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# Isomorphism of Two Bases of Exponentials in Weighted Lebesgue Spaces

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**Abstract.** This paper considers a double exponential system with complex-valued coefficients, which is a generalization of the well–known exponential system $\left\{e^{i(n-\alpha\, sign\, n)\,t}\right\}_{n\in Z}$, where $\alpha$ is in general some complex parameter. The study of basis properties of this system has a deep history which dates back to the works of Paley, Wiener and N. Levinson. The well–known Kadets 1/4 theorem also refers to this range of questions. In this paper, it is proved that the double exponential system forms a basis for the weighted Lebesgue space $L_{p,w}(-\pi,\pi)$, $1<p<+\infty$, if and only if it is isomorphic to the classical exponential system $\left\{e^{in\,t}\right\}_{n\in Z}$ in it and the weighted function $w(\cdot)$ satisfies the Muckenhoupt condition.

**Key Words and Phrases**: weighted Lebesgue spaces, exponential system, basis property, Muckenhoupt condition.

**2010 Mathematics Subject Classifications**: 33B10, 46E30, 54D70

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## 1. Introduction

Let us consider the following exponential system

$$\left\{e^{i(n-\alpha\, sign\, n)\,t}\right\}_{n\in Z}, \tag{1}$$

where $\alpha\in R$ is some real parameter. The study of basis properties of this system (completeness, minimality, basicity) in Lebesgue spaces has a deep history which dates back to the works of Paley, Wiener [12] and N. Levinson [13]. It follows from the results of Paley and Wiener [12] that the system (1) forms a Riesz basis for $|\alpha|<\frac{\ln 2}{\pi^2}$ in $L_2(-\pi,\pi)$ and therefore it is isomorphic to the classical exponential system $\left\{e^{in\,t}\right\}_{n\in Z}$ in $L_2(-\pi,\pi)$. This result was strengthened by M. Kadets [14] who obtained it for $|\alpha|<\frac{1}{4}$ and this is conclusive. A criterion was obtained for the basis property of the system (1) in $L_p(-\pi,\pi)$, $1<p<+\infty$, by A.M. Sedletsky [15]. Similar results concerning the sine and cosine systems

$$\{\sin(n-\alpha)\ t\}_{n\in N},\{\cos(n-\alpha)\ t\}_{n\in Z_+}, \tag{2}$$

were obtained in $L_p(-\pi,\pi)$, $1<p<+\infty$, by E.I. Moiseev [16]. These results were extended to the case of complex parameter $\alpha$ by G.G. Devdariani [20, 21]. Further development of these results belongs to B.T. Bilalov [1;2;4;5;10;17-19]. In [19], B.T.Bilalov considered the double exponential system

$$\left\{A(t)\,e^{int};B(t)\,e^{-i(n+1)t}\right\}_{n\in Z_+}, \tag{3}$$

with complex–valued coefficients $A;B:[-\pi,\pi]\to C$. He proved that if under condition $A^{\pm1};B^{\pm1}\in L_\infty(-\pi,\pi)$, the system (3) forms a basis for $L_p(-\pi,\pi)$, $1<p<+\infty$, then it is isomorphic to system $\left\{e^{int}\right\}_{n\in Z}$ in it. In particular, this implies that if the system (3) forms a basis for $L_2(0,\pi)$, then it is a Riesz basis in it. If we take $A(t)=e^{-i\alpha t}$ and $B(t)=e^{i\alpha t}$, then the system (3) will be a generalization of the system (1). The basis properties of the system (3) are well studied (see, e.g., [17, 18]) in Lebesgue spaces $L_p(-\pi,\pi)$,$1\le p\le+\infty$ $(L_\infty(-\pi,\pi)\equiv C[-\pi,\pi])$. It should be noted that the criterion for the basis property of sine (cosine) systems (2) is different from that of the exponentials system (1). Basis properties of the systems (2) in $L_p(0,\pi)$ have been first studied by E.I. Moiseev [16], and then analogous properties of the system (1) in $L_p(-\pi,\pi)$ have been obtained from them. B.T. Bilalov [26] proposed a method for obtaining a criterion for the basis property of the systems (2) from the criterion for basis property of (1). The sine and cosine systems of the form (2) are formed by solving a set of complex and elliptic type equations by the Fourier method (see, e.g., S.M. Ponomarev [22,23], E.I. Moiseev [24,25], etc.). Therefore, the study of the fundamental properties of the system (3) in various function spaces is of great scientific interest both in terms of the theory of partial differential equations and in terms of the theories of approximation and bases.

This paper considers a double exponential system with complex-valued coefficients, which is a generalization of the well-known exponential system $\left\{e^{i(n-\alpha\, sign\, n)\,t}\right\}_{n\in Z}$, where $\alpha$ is in general some complex parameter. The study of basis properties of this system has a deep history which dates back to the works of Paley, Wiener and N. Levinson. The well–known Kadets 1/4 theorem also refers to this range of questions. In this paper, it is proved that the double exponential system forms a basis for the weighted Lebesgue space $L_{p,w}(-\pi,\pi)$, $1<p<+\infty$, if and only if it is isomorphic to the classical exponential system $\left\{e^{int}\right\}_{n\in Z}$ in it and the weighted function $w(\cdot)$ satisfies the Muckenhoupt condition.

Note that the basis properties of perturbed system of exponents in weighted Lebesgue spaces was considered in [7-9].

## 2. Needful information

We will use the standard notation. By $N$ we will denote the natural numbers, by $Z$ the integers, $Z_+ = \{0\} \bigcup N$, by $C$ the complex numbers; by $\exists\,!$ the quantifier "exists and is unique". $KerT$ will denote the kernel of the operator $T$; $R_T$ the range of values of the operator $T$; $X/_M$ the factor space of the space $X$ over the subspace $M$; $\dim M$ the dimension of $M$; $\dot{+}$ the direct sum; $X^*$ the dual space of $X$. $B$-space will be a Banach space; $\|\cdot\|_X$ the norm in $X$, $[X]$ the algebra of bounded operators in $X$. The expression $f{\sim}g$ in $M$ will mean that the following inequality is true: $\exists\,\delta > 0 \ : \ \delta \leq \frac{f(x)}{g(x)} \leq \delta^{-1}$ , $\forall x \in M$. $(\bar{\cdot})$ will denote complex conjugation.

We will say that $\nu(\cdot)$ satisfies the Muckenhoupt condition (see, e.g., [6])$A_p$, if

$$\sup_{I\subset[-\pi,\pi]} \left(\frac{1}{|I|}\int_I \nu(t)\,dt\right)\left(\frac{1}{|I|}\int_I |\nu(t)|^{-\frac{1}{p-1}}\,dt\right)^{p-1} < +\infty.$$

We will denote this fact by $\nu \in A_p$.

We will consider the weighted Lebesgue space $L_{p,\nu} \equiv L_{p,\nu}(-\pi,\pi)$ with a norm

$$\|f\|_{p,\nu} = \left(\int_{-\pi}^{\pi} |f(t)|^p\,\nu(t)\,dt\right)^{\frac{1}{p}},$$

where $\nu : (-\pi,\pi) \to (0,+\infty)$ is some weight function. We need the following statement from [3] about the basis property of exponential system $\left\{ e^{i(n-\alpha\, sign\, n)t}\right\}_{n\in Z}$ in the weighted space $L_{p;w}(-\pi,\pi)$ , $1 < p < +\infty$.

**Proposition 1.** *[3] Let the condition*

$$\left\{w\,;\,\nu^{-p}w\right\} \subset A_p \& \,|\alpha| < 1 \tag{4}$$

*be satisfied, where*

$$\nu(t) = \left|\cos\frac{t}{2}\right|^{\alpha}, t \in (-\pi,\pi)\,.$$

*Then the exponential system*

$$\left\{e^{i(n-\alpha\, sign\, n)t}\right\}_{n\in Z}, \tag{5}$$

*forms a basis for* $L_{p;w}(-\pi,\pi)$ , $1 < p < +\infty$.

It is quite obvious that the relation

$$\nu(t) \sim |t-\pi|^{\alpha}\,|t+\pi|^{\alpha} \ , \ \ t \in (-\pi,\pi)\,,$$

holds. Consider the weight function in the following form:

$$w_0\left(t\right)=\prod_{k=0}^{r}\left|t-t_k\right|^{\alpha_k}\ ,\ \ t\in\left(-\pi,\pi\right),$$

where $-\pi=t_0<t_1<...<t_r=\pi$. Then,

$$\nu^{-p}\left(t\right)\ w_0\left(t\right)\sim\prod_{k=0}^{r}\left|t-t_k\right|^{\tilde{\alpha}_k},$$

where $\tilde{\alpha}_0=\alpha_0-p\alpha$; $\tilde{\alpha}_r=\alpha_r-p\alpha$; $\tilde{\alpha}_k=\alpha_k$, $\forall\,k=\overline{2,r-1}$. It is well known that the conditions $\left\{w_0;\ \nu^{-1}w_0\right\}\subset A_p$ are equivalent to the following inequalities (see, e.g., [6])

$$-1<\alpha_k<p-1\,,\ \ \forall\,k=\overline{0,r}\ ;$$

$$-1<\tilde{\alpha}_k<p-1\,,\ \ \forall\,k=\overline{0,r}\ ;$$

From these relations we immediately obtain the following

**Corollary 1.** *Let the inequalities*

$$-1<\alpha_0-p\alpha\,;\ \alpha_r-p\alpha<p-1\ ;$$

$$-1<\alpha_k<p-1\,,\ \ \forall\,k=\overline{2,r-1},$$

*hold. Then the exponential system* (5) *forms a basis for* $L_{p,w_0}\left(-\pi,\pi\right)\,,\ 1<p<+\infty$.

We will also use the concept of a “double” basis in $B$-space $X$.

Let us introduce the following

**Definition 1.** *A system* $\{x_n^+;\,x_n^-\}_{n\in N}\subset X$ *is called a double basis (or simply a basis) in* $B$*-space* $X$*, if*

$$\left\|\sum_{k=1}^{n_1}\lambda_k^+\,x_k^++\sum_{k=1}^{n_2}\lambda_k^-\,x_k^--x\right\|_X\to0,\quad as\ \ n_1,n_2\to\infty,$$

*for* $\forall\,x\in X\,;\ \ \exists!\ \{\lambda_n^+;\,\lambda_n^-\}_{n\in N}\subset C$.

Let us define weighted Hardy classes.

We denote the usual Hardy class of analytic functions inside (outside) the unit disc $\gamma=\{z:\ |z|=1\}$ on the complex plane by $H_p^+\left({}_mH_p^-\right)$. We define the weighted Hardy classes as follows:

$$H_{p,w}^+=\left\{f\in H_1^+\ :\ f^+\in L_{p,w}\left(\gamma\right)\right\},$$

$$ {}_mH^-_{p,w} = \left\{ f \in {}_mH^-_1 \ : \ f^- \in L_{p,w}(\gamma) \right\}, $$

where $f^+$ $(f^-)$ are the nontangential boundary values of the function $f$ on $\gamma$ from inside (outside) $\gamma$. The norms in these spaces are defined by the expressions

$$ \|f\|_{H^+_{p,w}} = \left\|f^+\right\|_{L_{p,w}(\gamma)}, \|f\|_{{}_mH^-_{p,w}} = \left\|f^-\right\|_{L_{p,w}(\gamma)}. $$

The restriction of the space $H^+_{p,w}$ $\left({}_mH^-_{p,w}\right)$ to $\gamma$ is denoted by $L^+_{p,w}(\gamma)$ $\left({}_mL^-_{p,w}(\gamma)\right)$. The following is true:

**Theorem 1.** *Let $w \in A_p(\gamma)$, $1 < p < +\infty$. Then: i) the system $\{z^n\}_{n\in Z_+}$ $\left(\left\{e^{int}\right\}_{n\in Z_+}\right)$ forms a basis for $H^+_{p,w}$ (for $L^+_{p,w}(\gamma)$) ; ii) the system $\{z^{-n}\}_{n\geq m}$ $\left(\left\{e^{-int}\right\}_{n\geq m}\right)$ forms a basis for ${}_mH^-_{p,w}$ (for ${}_mL^-_{p,w}(\gamma)$), where $L^+_{p,w}(\gamma) = H^+_{p,w}/_\gamma$ $({}_mL^-_{p,w}(\gamma) = {}_mH^-_{p,w}/_\gamma)$ – is the restriction to $\gamma$.*

For example, work [3] can be considered in relation to this result.

## 3. Isomorphism of two bases in $L_{p,w}(\gamma)$

Let us consider the following double exponential system

$$ \left\{ A(\tau)\,\tau^n ; \ \ B(\tau)\,\bar{\tau}^k \right\}_{n\in Z_+;\, k\in N}, \tag{6} $$

with complex-valued coefficients $A; B : \ \gamma \to C$. Let us prove that if, under the condition $A^{\pm 1}; B^{\pm 1} \in L_\infty(\gamma)$ and $w \in A_p(\gamma)$, $1 < p < +\infty$, the system (6) forms a basis for $L_{p,w}(\gamma)$, then it is isomorphic to the classical exponential system $\{\tau^n\}_{n\in Z}$ in $L_{p,w}(\gamma)$. Firstly, we prove that the system (6) forms a basis for $L_{p,w}(\gamma)$ if and only if the Riemann boundary value problem

$$ A(\tau)\,F^+(\tau) + B(\tau)\,F^-(\tau) = f(\tau), \tau \in \gamma, \tag{7} $$

is correctly solvable in the Hardy classes $H^+_{p,w} \times {}_{-1}H^-_{p,w}$. By the solution of the problem (7) we mean a pair of functions $(F^+; \ F^-) \in H^+_{p,w} \times {}_{-1}H^-_{p,w}$, whose boundary values on $\gamma$ satisfy the relation (7) a.e. on $\gamma$.

So, we first assume that the system (6) forms a basis for $L_{p,w}(\gamma)$, $1 < p < +\infty$. We also assume that the condition

$$ A^{\pm 1}; \, B^{\pm 1} \in L_\infty(\gamma) \ \& \ \ w \in A_p(\gamma). \tag{8} $$

is satisfied. Let $g \in L_{p,w}(\gamma)$ be an arbitrary function and let us expand it with respect to the basis (6):

$$ g(\tau) = A(\tau) \sum_{n=0}^{\infty} g_n \tau^n + B(\tau) \sum_{n=1}^{\infty} g_{-n} \tau^{-n}. \tag{9} $$

It is quite obvious that the series $\sum_{n=0}^{\infty} g_n \tau^n$ and $\sum_{n=1}^{\infty} g_{-n} \tau^{-n}$ represent some functions from $L_{p,w}(\gamma)$. Let us assume

$$f(\tau) = \sum_{n=-\infty}^{+\infty} g_n \tau^n \ , \ \tau \in \gamma.$$

We have $f \in L_{p,w}(\gamma)$. Let us show that $f^+ \in L_{p,w}^+$ and $f^- \in {}_{-1}L_{p,w}^-$, where

$$f^+(\tau) = \sum_{n=0}^{+\infty} g_n \tau^n, f^-(\tau) = \sum_{n=1}^{\infty} g_{-n} \tau^{-n} \ , \ \ \tau \in \gamma.$$

We have

$$\int_\gamma f^+(\tau)\, \tau^n d\tau = i \int_{-\pi}^{\pi} f^+(t)\, e^{i(n+1)t} dt = i \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} g_k e^{i(n+k+1)t} dt. \qquad (10)$$

It immediately follows from the Muckenhoupt condition that $w^{-\frac{p'}{p}} \in L_1(\gamma)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then it follows from the estimate

$$\int_{-\pi}^{\pi} |f|\, dt \le \|f\|_{L_{p,w}(\gamma)} \left( \int_{-\pi}^{\pi} |w|^{-\frac{p'}{p}}\, dt \right)^{\frac{1}{p'}}$$

that the series $\sum_{n=0}^{\infty} g_n \tau^n$ and $\sum_{n=1}^{\infty} g_{-n} \tau^{-n}$ converge in $L_1(\gamma)$. Therefore, (10) can be integrated term-by-term and, as a result, we obtain

$$\int_\gamma f^+(\tau)\, \tau^n d\tau = i \sum_{k=0}^{\infty} g_k \int_{-\pi}^{\pi} e^{i(n+k+1)t} dt = 0 \ \ \forall n \in Z_+.$$

By Privalov theorem (see, e.g., [11]), from this relation it follows that $f^+(\cdot)$ is a boundary value on $\gamma$ of some function $F^+ \in H_1^+ \ : \ F^+(\tau) = f^+(\tau)$, a.e. $\tau \in \gamma$. Moreover, it is clear that the Cauchy formula

$$F^+(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^+(\tau)\, d\tau}{\tau - z} \ , \ \ z \in \omega,$$

holds. Therefore $F^+ \in H_1^+$ and from $w \in A_p$ it follows that the Cauchy singular integral is bounded in $L_{p,w}(\gamma)$ (see, e.g. [27])

$$\& \ F^+ \in L_{p,w}(\gamma) \ \Rightarrow \ F^+ \in H_{p,w}^+.$$

We have

$$F^{+}(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f^{+}(\tau)}{\tau-z}d\tau=\frac{1}{2\pi i}\int_{\gamma}\sum_{n=0}^{\infty}g_n\frac{\tau^n d\tau}{\tau-z}=$$

$$=\sum_{n=0}^{\infty}g_n\frac{1}{2\pi i}\int_{\gamma}\frac{\tau^n d\tau}{\tau-z}=\sum_{n=0}^{\infty}g_n z^n, z\in\omega.$$

It can be proved in a completely similar way that the function

$$F^{-}(z)=\sum_{n=1}^{\infty}g_{-n}z^{-n}, z\in C\backslash\bar{\omega},$$

belongs to the class ${}_{-1}H^{-}_{p,w}$ and, moreover, its (nontangential) boundary values $F^{-}(\tau)$ on $\gamma$ a.e. coincide with $f^{-}(\tau): F^{-}(\tau)=f^{-}(\tau)$, a.e. $\tau\in\gamma$. We denote by $T^{\pm}$ the following multiplication operators $T^{\pm}$ in $L_{p,w}(\gamma)$:

$$T^{+}f=Af, T^{-}f=Bf.$$

Since $w\in A_p(\gamma)$, it immediately follows from the previous assertion that we have a direct decomposition

$$L_{p,w}(\gamma)=L^{+}_{p,w}(\gamma)\dot{+}{}_{-1}L^{-}_{p,w}(\gamma), 1<p<+\infty.$$

The projectors generated by this decomposition will be denoted by

$$P^{\pm}: P^{+}: L_{p,w}(\gamma)\to L^{+}_{p,w}(\gamma); P^{-}: L_{p,w}(\gamma)\to {}_{-1}L^{-}_{p,w}(\gamma).$$

Let us consider the operator $T=T^{+}P^{+}+T^{-}P^{-}$. We have

$$Tf=T^{+}P^{+}f+T^{-}P^{-}f=T^{+}f^{+}+T^{-}f^{-}=A(\cdot)F^{+}(\cdot)+B(\cdot)F^{-}(\cdot)=g(\cdot)$$

As a result, the equation

$$Tf=g, g\in L_{p,w}(\gamma), \tag{11}$$

has a solution for $\forall g\in L_{p,w}(\gamma)$, i.e. $R_T=L_{p,w}(\gamma)$. Let us show that $KerT=0$. Let $f\in KerT$. Let us expand $f$ with respect to the basis $\{\tau^n\}_{n\in Z}$:

$$f(\tau)=\sum_{n=-\infty}^{+\infty}f_n\tau^n, \tau\in\gamma,$$

in $L_{p,w}(\gamma)$. In exactly the same way as in the previous case, we show that

$$\sum_{n=0}^{\infty}f_n\tau^{-n}\in L^{+}_{p,w}(\gamma); \sum_{n=-\infty}^{-1}f_n\tau^n\in {}_{-1}L^{-}_{p,w}(\gamma),$$

and it is clear that

$$P^+ f = \sum_{n=0}^{\infty} f_n \tau^n ; P^- f = \sum_{n=-\infty}^{-1} f_n \tau^n .$$

Thus

$$0 = Tf = A(\tau) \sum_{n=0}^{\infty} f_n \tau^n + B(\tau) \sum_{n=1}^{\infty} f_{-n} \tau^{-n} .$$

Since the system (6) is fundamental in $L_{p,w}(\gamma)$, this relation implies $f_n = 0, \forall n \in Z \Rightarrow f = 0 \Rightarrow KerT = 0$. We have

$$\|Tf\|_{L_{p,w}(\gamma)} \le \left\|T^+ P^+ f\right\|_{L_{p,w}(\gamma)} + \left\|T^- P^- f\right\|_{L_{p,w}(\gamma)} \le$$

$$\le \|A\|_{L_\infty(\gamma)} \left\|P^+ f\right\|_{L_{p,w}(\gamma)} + \|B\|_{L_\infty(\gamma)} \left\|P^- f\right\|_{L_{p,w}(\gamma)} .$$

It is quite obvious that the projectors $P^{\pm}$ are continuous, and, as a result, from the previous inequality we obtain $T \in [L_{p,w}(\gamma)]$. Then it follows from Banach theorem that $T^{-1} \in L_{p,w}(\gamma)$, i.e. $T$ is an automorphism.

Now, on the contrary, suppose that the equation (11) is correctly solvable in $L_{p,w}(\gamma)$. Let $g \in L_{p,w}(\gamma)$ be an arbitrary function and let $f = T^{-1} g$. It is clear that $f \in L_{p,w}(\gamma)$. Let us expand $f$ with respect to the basis $\{\tau^n\}_{n \in Z}$ :

$$f(\tau) = \sum_{n=-\infty}^{+\infty} f_n \tau^n , \tau \in \gamma .$$

We have

$$P^+ f = \sum_{n=0}^{+\infty} f_n \tau^n ; P^- f = \sum_{n=-\infty}^{-1} f_n \tau^n .$$

Consequently

$$Tf = A(\tau) \sum_{n=0}^{\infty} f_n \tau^n + B(\tau) \sum_{n=1}^{\infty} f_{-n} \tau^{-n} = g(\tau) , \tau \in \gamma ,$$

and, as a result, an arbitrary element $g \in L_{p,w}(\gamma)$ can be expanded with respect to the system (6). Let us show that such a decomposition is unique. Let

$$A(\tau) \sum_{n=0}^{\infty} f_n \tau^n + B(\tau) \sum_{n=1}^{\infty} f_{-n} \tau^{-n} = 0 .$$

Also let

$$f(\tau) = \sum_{n=-\infty}^{+\infty} f_n \tau^n, \tau \in \gamma.$$

It is quite obvious that $f \in L_{p,w}(\gamma)$ and, as a result, it is clear that

$$f^+(\tau) = P^+ f = \sum_{n=0}^{\infty} f_n \tau^n \in L^+_{p,w}(\gamma);$$

$$f^-(\tau) = P^- f = \sum_{n=-\infty}^{-1} f_n \tau^n \in {}_{-1}L^-_{p,w}(\gamma).$$

Hence

$$Tf = A(\tau) P^+ f + B(\tau) P^- f = 0 \Rightarrow f = T^{-1} 0 = 0 \Rightarrow f_n = 0, \forall n \in Z.$$

The uniqueness of the decomposition is proved. So, the following is true

**Theorem 2.** *Let $A^{\pm 1}; B^{\pm 1} \in L_{p,w}(\gamma)$ & $w \in A_p(\gamma), 1 < p < +\infty$. The system* (6) *forms a basis for $L_{p,w}(\gamma)$ if and only if the equation* (11) *is correctly solvable in $L_{p,w}(\gamma)$, in other words, the Riemann problem*

$$A(\tau) F^+(\tau) + B(\tau) F^-(\tau) = f(\tau), \tau \in \gamma,$$

*has a unique solution in the Hardy classes $H^+_{p,w} \times {}_{-1}H^-_{p,w}$, for $\forall f \in L_{p,w}(\gamma)$.*

The following is also true

**Theorem 3.** *Let all conditions of Theorem 2 be satisfied. The system* (6) *forms a basis for $L_{p,w}(\gamma)$ if and only if it is isomorphic to the exponential system $\{\tau^n\}_{n\in Z}$ in $L_{p,w}(\gamma)$.*

*Proof.* It follows from $w \in A_p(\gamma), 1 < p < +\infty$, that the system $\{\tau^n\}_{n\in Z}$ forms a basis for $L_{p,w}(\gamma)$. Therefore, it is clear that if the system (6) is isomorphic to system $\{\tau^n\}_{n\in Z}$ in $L_{p,w}(\gamma)$, then it also forms a basis for $L_{p,w}(\gamma)$. Now let the system (6) form a basis for $L_{p,w}(\gamma)$. Then it follows from Theorem 2 that the equation (11) is correctly solvable in $L_{p,w}(\gamma)$ for $\forall g \in L_{p,w}(\gamma)$, in other words, the operator $T$ is an automorphism in $L_{p,w}(\gamma)$. We have

$$T[\tau^n] = A(\tau)\tau^n, \forall n \in Z_+, \ T\left[\tau^{-n}\right] = B(\tau)\tau^{-n}, \forall n \in N.$$

This implies that the system (6) also forms a basis for $L_{p,w}(\gamma)$.

The theorem is proved. ◀

This theorem immediately implies the following

**Corollary 2.** *Let $w \in A_p(\gamma)$, $1 < p < +\infty$. The exponential system*

$$\left\{\tau^{n-\alpha\, sign n}\right\}_{n \in Z}$$

*forms a basis for $L_{p,w}(\gamma)$ if and only if it is isomorphic to the system $\{\tau^n\}_{n \in Z}$ in it.*

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