0^\tau c^p(s)ds\right)^k}{k!}\left\|u_n-u_0\right\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,\tau}}d\tau\le$$

$$\le L^{pk}\left\|u_n\right\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,t}}\int_0^t \frac{\frac{d}{d\tau}\left(\int_0^\tau c^p(s)ds\right)^k}{k!}d\tau=L^{pk}\left\|u_n\right\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,t}}\frac{\left(\int_0^t c^p(\tau)d\tau\right)^k}{k!}.$$

Thus,

$$\left\|u_{n+k}-u_k\right\|^p_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}}\le\frac{L^{pk}R^p\left\|c\right\|^{pk}_{L_p(0,T)}}{k!}.$$

So the sequence $\{u_k\}_{k\in N}$ is fundamental in $B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}$. Let $u(t,x)$ be a limit of the sequence $\{u_k\}_{k\in N}$ in $B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}$. Obviously, $u(t,x)\in K$. Consider the sequence $\{Q(u_k)\}_{k\in N}$. From $u_{k+1}=Q(u_k)$ it follows that the sequence $\{Q(u_k)\}_{k\in N}$ converges to the function $u(t,x)$ in $B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}$. On the other hand, from

$$\left\|Q(u_k)-Q(u)\right\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}}=\left\|P(F(u_k)-F(u))\right\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}}\le$$

$$\le L\left\|F(u_k)-F(u)\right\|_{L_p([0,T],L_{p,|p-2|}(0,\pi))}\le L\left\|c\right\|_{L_p(0,T)}\left\|u_k-u\right\|_{B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}},$$

it follows that $Q(u_k)$ converges to $Q(u)$ in $B^{2+\frac{\alpha p}{p},1+\frac{\alpha p}{p}}_{p,p,T}$ as $k\to\infty$. Thus, $u=Q(u)$ and

$$u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx,$$

where $u_n(t)=\varphi_n+\frac{1}{\alpha n^2}\left(1-e^{-\alpha n^2 t}\right)\psi_n+\frac{1}{\alpha n^2}\int_0^t F_n(u,\tau)\left(1-e^{-\alpha n^2(t-\tau)}\right)d\tau$.

Let's show that $u(t,x)$ is a sought-for function. Obviously, condition (3) is fulfilled for $u(t,x)$. Further, we have

$$u(0,x)=\sum_{n=1}^{\infty}u_n(0)\sin nx=\sum_{n=1}^{\infty}\varphi_n\sin nx=\varphi(x),$$

$$u_t(0,x)=\sum_{n=1}^{\infty}u_n^{/}(0)\sin nx=\sum_{n=1}^{\infty}\psi_n\sin nx=\psi(x),$$

i.e. $u(t,x)$ satisfies the condition (2). It remains to show that the identity (4) is true for $u(t,x)$. Let for $\forall m \in N$

$$u_m(t,x) = \sum_{n=1}^{m} u_n(t) \sin nx,$$

$$J_m = \int_0^T \int_0^\pi \{u_{m,t}(t,x)v_t(t,x) - \alpha u_{m,xx}(t,x)v_t(t,x) + F(u)(t,x)v(t,x)\}\, dxdt-$$
$$-\alpha \int_0^\pi \varphi^{//}(x)v(0,x)dx + \int_0^\pi \psi(x)v(0,x)dx. \tag{12}$$

We have

$$\int_0^T \int_0^\pi u_{m,t}(t,x)v_t(t,x)dxdt = \int_0^T \int_0^\pi \sum_{n=1}^{m} u_n^{/}(t) \sin nx v_t(x,t)dxdt =$$
$$= \sum_{n=1}^{m} \int_0^\pi (\int_0^T u_n^{/}(t)v_t(t,x)dt) \sin nxdx =$$
$$= \sum_{n=1}^{m} \int_0^\pi (u_n^{/}(t)v(t,x)|_0^T - \int_0^T u_n^{//}(t)v(t,x)dt) \sin nxdx =$$
$$= -\int_0^\pi \sum_{n=1}^{m} \psi_n \sin nxv(0,x)dx - \int_0^T \int_0^\pi \sum_{n=1}^{m} u_n^{//}(t) \sin nxv(x,t)dxdt.$$

$$\int_0^T \int_0^\pi u_{m,xx}(t,x)v_t(t,x)dxdt = -\int_0^T \int_0^\pi \sum_{n=1}^{m} n^2 u_n(t)v_t(t,x) \sin nxdxdt =$$

$$= -\sum_{n=1}^{m} n^2 \left(\int_0^\pi (u_n(t)v(t,x)|_0^T - \int_0^T u_n^{/}(t)v(t,x)dt) \sin nxdx \right) =$$
$$= \int_0^\pi \sum_{n=1}^{m} n^2 \varphi_n \sin nxv(0,x)dx + \int_0^T \int_0^\pi \sum_{n=1}^{m} n^2 u_n^{/}(t)v(t,x) \sin nxdxdt.$$

By substituting these relationships into (12), we obtain

$$J_m = \int_0^T \left(\int_0^\pi \{F(u(t,x)) - \sum_{n=1}^{m} (u_n^{//}(t) \sin nx + \alpha n^2 u_n^{/}(t) \sin nx)\} v(t,x)dx \right) dt+$$

$$+\int_0^{\pi}(\psi(x)-\sum_{n=1}^{m}\psi_n\sin nx)v(0,x)dx-\alpha\int_0^{\pi}(\varphi^{//}(x)+\sum_{n=1}^{m}n^2\varphi_n\sin nx)v(0,x)dx=$$

$$=\int_0^T\left(\int_0^{\pi}\{F(u(t,x))-\sum_{n=1}^{m}F_n(u,t)\sin nx\}v(t,x)dx\right)dt+$$

$$+\int_0^{\pi}(\psi(x)-\sum_{n=1}^{m}\psi_n\sin nx)v(0,x)dx-\alpha\int_0^{\pi}(\varphi^{//}(x)+\sum_{n=1}^{m}n^2\varphi_n\sin nx)v(0,x)dx.$$

Finally, using Holder's inequality, we obtain

$$|J_m|\leq\left\|F(u)(t,x)-\sum_{n=1}^{m}F_n(u,t)\sin nx\right\|_{L_p(D)}\|v(t,x)\|_{L_q(D)}+$$

$$+\left\|\psi(x)-\sum_{n=1}^{m}\psi_n\sin nx\right\|_{L_p(0,\pi)}\|v(0,x)\|_{L_q(0,\pi)}+$$

$$+\left\|\varphi^{//}(x)+\sum_{n=1}^{m}n^2\varphi_n\sin nx\right\|_{L_p(0,\pi)}\|v(0,x)\|_{L_q(0,\pi)}.$$

Hence it follows that $J_m\to 0$ as $m\to\infty$, i.e. the identity (4) is true. The uniqueness of generalized solution follows from Theorem 1. The theorem is proved. ◀

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