Azerbaijan Journal of Mathematics
V. 12, No 1, 2022, January
ISSN 2218-6816

JS-Prešić Contractive Mappings in Extended Modular S-metric Spaces and Extended Fuzzy S-metric Spaces With an Application

M.M. Rezaee, S. Sedghi*, V. Parvaneh

Abstract. In this paper, we introduce the concept of extended modular S-metric spaces which induce the notion of extended fuzzy S-metric spaces and is a generalization of some classes of metric type spaces. We obtain some results for JS-Prešić contractive mappings in this new setting and in the related fuzzy setting. In fact, we obtain the Prešić fixed points via an easier way than the previous methods. An application in integral equations will support our results.

Key Words and Phrases: contractive mappings, S-metric spaces, modular metric spaces, extended b-metric spaces, fuzzy metric spaces, fixed point.

2010 Mathematics Subject Classifications: 47H10, 54H25

1. Introduction

Banach [1] proposed the Banach Fixed Point Theorem (BCP) to prove the existence of solutions for nonlinear operator equations and integral equations in 1922. It has since become a very reputable implement in tackling a range of problems, including control theory, economic theory, nonlinear analysis, and global analysis, due to its simplicity and utility. Later on, a tremendous amount of literature on applications, generalizations, and extensions of this theorem appeared. They are carried out in various ways by multiple researchers, for example, by weakening the contractive conditions, utilizing different settings, taking into account different mappings with different domains, and so on.

Sedghi et al [19] established the concept of a S-metric space and showed that it is a generalization of both G-metric spaces [21] and D^*-metric spaces [20] (D^*-metric is a modification of D-metric introduced by Dhage [6] and [7]). They've

*Corresponding author.

also proved several fixed point theorems for a self-map on a S-metric space, as well as features of S-metric spaces.

Modular metric spaces were introduced in [2, 3]. A modular on a set attributes a nonnegative field of velocities, and a metric on a set reflects nonnegative finite distances between any two points of the set, i.e., to each time $\lambda > 0$ an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the distance between the points $x, y \in \mho$ it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. The reader can refer to [12] and [13] for more details in this regard.

There is an important generalization of Banach's fixed point theorem called the Prešić generalization [22]. The proofs of the theorems that seek to generalize Prešić theorem, even Prešić result, are relatively long. In this article, in addition to combining Prešić's idea with Jelli's and Samet's idea, we are looking for brief proofs of these results. To do this, we first state and prove a hypothesis about the existence of fixed points in a generalized modular S-metric space. Then, with the help of it, we prove the results about the generalizations of Prešić theorem in these spaces. Modular metric spaces produce fuzzy metric spaces. Also, parametric metric spaces [9] induced, but in a different approach, the fuzzy type metric spaces. Therefore, we can express and prove Prešić-type theorems in this category of spaces by generating generalized fuzzy S-metric spaces.

Let $\mho$ be a nonempty set. Throughout this paper, for a function $\omega : (0, \infty) \times \mho \times \mho \longrightarrow [0, \infty]$, we will write

$$\omega_\lambda(\varrho, \varrho') = \omega(\lambda, \varrho, \varrho'),$$

for all $\lambda > 0$ and $\varrho, \varrho' \in \mho$.

Definition 1. *[2, 3] A function $\omega : (0, \infty) \times \mho \times \mho \longrightarrow [0, \infty]$ is said to be a modular metric on $\mho$ if:*

(i) $\varrho = \varrho'$ *if and only if* $\omega_\lambda(\varrho, \varrho') = 0$, *for all* $\lambda > 0$;

(ii) $\omega_\lambda(\varrho, \varrho') = \omega_\lambda(\varrho', \varrho)$, *for all* $\lambda > 0$, *and* $\varrho, \varrho' \in \mho$;

(iii) $\omega_{\lambda+\mu}(\varrho, \varrho') \leq \omega_\lambda(\varrho, \varrho'') + \omega_\mu(\varrho'', \varrho')$, *for all* $\lambda, \mu > 0$ *and* $\varrho, \varrho', \varrho'' \in \mho$.

Definition 2. *[19] Let $\mho$ be a nonempty set. An S-metric on $\mho$ is a function $S : \mho \times \mho \times \mho \to [0, \infty)$ such that:*

(1) $S(\varrho, \varrho', \varrho'') = 0$ *if and only if* $\varrho = \varrho' = \varrho''$,

(2) $S(\varrho, \varrho', \varrho'') \leq S(\varrho, \varrho, a) + S(\varrho', \varrho', a) + S(\varrho'', \varrho'', a)$,

for each $\varrho, \varrho', \varrho'', a \in \mho$. The pair $(\mho, S)$ is called an S-metric space.

Definition 3. *[10] Let $\mho$ be a non-empty set. A modular S-metric on $\mho$ is a function $\wp_\lambda : (0,\infty) \times \mho \times \mho \times \mho \longrightarrow [0,\infty]$ such that for all $\varrho, \varrho', \varrho'' \in \mho$ and $\lambda, \mu, \nu > 0$:*

(1) $\wp_\lambda(\varrho, \varrho', \varrho'') = 0$ if and only if $\varrho = \varrho' = \varrho''$;

(2) $\wp_{\lambda+\mu+\nu}(\varrho, \varrho', \varrho'') \le \wp_\lambda(\varrho, \varrho, a) + \wp_\mu(\varrho', \varrho', a) + \wp_\nu(\varrho'', \varrho'', a)$ for all $\lambda, \mu, \nu > 0$ and $a \in \mho$.

The pair $(\mho, S)$ is called a modular S-metric.

Definition 4. *[16] Let $\mho$ be a nonempty set. A function $\tilde{d} : \mho \times \mho \to [0,\infty)$ is called an extended b-metric (p-metric, for short) if there exists a strictly increasing continuous function $\pounds : [0,\infty) \to [0,\infty)$ with $\pounds^{-1}(t) \le t \le \pounds(t)$ such that for all $\varrho, \varrho', \varrho'' \in \mho$, the following conditions hold:*

(1) $\tilde{d}(\varrho, \varrho') = 0$ iff $\varrho = \varrho'$,

(2) $\tilde{d}(\varrho, \varrho') = \tilde{d}(\varrho', \varrho)$,

(3) $\tilde{d}(\varrho, \varrho'') \le \pounds(\tilde{d}(\varrho, \varrho') + \tilde{d}(\varrho', \varrho''))$.

In this case, the pair $(\mho, \tilde{d})$ is called an extended b-metric space, or, briefly, a p-metric space.

It's worth noting that the class of p-metric spaces is much bigger than the class of b-metric spaces, because a b-metric is a p-metric when $\pounds(t) = st$ for fixed $s \ge 1$, whereas a metric is a p-metric when $\pounds(t) = t$. To show that a p-metric does not have to be a b-metric, we refer to [16].

Definition 5. *[14] Let $\mho$ be a nonempty set and $b \ge 1$ be a given real number. Suppose that a mapping $S : \mho \times \mho \times \mho \to [0,\infty)$ satisfies for each $\varrho, \varrho', \varrho'', a \in \mho$,*

(1) $S(\varrho, \varrho', \varrho'') \ge 0$ with $\varrho \ne \varrho' \ne \varrho''$,

(2) $S(\varrho, \varrho', \varrho'') = 0$ if and only if $\varrho = \varrho' = \varrho''$,

(3) $S(\varrho, \varrho', \varrho'') \le b(S(\varrho, \varrho, a) + S(\varrho', \varrho', a) + S(\varrho'', \varrho'', a))$ for all $\varrho, \varrho', \varrho'', a \in \mho$.

Then S is called a S_b-metric and the pair $(\mho, S)$ is called a S_b-metric space.

Definition 6. *[15] Let $\mho$ be a nonempty set and $\pounds : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function such that $\pounds^{-1}(t) \le t \le \pounds(t)$ and $\pounds^{-1}(0) = 0 = \pounds(0)$. Suppose that a mapping $\wp : \mho \times \mho \times \mho \to [0,\infty)$ satisfies:*

(1) $\wp(\varrho, \varrho', \varrho'') \geq 0$,

(2) $\wp(\varrho, \varrho', \varrho'') = 0$ *if and only if* $\varrho = \varrho' = \varrho''$,

(3) $\wp(\varrho, \varrho', \varrho'') \leq \pounds(\wp(\varrho, \varrho, a) + \wp(\varrho', \varrho', a) + \wp(\varrho'', \varrho'', a))$ *for all* $\varrho, \varrho', \varrho'', a \in \mho$.

Then $\wp$ *is called an extended* S*-metric and the pair* $(\mho, \wp)$ *is called an extended* S*-metric space.*

Remark 1. *[15] In a* $\wp$*-metric space, we have* $\wp(\varrho, \varrho, \varrho') = \wp(\varrho', \varrho', \varrho)$*, for all* $\varrho, \varrho' \in \mho$.

Each S-metric space is a $\wp$-metric space with $\pounds(t) = t$ and every S_b-metric space with parameter $s \geq 1$ is a $\wp$-metric space with $\pounds(t) = st$.

Proposition 1. *[15] Let* $(\mho, S)$ *be a* S_b*-metric space with coefficient* $s \geq 1$ *and let* $\wp(\varrho, \varrho', \varrho'') = \xi(S(\varrho, \varrho', \varrho''))$*, where* $\xi : [0, \infty) \to [0, \infty)$ *is a strictly increasing continuous function with* $t \leq \xi(t)$ *for* $t \geq 0$ *and* $\xi(0) = 0$*. Then* $\wp$ *is an extended* S*-metric with* $\pounds(t) = \xi(st)$.

Motivated by the works in [8] and [10], we introduce the following new concept.

Definition 7. *Let* $\mho$ *be a nonempty set and* $\pounds : [0, \infty) \to [0, \infty)$ *be a strictly increasing continuous function such that* $\pounds^{-1}(t) \leq t \leq \pounds(t)$ *and* $\pounds^{-1}(0) = 0 = \pounds(0)$*. Suppose that a mapping* $\wp_\lambda : (0, \infty) \times \mho \times \mho \times \mho \to [0, \infty)$ *satisfies:*

(1) $\wp_\lambda(\varrho, \varrho', \varrho'') = 0$ *if and only if* $\varrho = \varrho' = \varrho''$,

(2) $\wp_{\lambda+\mu+\nu}(\varrho, \varrho', \varrho'') \leq \pounds(\wp_\lambda(\varrho, \varrho, a) + \wp_\mu(\varrho', \varrho', a) + \wp_\nu(\varrho'', \varrho'', a))$ *for all* $\lambda, \mu, \nu > 0$ *and* $\varrho, \varrho', \varrho'', a \in \mho$.

Then $\wp_\lambda$ *is called an extended modular* S*-metric (EMSM) and the pair* $(\mho, \wp_\lambda)$ *is called an EMSM space.*

Definition 8. *Let* $(\mho, \wp_\lambda)$ *be an EMSM space and* $A \subset \mho$.

(1) A sequence $\{\varrho_n\}$ *in* $\mho$ *converges to* ϱ *if and only if* $\wp_\lambda(\varrho_n, \varrho_n, \varrho) \to 0$ *as* $n \to \infty$*. That is for each* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\forall n \geq n_0 \Longrightarrow \wp_\lambda(\varrho_n, \varrho_n, \varrho) < \varepsilon,$$

and we denote it by $\lim_{n \longrightarrow \infty} \varrho_n = \varrho$.

(2) A sequence $\{\varrho_n\}$ *in* $\mho$ *is called a Cauchy sequence if for each* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $\wp_\lambda(\varrho_n, \varrho_n, \varrho_m) < \varepsilon$ *for each* $n, m \geq n_0$.

(3) The EMSM space $(\mho, \wp_\lambda)$ *is said to be complete if every Cauchy sequence in it is convergent.*

Theorem 1. *[11, Corollary 2.1] Let $(Ł, d)$ be a complete metric space and let $\Gamma : Ł \to Ł$ be a given map and*

$$\theta(d(\Gamma\varrho, \Gamma\varrho')) \leq \theta(d(\varrho, \varrho'))^k,$$

for all $\varrho, \varrho' \in Ł$ so that $d(\Gamma\varrho, \Gamma\varrho') \neq 0$, where $\theta : (0, \infty) \to (1, \infty)$ is increasing, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$, for each sequence $\{t_n\} \subseteq (0, \infty)$, and there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$ and $k \in (0, 1)$. Then there is a unique fixed point for Γ.

We use Jleli-Samet type contractions to achieve certain generalizations of Banach's fixed point theorem in this study. The results about fixed points of the Prešić type are presented. As a result, we provide a program for solving a system of functional integral equations. These findings generalize a number of similar ones seen in the literature.

2. Main Results

We start this section with the following essential lemma.

Lemma 1. *Let $(Ł, \wp_\lambda)$ be an EMSM space. If there exist sequences $\{\varrho_n\}$ and $\{\varrho'_n\}$ such that $\lim_{n\to\infty} \varrho_n = \varrho$ and $\lim_{n\to\infty} \varrho'_n = \varrho'$, then*

$$\pounds^{-2}(\wp_\lambda(\varrho, \varrho, \varrho')) \leq \limsup_{n\to\infty} \wp_{\frac{\lambda}{9}}(\varrho_n, \varrho_n, \varrho'_n)$$

and

$$\limsup_{n\to\infty} \wp_\lambda(\varrho_n, \varrho_n, \varrho'_n) \leq \pounds^2(\wp_{\frac{\lambda}{9}}(\varrho, \varrho, \varrho')).$$

In particular, if $\varrho = \varrho'$, then we have $\limsup_{n\to\infty} \wp_\lambda(\varrho_n, \varrho_n, \varrho'_n) = 0$. Moreover, suppose that $\{\varrho_n\}$ is convergent to ϱ and $\varrho' \in Ł$ is arbitrary. Then we have

$$\pounds^{-2}(\wp_\lambda(\varrho, \varrho, \varrho')) \leq \limsup_{n\to\infty} \wp_{\frac{\lambda}{9}}(\varrho_n, \varrho_n, \varrho')$$

and

$$\limsup_{n\to\infty} \wp_\lambda(\varrho_n, \varrho_n, \varrho') \leq \pounds^2(\wp_{\frac{\lambda}{9}}(\varrho, \varrho, \varrho')).$$

Proof. a) Using the triangular inequality, we have

$$\begin{aligned}\wp_\lambda(\varrho, \varrho, \varrho') &\leq \pounds(2\wp_{\frac{\lambda}{3}}(\varrho, \varrho, \varrho_n) + \wp_{\frac{\lambda}{3}}(\varrho', \varrho', \varrho_n)) \\ &\leq \pounds(2\wp_{\frac{\lambda}{3}}(\varrho, \varrho, \varrho_n) + \pounds(2\wp_{\frac{\lambda}{9}}(\varrho', \varrho', \varrho'_n)) + \wp_{\frac{\lambda}{9}}(\varrho_n, \varrho_n, \varrho'_n)))\end{aligned}$$

and

$$\begin{aligned}\wp_\lambda(\varrho_n, \varrho_n, \varrho_n') &\le \pounds(2\wp_{\frac{\lambda}{3}}(\varrho_n, \varrho_n, \varrho) + \wp_{\frac{\lambda}{3}}(\varrho_n', \varrho_n', \varrho)) \\ &\le \pounds(2\wp_{\frac{\lambda}{3}}(\varrho_n, \varrho_n, \varrho) + \pounds(2\wp_{\frac{\lambda}{9}}(\varrho_n', \varrho_n', \varrho')) + \wp_{\frac{\lambda}{9}}(\varrho, \varrho, \varrho'))).\end{aligned}$$

As $n \to \infty$, we have

$$\wp_\lambda(\varrho, \varrho, \varrho') \le \pounds^2(\limsup_{n\to\infty} \wp_{\frac{\lambda}{9}}(\varrho_n, \varrho_n, \varrho_n'))$$

and

$$\limsup_{n\to\infty} \wp_\lambda(\varrho_n, \varrho_n, \varrho_n') \le \pounds^2(\wp_{\frac{\lambda}{9}}(\varrho, \varrho, \varrho')).$$

b) Using the triangular inequality, we get

$$\begin{aligned}\wp_\lambda(\varrho, \varrho, \varrho') &\le \pounds(2\wp_{\frac{\lambda}{3}}(\varrho, \varrho, \varrho_n) + \wp_{\frac{\lambda}{3}}\varrho', \varrho', \varrho_n)) \\ &\le \pounds(2\wp_{\frac{\lambda}{3}}(\varrho, \varrho, \varrho_n) + \pounds(2\wp_{\frac{\lambda}{9}}(\varrho', \varrho', \varrho') + \wp_{\frac{\lambda}{9}}(\varrho_n, \varrho_n, \varrho')))\end{aligned}$$

and

$$\begin{aligned}\wp_\lambda(\varrho_n, \varrho_n, \varrho') &\le \pounds(2\wp_{\frac{\lambda}{3}}(\varrho_n, \varrho_n, \varrho) + \wp_{\frac{\lambda}{3}}(\varrho', \varrho', \varrho)) \\ &\le \pounds(2\wp_{\frac{\lambda}{3}}(\varrho_n, \varrho_n, \varrho) + \pounds(2\wp_{\frac{\lambda}{9}}(\varrho', \varrho', \varrho') + \wp_{\frac{\lambda}{9}}(\varrho, \varrho, \varrho'))).\end{aligned}$$

As $n \to \infty$, we have

$$\pounds^{-2}(\wp_\lambda(\varrho, \varrho, \varrho')) \le \limsup_{n\to\infty} \wp_{\frac{\lambda}{9}}(\varrho_n, \varrho_n, \varrho')$$

and

$$\limsup_{n\to\infty} \wp_\lambda(\varrho_n, \varrho_n, \varrho') \le \pounds^2(\wp_{\frac{\lambda}{9}}(\varrho, \varrho, \varrho')). \qquad \blacktriangleleft$$

Denote by Θ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ such that:

(θ1) θ is continuous and increasing;

(θ2) $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} \theta(t_n) = 1$ for all $\{t_n\} \subseteq (0, \infty)$.

Denote by Ψ the set of all functions $\psi : (1, \infty) \to (1, \infty)$ such that:

(ψ_1) ψ is continuous and increasing;

(ψ_2) $\lim_{n\to\infty} \psi^n(t) = 1$ for all $t \in (1, \infty)$.

Now we'll present the study's primary result, which expands and generalizes Banach's well-known fixed point theorem.

Theorem 2. *Let* $(Ł, \wp_\lambda)$ *be a complete EMSM space and let* $\Re : Ł \longrightarrow Ł$ *be a self-mapping such that*

$$\theta(\pounds^2(\wp_\lambda(\Re\varrho, \Re\varrho', \Re\varrho''))) \leq \psi(\theta(\wp_\lambda(\varrho, \varrho', \varrho''))) \tag{1}$$

for all $\varrho, \varrho', \varrho'' \in Ł$, *where* $\theta \in \Theta$ *and* $\psi \in \Psi$. *Then* $\Re$ *has a unique fixed point in* $Ł$.

Proof. Let $\varrho_0 \in Ł$ be arbitrary. We define the sequence $\{\varrho_n\}$ by $\varrho_n = \Re\varrho_{n-1}$. For all $n \in \mathbb{N}$, using (1) we have

$$\begin{aligned} \theta(\wp_\lambda(\varrho_n, \varrho_n, \varrho_{n+1})) &= (\theta(\wp_\lambda(\Re\varrho_{n-1}, \Re\varrho_{n-1}, \Re\varrho_n))) \\ &\leq \psi(\theta(\wp_\lambda(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)) \\ &\leq \psi^2(\theta(\wp_\lambda(\varrho_{n-2}, \varrho_{n-2}, \varrho_{n-1}))) \\ &\leq \cdots \\ &\leq \psi^n(\theta(\wp_\lambda(\varrho_0, \varrho_0, \varrho_1))). \end{aligned} \tag{2}$$

From given assumptions on θ and ψ, we have

$$\lim_{n\longrightarrow\infty} \theta(\wp_\lambda(\varrho_n, \varrho_n, \varrho_{n+1})) = 1, \tag{3}$$

which implies that

$$\lim_{n\longrightarrow\infty} \wp_\lambda(\varrho_n, \varrho_n, \varrho_{n+1}) = 0. \tag{4}$$

Now, we show that $\{\varrho_n\}$ is a Cauchy sequence. Let there exists $\varepsilon > 0$ such that for all $i \in \mathbb{N}$ there are m_i, n_i with $i < m_i < n_i$ and

$$\wp_\lambda(\varrho_{m_i}, \varrho_{m_i}, \varrho_{n_i}) \geq \varepsilon, \tag{5}$$

that is,

$$\wp_\lambda(\varrho_{m_i}, \varrho_{m_i}, \varrho_{n_i-1}) < \varepsilon. \tag{6}$$

From (5), we have

$$\wp_{9\lambda}(\varrho_{m_i}, \varrho_{m_i}, \varrho_{n_i}) \leq \pounds(2\wp_{3\lambda}(\varrho_{m_i}, \varrho_{m_i}, \varrho_{m_i+1}) + \wp_{3\lambda}(\varrho_{n_i}, \varrho_{n_i}, \varrho_{m_i+1}))$$

$$\leq \pounds(2\wp_{3\lambda}(\varrho_{m_i}, \varrho_{m_i}, \varrho_{m_i+1}) + \pounds(\wp_\lambda(\varrho_{m_i+1}, \varrho_{m_i+1}, \varrho_{n_i}))).$$

Letting $i \to \infty$ and using (4), we get

$$\pounds^{-2}(\varepsilon) \leq \limsup_{i\to\infty} \wp_\lambda(\varrho_{m_i+1}, \varrho_{m_i+1}, \varrho_{n_i}). \tag{7}$$

On the other hand, we have

$$\begin{aligned}\theta(\pounds^2(\wp_\lambda(\varrho_{m_i+1}, \varrho_{m_i+1}, \varrho_{n_i}))) &= \theta(\pounds^2(\wp_\lambda(\Re\varrho_{m_i}, \Re\varrho_{m_i}, \Re\varrho_{n_i-1}))) \\ &\leq \psi[\theta(\wp_\lambda(\varrho_{m_i}, \varrho_{m_i} \cdot \varrho_{n_i-1}))]. \end{aligned} \tag{8}$$

Using now ($\theta 1$) and (4)-(7), we have

$$\begin{aligned}\theta(\varepsilon) &= \theta(\pounds^2 \cdot \pounds^{-2}(\varepsilon)) \\ &\leq \theta(\pounds^2(\limsup_{i\to\infty} \wp_\lambda(\varrho_{m_i+1}, \varrho_{m_i+1}, \varrho_{n_i}))) \qquad (9) \\ &\leq \psi[\theta(\limsup_{i\to\infty}(\wp_\lambda(\varrho_{m_i}, \varrho_{m_i}, \varrho_{n_i-1}))] \\ &\leq \psi(\theta(\varepsilon)). \qquad (10)\end{aligned}$$

This implies that

$$1 \leq \theta(\varepsilon) \leq \psi(\theta(\varepsilon)) < \theta(\varepsilon),$$

which is a contradiction.

Thus, $\{\varrho_n\}$ is a $\wp_\lambda$-Cauchy sequence in the EMSM space $(Ł, \wp_\lambda)$. $\wp_\lambda$-completeness of Ł yields that $\{\varrho_n\}$ $\wp_\lambda$-converges to a point $\varrho^* \in Ł$. Now, we show that ϱ^* is a fixed point of $\Re$. First, let $\Re$ be continuous. Then we have

$$\varrho^* = \lim_{n\to\infty} \varrho_{n+1} = \lim_{n\to\infty} \Re\varrho_n = \Re\varrho^*.$$

Let $\Re$ be not continuous. Now, by Lemma 1,

$$\begin{aligned}\theta(\pounds^2 \cdot \pounds^{-2}(\wp_\lambda(u, u, \Re u))) &\leq \theta(\pounds^2 \limsup_{n\to\infty} \tilde{\wp_{\frac{\lambda}{9}}}(\Re\varrho_{n+1}, \Re\varrho_{n+1}, \Re u)) \\ &\leq \psi(\theta(\limsup_{n\to\infty} \tilde{\wp_{\frac{\lambda}{81}}}(\varrho_{n+1}, \varrho_{n+1}, u))) \to 0 \ as \ n \to \infty. \end{aligned} \tag{11}$$

Therefore, we deduce that $\wp_\lambda(u, u, \Re u) = 0$, so $u = \Re u$. Finally, suppose that u and v are two fixed points of $\Re$ such that $u \neq v$. Then, by (1), we have

$$\begin{aligned}\theta(\wp_\lambda(u, u, v)) &\leq \theta(\pounds^2(\wp_\lambda(\Re u, \Re u, \Re v))) \\ &\leq \psi[\theta(\wp_\lambda(u, u, v))] < \theta(\wp_\lambda(u, u, v)). \end{aligned} \tag{12}$$

It is a contradiction. So, we deduce that $u = v$. Therefore, $\Re$ has a unique fixed point. ◀

Let ζ denote the class of all functions $\varphi : [0,\infty) \to [0, \pounds^{-1}(1))$ satisfying the following condition:

$$\limsup_{n\to\infty} \varphi(t_n) = \pounds^{-1}(1) \text{ implies that } t_n \to 0, \text{ as } n \to \infty.$$

Taking $\psi(t) = t^{\varphi(t)}$ so that $\lim_{n\to\infty} (\varphi(t))^n = 0$ for all $t \in [0,\infty)$ in Theorem 2, we have,

Corollary 1. *Let $(Ł, \wp_\lambda)$ be a complete EMSM space and let $\Re : Ł \longrightarrow Ł$ be a self-mapping such that*

$$\theta(\pounds^2(\wp_\lambda(\Re\varrho, \Re\varrho', \Re\varrho''))) \le (\theta(\wp_\lambda(\varrho, \varrho', \varrho'')))^{\varphi(\wp_\lambda(\varrho,\varrho',\varrho''))} \tag{13}$$

for all $\varrho, \varrho', \varrho'' \in Ł$, where $\theta \in \Theta$ and $\varphi \in \zeta$. Then $\Re$ has a unique fixed point in $Ł$.

Taking $\psi(t) = t^k$, $k \in (0,1)$ in Theorem 2, we have,

Corollary 2. *Let $(Ł, \wp_\lambda)$ be a complete EMSM space and let $\Re : Ł \longrightarrow Ł$ be a self-mapping such that*

$$\theta(\pounds^2(\wp_\lambda(\Re\varrho, \Re\varrho', \Re\varrho''))) \le \theta(\wp_\lambda(\varrho, \varrho', \varrho''))^k \tag{14}$$

for all $\varrho, \varrho', \varrho'' \in Ł$, where $\theta \in \Theta$. Then $\Re$ has a unique fixed point in $Ł$.

Taking $\theta(t) = 1 + \ln(1 + t)$ and $\psi(t) = 1 + \ln(t)$ in Theorem 2, we have,

Corollary 3. *Let $(Ł, \wp_\lambda)$ be a complete EMSM space and let $\Re : Ł \longrightarrow Ł$ be a self-mapping such that*

$$1 + \ln(1 + \pounds^2(\wp_\lambda(\Re\varrho, \Re\varrho', \Re\varrho''))) \le 1 + \ln(1 + \ln(1 + \wp_\lambda(\varrho, \varrho', \varrho''))) \tag{15}$$

for all $\varrho, \varrho', \varrho'' \in L$. Then $\Re$ has at least one fixed point in L.

Taking $\theta(t) = e^t$ in Corollary 2, we have,

Remark 2. *Let $(Ł, \wp_\lambda)$ be a complete modular S_b metric space and let $\Re : Ł \longrightarrow Ł$ be a self-mapping such that*

$$\wp_\lambda(\Re\varrho, \Re\varrho, \Re\varrho') \le \frac{k}{s^2}(\wp_\lambda(\varrho, \varrho', \varrho'')) \tag{16}$$

for all $\varrho, \varrho', \varrho'' \in Ł$. Then $\Re$ has a unique fixed point in $Ł$.

Example 1. *Let* $\mho = [0, 8]$ *be equipped with the* $\wp$*-metric*

$$\wp(\varrho, \varrho', \varrho'') = \sinh(\frac{|\varrho' - \varrho''| + |\varrho'' - \varrho|}{\lambda})$$

for all $\varrho, \varrho', \varrho'' \in \mho$*, where* $\pounds(\varrho) = \sinh \varrho$ *with* $\pounds^{-1}(\varrho) = \sinh^{-1}(\varrho)$*.*

Define the function $f : [0, 8] \to [0, 2]$ *by*

$$f\varrho = \sqrt{2 + \frac{\varrho}{100}}$$

and the function β *by* $\beta(t) = e^{-t}$*.*

For all $\varrho, \varrho' \in \mho$*, we have,*

$$\begin{aligned}
\pounds^2(\wp(f\varrho, f\varrho', f\varrho'')) &= \sinh(\sinh(\sinh(\frac{|\sqrt{2 + \frac{\varrho'}{100}} - \sqrt{2 + \frac{\varrho''}{100}}| + |\sqrt{2 + \frac{\varrho''}{100}} - \sqrt{2 + \frac{\varrho}{100}}|}{\lambda})))\\
&\leq \sinh(\sinh(\sinh(\frac{|\frac{\varrho'}{100} - \frac{\varrho''}{100}| + |\frac{\varrho''}{100} - \frac{\varrho}{100}|}{\lambda})))\\
&\leq \sinh(\sinh(\frac{\wp(\varrho, \varrho', \varrho'')}{100}))\\
&\leq e^{-\wp(\varrho, \varrho', \varrho'')}\wp(\varrho, \varrho', \varrho'') = \beta(\wp(\varrho, \varrho', \varrho''))\wp(\varrho, \varrho', \varrho'').
\end{aligned}$$

So, by taking the exp and using Theorem 2, we arrive at the conclusion that f *has a fixed point.*

3. Prešić type fixed point results

We recall one of the most powerful results in nonlinear analysis, the Banach contraction principle (BCP), [1]. In the background of ODE and PDE, it has numerous applications.

Theorem 3. *[1] Let* (Δ, d) *be a complete metric space and let* $\Re : \Delta \to \Delta$ *be such that*

$$d(\Re\iota, \Re\kappa) \leq \gamma d(\iota, \kappa) \text{ for all } \iota, \kappa \in \Delta,$$

where $\gamma \in [0, 1)$*. Then, there is a unique* σ *in* Δ *such that* $\sigma = \Re\sigma$*. Also, for each* $\zeta_0 \in \Delta$*, the sequence* $\zeta_{n+1} = \Re\zeta_n$ *converges to* σ*.*

The BCP has been expanded and generalized in a variety of ways. Prešić [22] came up with the following outcome.

Theorem 4. *[22] Let* (Δ, d) *be a complete metric space and let* $\Re : \Delta^k \to \Delta$ *(*k *is a positive integer). Suppose that*

$$d(\Re(\zeta_1, ..., \zeta_k), \Re(\zeta_2, ..., \zeta_{k+1})) \leq \sum_{i=1}^{k} \lambda_i d(\zeta_i, \zeta_{i+1}) \tag{17}$$

for all $\zeta_1, ..., \zeta_{k+1}$ *in* Δ, *where* $\lambda_i \geq 0$ *and* $\sum_{i=1}^{k} \lambda_i \in [0,1)$. *Then* $\Re$ *has a unique fixed point* ζ^* *(that is,* $\Upsilon(\zeta^*, ..., \zeta^*) = \zeta^*$*). Moreover, for all arbitrary points* $\zeta_1, ..., \zeta_{k+1}$ *in* Δ, *the sequence* $\{\zeta_n\}$ *defined by* $\zeta_{n+k} = \Re(\zeta_n, \zeta_{n+1}, ..., \zeta_{n+k-1})$, *converges to* ζ^*.

It is obvious that for $k = 1$, Theorem 4 coincides with the BCP.

The above theorem has been generalized by Ćirić and Presić [5] as follows.

Theorem 5. *[5] Let* (Δ, d) *be a complete metric space and* $\Re : \Delta^k \to \Delta$ *(k is a positive integer). Suppose that*

$$d(\Re(\zeta_1, ..., \zeta_k), \Re(\zeta_2, ..., \zeta_{k+1})) \leq \lambda \max\{d(\zeta_i, \zeta_{i+1}) : 1 \leq i \leq k\}, \tag{18}$$

for all $\zeta_1, ..., \zeta_{k+1}$ *in* Δ, *where* $\lambda \in [0,1)$. *Then* $\Re$ *has a fixed point* $\zeta^* \in \Delta$. *Also, for all points* $\zeta_1, ..., \zeta_{k+1} \in \Delta$, *the sequence* $\{\zeta_n\}$ *defined by* $\zeta_{n+k} = \Re(\zeta_n, \zeta_{n+1}, ..., \zeta_{n+k-1})$, *converges to* ζ^*. *The fixed point of* $\Re$ *is unique if*

$$d(\Re(\rho, ..., \rho), \Re(\varrho, ..., \varrho)) < d(\rho, \varrho),$$

for all $\rho, \varrho \in \Delta$ *with* $\rho \neq \varrho$.

For more details on Presić type contractions, we refer the reader to [4, 22].

Theorem 6. *Suppose that* $\wp_{\lambda 1}, \wp_{\lambda 2}, \ldots, \wp_{\lambda n}$ *are some EMSM on nonempty sets* $Ł_1, Ł_2, \ldots, Ł_n$, *respectively, and let* $\upsilon : [0,\infty)^n \longrightarrow [0,\infty)$ *be such* $\upsilon(\sigma_1, \ldots, \sigma_n) = 0$ *if and only if* $\sigma_i = 0$ *for all* $i = 1, 2, 3, \ldots, n$ *and*

$$\begin{aligned} &\upsilon(a_{11} + a_{12} + a_{13}, a_{21} + a_{22} + a_{23}, ..., a_{n1} + a_{n2} + a_{n3}) \\ &\leq £[\upsilon(a_{11}, a_{21}, \ldots, a_{n1}) + \upsilon(a_{12}, a_{22}, \ldots, a_{n2}) + \upsilon(a_{13}, a_{23}, \ldots, a_{n3})] \end{aligned} \tag{19}$$

for all $a_{ij} \in [0,\infty)$. *Then*

$$\begin{aligned} &\widetilde{\wp}_\lambda((\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), (\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), (\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})) \\ &= \upsilon(\wp_{\lambda 1}(\varrho_{11}, \varrho_{21}, \varrho_{31}), \wp_{\lambda 2}(\varrho_{12}, \varrho_{22}, \varrho_{32}), \ldots, \wp_{\lambda n}(\varrho_{1n}, \varrho_{2n}, \varrho_{3n})), \end{aligned}$$

is an EMSM in $[Ł_1 \times Ł_2 \times \ldots \times Ł_n]^3$.

Proof. We only show the triangular inequality in an EMSM space. Let $\varrho_{ij} \in Ł_j$ for all $1 \leq i \leq 3$ and $1 \leq j \leq n$. So,

$$\widetilde{\wp}_{\lambda_1+\lambda_2+\lambda_3}((\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), (\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), (\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}))$$
$$= \upsilon(\wp_{(\lambda_1+\lambda_2+\lambda_3)_1}(\varrho_{11}, \varrho_{21}, \varrho_{31}), \wp_{(\lambda_1+\lambda_2+\lambda_3)_2}(\varrho_{12}, \varrho_{22}, \varrho_{32}), \ldots, \wp_{(\lambda_1+\lambda_2+\lambda_3)_n}(\varrho_{1n}, \varrho_{2n}, \varrho_{3n}))$$

$$\leq \upsilon(\wp_{\lambda_1 1}(\varrho_{11},\varrho_{11},a_1)+\wp_{\lambda_2 1}(\varrho_{21},\varrho_{21},a_1)+\wp_{\lambda_3 1}(\varrho_{31},\varrho_{31},a_1)),$$
$$\wp_{\lambda_1 2}(\varrho_{12},\varrho_{12},a_2)+\wp_{\lambda_2 2}(\varrho_{22},\varrho_{22},a_2)+\wp_{\lambda_3 2}(\varrho_{32},\varrho_{32},a_2)),$$
$$...,\wp_{\lambda_1 n}(\varrho_{1n},\varrho_{1n},a_n)+\wp_{\lambda_2 n}(\varrho_{2n},\varrho_{2n},a_n)+\wp_{\lambda_3 n}(\varrho_{3n},\varrho_{3n},a_n))$$
$$\leq \pounds[\upsilon(\wp_{\lambda_1 1}(\varrho_{11},\varrho_{11},a_1),\wp_{\lambda_1 2}(\varrho_{12},\varrho_{12},a_2),\dots,\wp_{\lambda_1 n}(\varrho_{1n},\varrho_{1n},a_n))$$
$$+\upsilon(\wp_{\lambda_2 1}(\varrho_{21},\varrho_{21},a_1),\wp_{\lambda_2 2}(\varrho_{22},\varrho_{22},a_2),\dots,\wp_{\lambda_2 n}(\varrho_{2n},\varrho_{2n},a_n))$$
$$+\upsilon(\wp_{\lambda_3 1}(\varrho_{31},\varrho_{31},a_1),\wp_{\lambda_3 2}(\varrho_{32},\varrho_{32},a_2),\dots,\wp_{\lambda_3 n}(\varrho_{3n},\varrho_{3n},a_n))]$$
$$\leq \pounds(\widetilde{\wp}_{\lambda_1}((\varrho_{11},\varrho_{12},...,\varrho_{1n}),(\varrho_{11},\varrho_{12},...,\varrho_{1n}),(a_1,a_2,...,a_n))$$
$$+\widetilde{\wp}_{\lambda_2}((\varrho_{21},\varrho_{22},...,\varrho_{2n}),(\varrho_{21},\varrho_{22},...,\varrho_{2n}),(a_1,a_2,...,a_n))$$
$$+\widetilde{\wp}_{\lambda_3}((\varrho_{31},\varrho_{32},...,\varrho_{3n}),(\varrho_{31},\varrho_{32},...,\varrho_{3n}),(a_1,a_2,...,a_n))). \quad \blacktriangleleft$$

From now on, let θ be a subadditive mapping.

Theorem 7. *Let $(\L,\wp_\lambda)$ be an EMSM space and $\Re : \L^n \to \L$ be a continuous function such that*

$$\theta\Big(\pounds^2\Big[\wp_\lambda\Big(\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n})\Big)\Big]\Big)$$
$$\leq \frac{1}{n}\psi\Big(\Big[\theta\Big(\wp_\lambda(\varrho_{11},\varrho_{21},\varrho_{31})+\wp_\lambda(\varrho_{12},\varrho_{22},\varrho_{32})+\dots+\wp_\lambda(\varrho_{1n},\varrho_{2n},\varrho_{3n})\Big)\Big]\Big) \quad (20)$$

for all $\varrho_{ij} \subseteq \L$, where $\theta \in \Theta$ is a subadditive mapping and $\psi \in \Psi$. Then $\Re$ has at least a Presić type fixed point.

Proof. We define the mapping $\widetilde{\Re} : \L^n \to \L^n$ by

$$\widetilde{\Re}(\sigma_1,...,\sigma_n) = (\Re(\sigma_1,...,\sigma_n),...,\Re(\sigma_1,...,\sigma_n)).$$

Clearly, $\widetilde{\Re}$ is continuous. We demonstrate that $\widetilde{\Re}$ satisfies all the conditions of Theorem 2. We know that

$$\widehat{S}((\varrho_{11},\varrho_{12},...,\varrho_{1n}),(\varrho_{21},\varrho_{22},...,\varrho_{2n}),(\varrho_{31},\varrho_{32},...,\varrho_{3n}))$$
$$= \wp_\lambda(\varrho_{11},\varrho_{21},\varrho_{31})+\wp_\lambda(\varrho_{12},\varrho_{22},\varrho_{32})+\dots+\wp_\lambda(\varrho_{1n},\varrho_{2n},\varrho_{3n})$$

is an EMSM. From (20) we have

$$\theta\Big(\pounds^2(\widehat{S}(\widetilde{\Re}(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\widetilde{\Re}(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\widetilde{\Re}(\varrho_{31},\varrho_{32},...,\varrho_{3n})))\Big)$$

$$= \theta\Big(\widehat{S}((\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),...,\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n})),$$

$$(\Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), ..., \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n})),$$
$$(\Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}), ..., \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}))\Big)$$
$$= \theta\Big(n\wp_\lambda(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}))\Big)$$
$$\leq n\theta\Big(\wp_\lambda(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}))\Big)$$
$$\leq \psi\Big(\theta\Big(\wp_\lambda(\varrho_{11}, \varrho_{21}, \varrho_{31}) + \wp_\lambda(\varrho_{12}, \varrho_{22}, \varrho_{32}) + \ldots + \wp_\lambda(\varrho_{1n}, \varrho_{2n}, \varrho_{3n})\Big)\Big)$$
$$= \psi\Big(\theta\Big(\widehat{S}((\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), (\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), (\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}))\Big)\Big).$$

Now, according to Theorem 2 we deduce that $\widetilde{\Re}$ admits at least a fixed point which implies that there exist $\sigma_1, ..., \sigma_n$ such that $\Re(\sigma_1, ..., \sigma_n) = \sigma_1 = ... = \sigma_n$, that is, $\Re$ possesses at least a Presić type fixed point. ◄

Theorem 8. *Let $(L, \wp_\lambda)$ be an EMSM space and $\Re : L^n \to L$ be a continuous function such that*

$$\theta\Big(\pounds^2(\wp_\lambda(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})))\Big)$$
$$\leq \psi\Big(\theta\Big(\max\{\wp_\lambda(\varrho_{11}, \varrho_{21}, \varrho_{31}), \wp_\lambda(\varrho_{12}, \varrho_{22}, \varrho_{32}), \ldots, \wp_\lambda(\varrho_{1n}, \varrho_{2n}, \varrho_{3n})\}\Big)\Big) \quad (21)$$

for all $\varrho_{ij} \in L$, where $\theta \in \Theta$ and $\psi \in \Psi$. Then $\Re$ has at least a Presić type fixed point.

Proof. Let $\widetilde{\Re} : L^n \to L^n$ be defined by

$$\widetilde{\Re}(\sigma_1, ..., \sigma_n) = (\Re(\sigma_1, ..., \sigma_n), ..., \Re(\sigma_1, ..., \sigma_n)).$$

Evidently, $\widetilde{\Re}$ is continuous. It will be demonstrated that $\widetilde{\Re}$ satisfies all the conditions of Theorem 2. We know that

$$\widehat{S}((\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), (\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), (\varrho_{31}, \varrho_{32}, ..., \varrho_{3n}))$$
$$= \max\{\wp_\lambda(\varrho_{11}, \varrho_{21}, \varrho_{31}), \wp_\lambda(\varrho_{12}, \varrho_{22}, \varrho_{32}), \ldots, \wp_\lambda(\varrho_{1n}, \varrho_{2n}, \varrho_{3n})\}$$

is an EMSM.

According to (21), we have

$$\theta\Big(\pounds^2(\widehat{S}(\widetilde{\Re}(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \widetilde{\Re}(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \widetilde{\Re}(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})))\Big)$$
$$= \theta\Big(\widehat{S}((\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), ..., \Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n})),$$

$$(\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}))$$
$$(\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n})))\Big)$$
$$=\theta\Big(\max\{\wp_\lambda(\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n})),$$
$$\wp_\lambda(\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n})),$$
$$\ldots,\wp_\lambda(\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n}))\}\Big)$$
$$=\theta\Big(\wp_\lambda(\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n}))\Big)$$
$$\leq\psi\Big(\theta\Big(\max\{\wp_\lambda(\varrho_{11},\varrho_{21},\varrho_{31}),\wp_\lambda(\varrho_{12},\varrho_{22},\varrho_{32}),\ldots,\wp_\lambda(\varrho_{1n},\varrho_{2n},\varrho_{3n})\}\Big)\Big)$$
$$=\psi\Big(\theta\Big(\widehat{S}((\varrho_{11},\varrho_{12},...,\varrho_{1n}),(\varrho_{21},\varrho_{22},...,\varrho_{2n}),(\varrho_{31},\varrho_{32},...,\varrho_{3n}))\Big)\Big).$$

So, from Theorem 2 we obtain that $\widetilde{\Re}$ has at least a fixed point which implies that $\Re$ has at least a Presić type fixed point. ◄

Corollary 4. *Let $(Ł,\wp_\lambda)$ be an EMSM space and $\Re: Ł^n \to Ł$ be a continuous function such that*

$$\theta\Big(\pounds^2(\wp_\lambda(\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n})))\Big)$$
$$\leq\psi(\theta(\textstyle\sum_{i=1}^{n}\lambda_i\wp_\lambda(\varrho_{1i},\varrho_{2i},\varrho_{3i}))) \tag{22}$$

for all $\varrho_{ij}\in Ł$, where $\theta\in\Theta$, $\psi\in\Psi$, $\lambda_i\geq 0$ and $\sum\limits_{i=1}^{k}\lambda_i\in[0,1)$. Then $\Re$ has at least a Presić type fixed point.

Corollary 5. *Let $(Ł,\wp_\lambda)$ be an EMSM space and $\Re: Ł^n \to Ł$ be a continuous function such that*

$$\theta\Big(\pounds^2(\wp_\lambda(\Re(\varrho_{11},\varrho_{12},...,\varrho_{1n}),\Re(\varrho_{21},\varrho_{22},...,\varrho_{2n}),\Re(\varrho_{31},\varrho_{32},...,\varrho_{3n})))\Big)$$
$$\leq(\theta(\textstyle\sum_{i=1}^{n}\lambda_i\wp_\lambda(\varrho_{1i},\varrho_{2i},\varrho_{3i})))^{\varphi(\sum_{i=1}^{n}\lambda_i\wp_\lambda(\varrho_{1i},\varrho_{2i},\varrho_{3i}))} \tag{23}$$

for all $\varrho_{ij}\in Ł$, where $\theta\in\Theta$, $\varphi:[0,\infty)\to[0,1)$ so that $\limsup\limits_{n\to\infty}\varphi(t_n)=\pounds^{-1}(1)$ implies that $t_n\to 0$, as $n\to\infty$ and $\lim\limits_{n\to\infty}\varphi^n(t)=0$ for all $t\in[0,\infty)$, $\lambda_i\geq 0$ and $\sum\limits_{i=1}^{k}\lambda_i\in[0,1)$. Then $\Re$ has at least a Presić type fixed point.

Corollary 6. *Let* $(Ł, \wp_\lambda)$ *be an EMSM space and* $\Re : Ł^n \to Ł$ *be a continuous function such that*

$$\theta\Big(\pounds^2(\widetilde{\wp}_\lambda(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})))\Big) \\ \leq (\theta(\textstyle\sum_{i=1}^{n} \lambda_i \wp_\lambda(\varrho_{1i}, \varrho_{2i}, \varrho_{3i})))^k \tag{24}$$

for all $\varrho_{ij} \in Ł$, *where* $\theta \in \Theta$, $k \in [0, 1)$, $\lambda_i \geq 0$ *and* $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. *Then* $\Re$ *has at least a Presić type fixed point.*

4. Extended Fuzzy S-metric spaces

In this section, we introduce the concept of extended fuzzy S-metric space and study the relationship between EMSM and extended fuzzy S-metric. Also, we present some new fixed point results in extended fuzzy S-metric spaces.

Definition 9. *(Schweizer and Sklar [18]) A binary operation* $\star : [0, 1] \times [0, 1] \to [0, 1]$ *is called a continuous t-norm if:*

(T1) $\star$ *is commutative and associative;*

(T2) $\star$ *is continuous;*

(T3) $\sigma \star 1 = \sigma$ *for all* $\sigma \in [0, 1]$;

(T4) $\sigma \star \varsigma \leq \tau \star d$ *when* $\sigma \leq \tau$ *and* $\varsigma \leq d$, *with* $\sigma, \varsigma, \tau, \upsilon \in [0, 1]$.

Definition 10. *[17] A 3-tuple* $(Ł, M, *)$ *is said to be a fuzzy metric space if* $Ł$ *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $Ł^2 \times (0, \infty)$ *such that, for all* $\varrho, \varrho', \varrho'' \in Ł$ *and* $t, s > 0$,

(i) $M(\varrho, \varrho', t) > 0$;

(ii) $M(\varrho, \varrho', t) = 1$ *for all* $t > 0$ *if and only if* $\varrho = \varrho'$;

(iii) $M(\varrho, \varrho', t) = M(\varrho', \varrho, t)$;

(iv) $M(\varrho, \varrho', t) * M(\varrho', \varrho'', s) \leq M(\varrho, \varrho'', t + s)$;

(v) $M(\varrho, \varrho', \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*

The degree of nearness between ϱ *and* ϱ' *with respect to* t *is denoted by the values of the function* $M(\varrho, \varrho', t)$.

Definition 11. *A 3-tuple* $(Ł, S, *)$ *is said to be a fuzzy* S*-metric space if* $Ł$ *is an arbitrary set,* $*$ *is a continuous t-norm and* S *is a fuzzy set on* $Ł^3 \times (0, \infty)$ *such that, for all* $\varrho, \varrho', \varrho'' \in Ł$ *and* $t, s > 0$,

(i) $S(\varrho, \varrho', \varrho'', t) > 0$;

(ii) $S(\varrho, \varrho', \varrho'', t) = 1$ *for all* $t > 0$ *if and only if* $\varrho = \varrho' = \varrho''$;

(iii) $S(\varrho, \varrho, \varrho', t) = S(\varrho', \varrho', \varrho, t)$;

(iv) $S(\varrho, \varrho, a, t) * S(\varrho', \varrho', a, s) * S(\varrho'', \varrho'', a, r) \leq S(\varrho, \varrho', \varrho'', t + s + r)$;

(v) $S(\varrho, \varrho', \varrho'', \cdot) : (0, \infty) \to [0, 1]$ *is left continuous.*

Definition 12. *An EFSM space is an ordered quadruple* $(Ł, S, \star, £)$ *such that* $Ł$ *is a nonempty set,* $\star$ *is a continuous t-norm and* B *is a fuzzy set on* $Ł^3 \times (0, \infty)$ *such that, for all* $\varrho, \varrho', \varrho'' \in Ł$ *and* $t, s, r > 0$*,*

(i) $B(\varrho, \varrho, \varrho', t) > 0$*;*

(ii) $B(\varrho, \varrho', \varrho'', t) = 1$ *for all* $t > 0$ *if and only if* $\varrho = \varrho' = \varrho''$*;*

(iii) $B(\varrho, \varrho, a, t) * B(\varrho', \varrho', a, s) * B(\varrho'', \varrho'', a, r) \leq B(\varrho, \varrho', \varrho'', £(t + s + r))$*;*

(iv) $B(\varrho, \varrho', \varrho'', \cdot) : (0, \infty) \to [0, 1]$ *is left continuous.*

Definition 13. *Let* $(Ł, B, \star, £)$ *be an EFSM space and* $\{\varrho_n\}$ *be a sequence in* $Ł$ *and* $\varrho \in Ł$*.*

(i) $\{\varrho_n\}$ *is said to be convergent and converges to* ϱ *if* $\lim_{n,m\to\infty} B(\varrho_n, \varrho_n, \varrho, t) = 1$ *for all* $t > 0$*.*

(ii) $\{\varrho_n\}$ *is said to be a Cauchy sequence if and only if, for all* $\epsilon \in (0, 1)$ *and for all* $t > 0$*, there exists* n_0 *such that* $B(\varrho_n, \varrho_n, \varrho_m, t) > 1 - \epsilon$ *for all* $m, n \geq n_0$*;*

(iii) $(Ł, B, \star, £)$ *is said to be complete if every Cauchy sequence is a convergent sequence.*

Definition 14. *Let* $(Ł, B, \star, £)$ *be an EFSM space. The EFSM* B *is called triangular whenever*

$$\frac{1}{B(\varrho, \varrho', \varrho'', t)} - 1 \leq £[\frac{1}{B(\varrho, \varrho, a, t)} - 1 + \frac{1}{B(\varrho', \varrho', a, t)} - 1 + \frac{1}{B(\varrho'', \varrho'', a, t)} - 1].$$

for all $\varrho, \varrho', \varrho'', a \in Ł$ *and for all* $t > 0$*.*

Remark 3. *Notice that* $\wp_\lambda : Ł^3 \times (0, \infty) \to [0, \infty)$ *with* $\wp_\lambda(\varrho, \varrho', \varrho'', t) = \dfrac{1}{B(\varrho, \varrho', \varrho'', t)} - 1$ *is an EMSM whenever* B *is a triangular EFSM.*

We may deduce the following consequences in EFSM spaces using Remark 3 and the results obtained in Section 2.

Theorem 9. *Let* $(Ł, B, \star, £)$ *be a triangular complete EFSM space and let* $\Re : Ł \longrightarrow Ł$ *be a self-mapping such that*

$$\theta(£^2[\frac{1}{B(\Re\varrho, \Re\varrho', \Re\varrho'', t)} - 1]) \leq \psi(\theta(\frac{1}{B(\varrho, \varrho', \varrho'', t)} - 1)) \tag{25}$$

for all $\varrho, \varrho', \varrho'' \in Ł$*, where* $\theta \in \Theta$ *and* $\psi \in \Psi$*. Then* $\Re$ *has a unique fixed point in* $Ł$*.*

Corollary 7. *Let* $(Ł, B, \star, £)$ *be a triangular complete EFSM space and* $\Re : Ł \longrightarrow Ł$ *be a self-mapping such that*

$$\theta(£^2[\frac{1}{B(\Re\varrho, \Re\varrho', \Re\varrho'', t)} - 1]) \leq (\theta(\frac{1}{B(\varrho, \varrho', \varrho'', t)} - 1))^{\varphi(\frac{1}{B(\varrho, \varrho', \varrho'', t)} - 1)} \tag{26}$$

for all $\varrho, \varrho', \varrho'' \in Ł$, where $\theta \in \Theta$ and $\varphi \in \zeta$. Then $\Re$ has a unique fixed point in $Ł$.

Corollary 8. *Let $(Ł, B, \star, £)$ be a triangular complete EFSM space and $\Re : Ł \longrightarrow Ł$ be a self-mapping such that*

$$\theta(£^2[\frac{1}{B(\Re\varrho, \Re\varrho', \Re\varrho'', t)} - 1]) \leq (\theta(\frac{1}{B(\varrho, \varrho', \varrho'', t)} - 1))^k \tag{27}$$

for all $\varrho, \varrho', \varrho'' \in Ł$, where $\theta \in \Theta$ and $k \in [0, 1)$. Then $\Re$ has a unique fixed point in $Ł$.

Theorem 10. *Let $(Ł, B, \star, £)$ be an EFSM space and $\Re : Ł^n \to Ł$ be a continuous function such that*

$$\theta\Big(£^2\Big[\frac{1}{B\Big(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})} - 1\Big)\Big]\Big)$$

$$\leq \frac{1}{n}\psi\Big(\Big[\theta\Big(\frac{1}{B(\varrho_{11}, \varrho_{21}, \varrho_{31})} - 1 + \frac{1}{B(\varrho_{12}, \varrho_{22}, \varrho_{32})} - 1 + ... + \frac{1}{B(\varrho_{1n}, \varrho_{2n}, \varrho_{3n})} - 1\Big)\Big]\Big)$$

for all $\varrho_{ij} \subseteq Ł$, where $\theta \in \Theta$ is a subadditive mapping and $\psi \in \Psi$. Then $\Re$ has at least a Presić type fixed point.

Theorem 11. *Let $(Ł, B, \star, £)$ be a complete triangular EFSM space and $\Re : Ł^n \to Ł$ be a continuous function such that*

$$\theta\Big(£^2(\frac{1}{B\Big(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})} - 1)\Big)$$

$$\leq \psi\Big(\theta\Big(\max\{\frac{1}{B(\varrho_{11}, \varrho_{21}, \varrho_{31})} - 1, \frac{1}{B(\varrho_{12}, \varrho_{22}, \varrho_{32})} - 1, ..., \frac{1}{B(\varrho_{1n}, \varrho_{2n}, \varrho_{3n})} - 1\}\Big)\Big)$$

for all $\varrho_{ij} \in Ł$, where $\theta \in \Theta$ and $\psi \in \Psi$. Then $\Re$ has at least a Presić type fixed point.

Corollary 9. *Let $(Ł, B, \star, £)$ be a complete triangular EFSM space and $\Re : Ł^n \to Ł$ be a continuous function such that*

$$\theta\Big(£^2(\frac{1}{B\Big(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})} - 1)\Big)$$

$$\leq \psi(\theta\big(\sum_{i=1}^{n} \lambda_i \frac{1}{B(\varrho_{1i}, \varrho_{2i}, \varrho_{3i})} - 1))$$

for all $\varrho_{ij} \in Ł$, where $\theta \in \Theta$, $\psi \in \Psi$, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. Then $\Re$ has at least a Presić type fixed point.

Corollary 10. *Let $(Ł, B, \star, \pounds)$ be a complete triangular EFSM space and $\Re : Ł^n \to Ł$ be a continuous function such that*

$$\theta\Big(\pounds^2(\frac{1}{B\Big(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})} - 1)\Big)$$

$$\leq (\theta(\sum_{i=1}^{n} \lambda_i \frac{1}{B(\varrho_{1i}, \varrho_{2i}, \varrho_{3i})} - 1))^{\varphi(\sum_{i=1}^{n} \lambda_i \frac{1}{B(\varrho_{1i}, \varrho_{2i}, \varrho_{3i})} - 1)}$$

for all $\varrho_{ij} \in Ł$, where $\theta \in \Theta$, $\varphi : [0, \infty) \to [0, 1)$ so that $\limsup_{n\to\infty} \varphi(t_n) = 1$ implies that $t_n \to 0$, as $n \to \infty$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0, \infty)$, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. Then $\Re$ has at least a Presić type fixed point.

Corollary 11. *Let $(Ł, B, \star, \pounds)$ be a complete triangular EFSM space and $\Re : Ł^n \to Ł$ be a continuous function such that*

$$\theta\Big(\pounds^2(\frac{1}{B\Big(\Re(\varrho_{11}, \varrho_{12}, ..., \varrho_{1n}), \Re(\varrho_{21}, \varrho_{22}, ..., \varrho_{2n}), \Re(\varrho_{31}, \varrho_{32}, ..., \varrho_{3n})} - 1)\Big)$$

$$\leq (\theta(\sum_{i=1}^{n} \lambda_i \frac{1}{B(\varrho_{1i}, \varrho_{2i}, \varrho_{3i})} - 1))^{k}$$

for all $\varrho_{ij} \in Ł$, where $\theta \in \Theta$, $k \in [0, 1)$, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. Then $\Re$ has at least a Presić type fixed point.

5. Application

In this section we present an application of Theorem 2.

Let $Ł = C([a, b], \mathbb{R})$ be the set of real continuous functions defined on $[a, b]$. Let

$$d(\varrho, \varrho') = \max_{t\in[a,b]} \mid \varrho(t) - \varrho'(t) \mid$$

for all $\varrho, \varrho' \in Ł$, and $\wp_\lambda : (0, +\infty) \times Ł \times Ł \times Ł \to [0, +\infty]$ be defined by

$$\wp_\lambda(\varrho, \varrho', \varrho'') = \frac{d(\varrho, \varrho'') + d(\varrho', \varrho'')}{\lambda} \text{ for all } \varrho, \varrho', \varrho'' \in Ł.$$

Then $(Ł, \wp_\lambda)$ is a complete EMSM space. Now, let us consider the integral equation

$$\varrho(t) = \int_a^b \mathcal{K}(t, s, \varrho(s))ds, \text{ for all } t, s \in [a, b], \tag{28}$$

where $\mathcal{K} : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 12. *Assume that for all $t, s \in [a,b]$ we have*

$$\int_a^b |\mathcal{K}(t,s,\varrho(s)) - \mathcal{K}(t,s,\varrho''(s))|ds \leq \lambda\theta^{-1}(\psi(\theta(\frac{|\varrho(t)-\varrho''(t)|}{\lambda}))).$$

Then, the integral equation (28) *has a unique solution $u \in Ł$ provided that the combination function $\theta^{-1}(\psi(\theta))$ is superadditive.*

Proof. Let us define $\Re : Ł \to Ł$ by

$$\Re(\varrho)(t) = \int_a^b \mathcal{K}(t,s,\varrho(s))ds, \ \ \varrho \in Ł, t,s \in [a,b].$$

Let $\varrho, \varrho'' \in Ł$. Then we have

$$\begin{aligned} |\Re\varrho(t) - \Re\varrho''(t)| &= |\int_a^b \mathcal{K}(t,s,\varrho(s))ds - \int_a^b \mathcal{K}(t,s,\varrho''(s))ds| \\ &\leq \int_a^b |\mathcal{K}(t,s,\varrho(s)) - \mathcal{K}(t,s,\varrho''(s))|ds \\ &\leq \lambda\theta^{-1}(\psi(\theta(\frac{|\varrho(t)-\varrho''(t)|}{\lambda}))) \\ &\leq \lambda\theta^{-1}(\psi(\theta(\frac{d(\varrho,\varrho'')}{\lambda}))). \end{aligned}$$

Therefore,

$$\begin{aligned} p_\lambda(\Re\varrho, \Re\varrho', \Re\varrho'') &= \frac{d(\Re\varrho, \Re\varrho'') + d(\Re\varrho', \Re\varrho'')}{\lambda} \\ &\leq \theta^{-1}(\psi(\theta(\frac{d(\varrho,\varrho'')}{\lambda}))) + \theta^{-1}(\psi(\theta(\frac{d(\varrho',\varrho'')}{\lambda}))) \leq \theta^{-1}(\psi(\theta(\wp_\lambda(\varrho,\varrho',\varrho'')))) \end{aligned}$$

for all $\varrho, \varrho', \varrho'' \in Ł$.

As a result, the criteria of Theorem 2 are satisfied, and $\Re$ has a unique fixed point in Ł, indicating that the nonlinear integral equation (28) has a unique solution in $C([a,b],\mathbb{R})$. ◄

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intègrales*, Fund. Math., **3**, 1922, 133-181.

[2] V.V. Chistyakov, *Modular metric space, I: Basic concepts*, Nonlinear Anal., **72(1)**, 2010, 1-14.

[3] V.V. Chistyakov, *Modular metric space, II: Application to superposition operators*, Nonlinear Anal., **72(1)**, 2010, 15-30.

[4] Y.Z. Chen, *A Presić type contractive condition and its applications*, Nonlinear Anal., **71**, 2009, 2012-2017.

[5] L.B. Ćirić, S.B. Presić, *On Presić type generalisation of Banach contraction mapping principle*, Acta. Math. Univ. Com., **LXXVI(2)**, 2007, 143-147.

[6] B. C. Dhage, *A common fixed point principle in D-metric spaces*, Bulletin of the Calcutta Mathematical Society, **91(6)**, 1999, 475–480.

[7] B. C. Dhage, *Generalised metric spaces and mappings with fixed point*, Bulletin of the Calcutta Mathematical Society, **84(4)**, 1992, 329–336.

[8] M.E. Ege, C. Alaca, *Some results for modular b-metric spaces and an application to system of linear equations*, Azerbaijan Journal of Mathematics, **8(1)**, 2018, 3-14.

[9] O. Ege, I. Karaca, *Fixed Point Theorems and an Application in Parametric Metric Spaces*, Azerbaijan Journal of Mathematics, **7(1)**, 2017, 27-39.

[10] M.E. Ege, C. Alaca, *Some properties of modular S-metric spaces and its fixed point results*, Journal of Computational Analysis and Applications, **20(1)**, 2016.

[11] M. Jleli, B. Samet, *A generalized metric spaces and related fixed Point theorems*, Fixed point Theory and Applications, **61**, 2015, doi:10.1186/s13663-015-0312-7.

[12] M.A. Khamsi, W.K. Kozlowski, S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Analysis, **14**, 1990, 935-953.

[13] W.M. Kozlowski, *Modular Function Spaces*, Series of Monographs and Textbooks in Pure and Applied Mathematics, **122**, Dekker, New York/Basel, 1988.

[14] N. Mlaiki, *Extended S_b-metric spaces*, J. Math. Anal., **9**, 2018, 124–135.

[15] Z. Mustafa, R.J. Shahkoohi, V. Parvaneh, Z. Kadelburg, M.M.M. Jaradat, *Modified ordered S-metric spaces and some fixed point theorems for contractive mappings with application to periodic boundary value problems*, Fixed point theory and applications, **16(2019)**, 2019.

[16] V. Parvaneh, S.J. Hosseini Ghoncheh, *Fixed points of* (ψ, φ)*-contractive mappings in ordered p-metric spaces*, Global Analysis and Discrete Mathematics, **4(1)**, 2019, 15-29.

[17] V. Parvaneh, N. Hussain, M.A. Kutbi, M. Khorshdi, *Some fixed point results in extended parametric b-metric spaces with application to integral equations*, Journal of Mathematical Analysis, **10(5)**, 2019, 14-33.

[18] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math., **10**, 1960, 314-334

[19] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in* S*-metric spaces*, Math. Vesnik, **64**, 2012, 258-266.

[20] S. Sedghi, N. Shobe and H. Zhou, *A Common Fixed Point Theorem in* D^**-Metric Spaces*, **2007**, Article ID 27906, 13 pages, doi:10.1155/2007/27906.

[21] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear and Convex Analysis, **7**, 2006, 289-297.

[22] S.B. Presić, *Sur une classe d'inèquations aux differences finies et sur la convergence de certaines suites*, Pub. de. l'Institut Math. Belgrade, **5(15)**, 1965, 75-78.

Mohammad Mahdi Rezaee
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
E-mail: mohammad.m.rezaee@gmail.com

Shaban Sedghi
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
E-mail: sedghi_gh@yahoo.com

Vahid Parvaneh
Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
E-mail: zam.dalahoo@gmail.com

Received 10 September 2021
Accepted 10 October 2021