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On the Regularization Cauchy Problem for Matrix Factorizations of the Helmholtz Equation in a Multidimensional Bounded Domain

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Abstract. In this paper, the problem of continuation of the solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain is considered. It is assumed that the solution to the problem exists and is continuously differentiable in a closed domain with exactly given Cauchy data. For this case, an explicit formula for the continuation of the solution is established, as well as a regularization formula for the case where, under the indicated conditions, instead of Cauchy data their continuous approximations with a given error in uniform metric are given. A stability estimate for the solution of the Cauchy problem in the classical sense is obtained.
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1. Introduction

This paper studies the construction of exact and approximate solutions to the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation. Such problems naturally arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.). In general, the theory of ill-posed problems for systems of elliptic equations has been sufficiently formed thanks to the works of A.N. Tikhonov, V.K. Ivanov, M.M. Lavrent'ev, N.N. Tarkhanov and many other mathematicians. Among them, the most important for applications are the so-called conditionally well-posed problems, characterized by stability in the presence of additional information about the nature of the problem data. One of the

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most effective ways to study such problems is to construct regularizing operators. For example, this can be the Carleman-type formulas (as in complex analysis) or iterative processes (the Kozlov-Maz'ya-Fomin algorithm, etc.).

This work is dedicated to the main problem for partial differential equations, which is the Cauchy problem. There are classes of equations for which this problem behaves well - the so-called hyperbolic equations. The main attention is paid to the regularization formulas for solutions of the Cauchy problem. The question of the existence of a solution to this problem is not considered - it is assumed a priori. At the same time, it should be noted that any regularization formula leads to an approximate solution of the Cauchy problem for all data, even if there is no solution in the usual classical sense. Moreover, for explicit regularization formulas, one can indicate in what sense the approximate solution turns out to be optimal. From this point of view, exact regularization formulas are very useful for real numerical calculations. There is a good reason to hope that numerous practical applications of regularization formulas are still ahead.

This problem belongs to the class of ill-posed problems, i.e., it is unstable. It is known that the Cauchy problem for elliptic equations is ill-posed because it is unstable for relatively small changes in the data, i.e., incorrect (Hadamard's example, see, for instance [21], p. 39). There is a sizable literature on the subject (see, e.g., [22], [3, 34], [24], [25] and [4]. N.N. Tarkhanov [30] has found a criterion for the solvability of a larger class of boundary value problems for elliptic systems. In unstable problems, the image of the operator is not closed, therefore, the solvability condition can not be written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique, the problem is solvable for everywhere dense set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see, for instance, [31]).

The uniqueness of the solution follows from Holmgren's general theorem (see [4]). The conditional stability of the problem follows from the work of A.N. Tikhonov (see [4]), if we restrict the class of possible solutions to a compactum.

We note that when solving applied problems, one should find the approximate values of $U(x)$ and $\dfrac{\partial U(x)}{\partial x_j},\ x \in G,\ j = 1, ..., m.$

In this paper we construct a family of vector-functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U(x, f_\delta)}{\partial x_j} = \dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j},\ j = 1, ..., m$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ converges in the usual sense to a solution $U(x)$ and its derivative $\dfrac{\partial U(x)}{\partial x_j}$, $x \in G$ at the point $x \in G$.

Following A.N. Tikhonov (see [4]), a family of vector-valued functions $U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Formulas that allow finding a solution to an elliptic equation in the case where the Cauchy data are known only on a part of the boundary of the domain are called Carleman type formulas. In [34], Carleman established a formula giving a solution to the Cauchy-Riemann equations in a domain of special form. Developing his idea, G.M. Goluzin and V.I. Krylov [17] derived a formula for determining the values of analytic functions from data known only on a portion of the boundary for arbitrary domains. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in [22]. A formula of Carleman type, in which the fundamental solution of a differential operator with special properties (the Carleman function) is used, was obtained by M.M. Lavrent'ev (see, for instance, [23, 24]). Using this method, Sh. Ya. Yarmukhamedov (see, for instance, [35, 36, 37, 38]) constructed the Carleman functions for the Laplace and Helmholtz operators with $n(x, y) \equiv 1$ for spatial domains of a special form, when the part of the boundary of the domain where the data are unknown is a conical surface or a hypersurface $\{x_3 = 0\}$. In [31] an integral formula is proved for the systems of equations of elliptic type of the first order with constant coefficients in a bounded domain. Using the methodology of works [35, 36, 37, 38]), Ikehata [26] considered the probe method and Carleman functions for the Laplace and Helmholtz equations in the three-dimensional domain. Using exponentially growing solutions, Ikehata [27] obtained a formula for solving the Helmholtz equation with a variable coefficient for regions in space where the unknown data are located on a section of the hypersurface $\{x \cdot s = t\}$. Carleman type formulas for various elliptic equations and systems were also obtained in [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 26, 27, 28, 29, 30, 31]. In [16] the Cauchy problem for the Helmholtz equation was considered in an arbitrary bounded plane domain with Cauchy data, known only on the boundary. The solvability criterion for the Cauchy problem for the Laplace equation in the space $\mathbb{R}^m$ was considered by Shlapunov in [1]. In [18], the problem for the Helmholtz equation was investigated and the results of numerical experiments were presented.

The construction of the Carleman matrix for elliptic systems was carried out by: Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov, D.A.

Juraev and others (see, for instance, [35, 36, 37, 38], [1, 2], [19, 20] and [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]). The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (see, for instance, [31]).

In many well-posed problems for the systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of the systems of first order equations of elliptic type with constant coefficients, factorizing the Helmholtz operator (see, for instance, [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]), is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in [23, 24], [35, 36, 37, 38] and subsequently developed in [16], [17], [28, 29, 30, 31], [26, 27], [19, 20, 33], [1, 2] and [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

In this paper, we present an explicit formula for the approximate solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation in a multidimensional bounded domain. The odd dimensional case requires special consideration in contrast to even dimensions in many mathematical problems. The odd-dimensional case will be further considered in future works of the authors. Our formula for an approximate solution also includes the construction of a family of fundamental solutions for the Helmholtz operator in a multidimensional bounded domain. This family is parametrized by some entire function $K(w)$, the choice of which depends on the dimension of the space. This motivates a separate study of regularization formulas in odd dimensional spatial domains.

Let $\mathbb{R}^m$ $(m = 2k+1,\ k \geq 1)$ be the $m-$dimensional real Euclidean space,

$$x = (x_1, ..., x_m) \in \mathbb{R}^m,\ y = (y_1, ..., y_m) \in \mathbb{R}^m,$$

$$x' = (x_1, ..., x_{m-1}) \in \mathbb{R}^{m-1},\ y' = (y_1, ..., y_{m-1}) \in \mathbb{R}^{m-1}.$$

We introduce the following notation:

$$r = |y - x|\,, \alpha = |y' - x'|\,, w = i\sqrt{u^2 + \alpha^2} + y_m,\ u \geq 0,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\right)^T,\ \frac{\partial}{\partial x} = \xi^T,\ \xi^T = \begin{pmatrix} \xi_1 \\ ... \\ \xi_m \end{pmatrix} - \text{transposed vector}\, x,$$

$$U(x) = (U_1(x), ..., U_n(x))^T,\ u^0 = (1, ..., 1) \in \mathbb{R}^n,\ n = 2^m,\ m \geq 3,$$

$$E(z) = \left\| \begin{matrix} z_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots z_n \end{matrix} \right\| - \text{diagonal matrix}, z = (z_1, \ldots, z_n) \in \mathbb{R}^n.$$

Let $G \subset \mathbb{R}^m$ be a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_m = 0$ and a smooth surface S, lying in the half-space $y_m > 0$, i.e., $\partial G = S \bigcup T$.

Let $D(\xi^T)$ be an $(n \times n)-$dimensional matrix with elements consisting of a set of linear functions with constant coefficients in the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$, λ is a real number.

We consider in G a system of differential equations

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is a matrix of first-order differential operators.

We denote by $A(G)$ the class of vector functions in G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (1).

2. Construction of the Carleman matrix and the Cauchy problem

2.1. General

Formulation of the problem. Suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y),\ y \in S. \tag{2}$$

Here, $f(y)$ is a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G, based on its values $f(y)$ on S.

If $U(y) \in A(G)$, then the following Cauchy-type integral formula is valid:

$$U(x) = \int\limits_{\partial G} N(y, x; \lambda)U(y)ds_y,\ \ x \in G, \tag{3}$$

where

$$N(y, x; \lambda) = \left(E\left(\varphi_m(\lambda r)u^0\right) D^*\left(\frac{\partial}{\partial y}\right)\right) D(t^T).$$

Here $t = (t_1, \dots, t_m)$ is the outer unit normal drawn at the point y, the surface ∂G, $\varphi_m(\lambda r)$ is a fundamental solution of the Helmholtz equation in $\mathbb{R}^m$ ($m = 2k+1$, $k \geq 1$), defined by the following formula (see [32]):

$$\varphi_m(\lambda r) = P_m \lambda^{(m-2)/2} \frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}},$$
$$P_m = \frac{1}{2i(2\pi)^{(m-2)/2}}, \; m = 2k+1, \; k \geq 1. \tag{4}$$

We denote by $K(w)$ an entire function taking real values for real w ($w = u + iv$, $u, v-$real numbers) and satisfying the following conditions:

$$K(u) \neq 0, \; \sup_{v \geq 1} \left| v^p K^{(p)}(w) \right| = B(u, p) < \infty,$$
$$-\infty < u < \infty, \; p = 0, 1, ..., m. \tag{5}$$

We define the function $\Phi(y, x; \lambda)$ for $y \neq x$ by the following equality:

$$\Phi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im}\left[\frac{K(w)}{w - x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du,$$
$$m = 2k+1, \; k \geq 1, \tag{6}$$

where $c_m = (-1)^k 2^{-k} (2k-1)!(m-2)\pi\omega_m$; ω_m is an area of a unit sphere in the space $\mathbb{R}^m$.

In the formula (6), choosing

$$K(w) = \exp(\sigma w), \; K(x_m) = \exp(\sigma x_m), \; \sigma > 0, \tag{7}$$

we get

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{-\sigma x_m}}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im}\left[\frac{\exp(\sigma w)}{w - x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \tag{8}$$

The formula (3) is true if we substitute $\varphi_m(\lambda r)$ with the function

$$\Phi_\sigma(y, x; \lambda) = \varphi_m(\lambda r) + g_\sigma(y, x; \lambda), \tag{9}$$

where $g_\sigma(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula becomes:

$$U(x) = \int\limits_{\partial G} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G, \tag{10}$$

where

$$N_\sigma(y, x; \lambda) = \left(E \left(\Phi_\sigma(y, x; \lambda) u^0 \right) D^* \left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

3. The continuation formula and regularization according to M.M. Lavrent'ev

Theorem 1. *Let $U(y) \in A(G)$ and the inequality*

$$|U(y)| \leq M, \; y \in T, \tag{11}$$

hold. If

$$U_\sigma(x) = \int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y, \; x \in G, \tag{12}$$

then the following estimates are true:

$$|U(x) - U_\sigma(x)| \leq MC(x)\sigma^{k+1} e^{-\sigma x_m}, \; \sigma > 1, \; x \in G. \tag{13}$$

$$\left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j} \right| \leq C(x)\sigma^{k+1} M e^{-\sigma x_m}, \; \sigma > 1, \; x \in G, \; j = 1, .., m. \tag{14}$$

Here and below, the functions bounded on compact subsets of the domain G are denoted by $C(x)$.

Proof. Let us first estimate the inequality (13). Using the integral formula (10) and the equality (12), we obtain

$$U(x) = \int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y + \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y =$$

$$= U_\sigma(x) + \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y, \; x \in G.$$

Taking into account the inequality (11), we have

$$|U(x) - U_\sigma(x)| \le \left| \int_T N_\sigma(y, x; \lambda) U(y) ds_y \right| \le$$
$$\le \int_T |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \le M \int_T |N_\sigma(y, x; \lambda)| \, ds_y, \;\; x \in G. \tag{15}$$

Now let's estimate the integrals $\int_T |\Phi_\sigma(y, x; \lambda)| \, ds_y$, $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} \right| ds_y$, $(j = 1, 2, ..., m-1)$ and $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_m} \right| ds_y$ on the part T of the plane $y_m = 0$.

Separating the imaginary part of (8), we obtain

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{\sigma(y_m - x_m)}}{c_m} \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty - \frac{\cos \sigma \sqrt{u^2 + \alpha^2}}{u^2 + r^2} \; \cos(\lambda u) \, du + \right.$$
$$\left. + \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \frac{(y_m - x_m) \sin \sigma \sqrt{u^2 + \alpha^2}}{u^2 + r^2} \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du \right], \;\; x_m > 0. \tag{16}$$

Taking into account the equality (16), we have

$$\int_T |\Phi_\sigma(y, x; \lambda)| \, ds_y \le C(x) \sigma^{k+1} M e^{-\sigma x_m}, \;\; \sigma > 1, \;\; x \in G, \tag{17}$$

To estimate the second integral, we use the equality

$$\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} = \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s} \frac{\partial s}{\partial y_j} = 2(y_j - x_j) \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s},$$
$$s = \alpha^2, \;\; j = 1, 2, ..., m-1. \tag{18}$$

Using the equalities (16) and (18), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} \right| ds_y \le C(x) \sigma^{k+1} M e^{-\sigma x_m}, \;\; \sigma > 1, \;\; x \in G,$$
$$j = 1, 2, ..., m-1. \tag{19}$$

Now, we estimate the integral $\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$.

Taking into account the equality (16), we obtain

$$\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y \leq C(x)\sigma^{k+1}Me^{-\sigma x_m},\ \sigma > 1,\ x \in G, \tag{20}$$

From the inequalities (17), (19) and (20), we obtain an estimate (13).

Now let us prove the inequality (14). To do this, we take the derivatives of equalities (10) and (12) with respect to $x_j,\ j = 1, ..., m$. Then we obtain the following:

$$\begin{gathered} \frac{\partial U(x)}{\partial x_j} = \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y)ds_y + \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y)ds_y, \\ \frac{\partial U_\sigma(x)}{\partial x_j} = \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y)ds_y,\ \ x \in G,\ j = 1, ..., m. \end{gathered} \tag{21}$$

Taking into account (21) and the inequality (11), we have

$$\begin{gathered} \left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial_\sigma U(x)}{\partial x_j}\right| \leq \left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y)ds_y\right| \leq \\ \leq \int\limits_T \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| |U(y)|\, ds_y \leq M\int\limits_T \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y, \\ x \in G,\ j = 1, ..., m. \end{gathered} \tag{22}$$

Now let's estimate the integrals $\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y\ (j = 1, 2, ..., m-1)$ and $\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y$ on the part T of the plane $y_m = 0$.

To estimate the first integral, we use the equality

$$\begin{gathered} \frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_j} = \frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial x_j} = -2(y_j - x_j)\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s}, \\ s = \alpha^2,\ j = 1, 2, ..., m-1. \end{gathered} \tag{23}$$

Using the equalities (16) and (23), we obtain

$$\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y \le C(x)\sigma^{k+1}Me^{-\sigma x_m},\ \sigma > 1,\ x \in G, \tag{24}$$
$$j = 1, 2, ..., m-1.$$

Now, we estimate the integral $\int\limits_T \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y$.

Taking into account the equality (16), we obtain

$$\int_T \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y \le C(x)\sigma^{k+1}Me^{-\sigma x_m},\ \sigma > 1,\ x \in G, \tag{25}$$

From the inequalities (22), (24) and (25), we obtain an estimate (14). Theorem 1 is proved. ◄

Corollary 1. *For every $x \in G$, the following equalities are true:*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x),\ \lim_{\sigma\to\infty}\frac{\partial U_\sigma(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j},\ j = 1, ..., m.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon = \left\{(x_1, ..., x_m) \in G,\ a > x_m \ge \varepsilon,\ a = \max_T \psi(x'),\ 0 < \varepsilon < a\right\}.$$

It is easy to see that the set $\overline{G}_\varepsilon \subset G$ is compact.

Corollary 2. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{\frac{\partial U_\sigma(x)}{\partial x_j}\right\}$ converge uniformly as $\sigma \to \infty$, i.e.*

$$U_\sigma(x) \rightrightarrows U(x),\ \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1, ..., m.$$

It should be noted that the set $E_\varepsilon = G\backslash\overline{G}_\varepsilon$ serves as a boundary layer for this problem as in the theory of singular perturbations, where there is no uniform convergence.

4. Estimation of the stability of the solution to the Cauchy problem

Suppose that the surface S (or the curve for $m = 2$) is given by the equation

$$y_m = \psi(y'),\ y' \in \mathbb{R}^{m-1},$$

where $\psi(y')$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$a = \max_T \psi(y'),\ b = \max_T \sqrt{1 + \psi'^2(y')}.$$

Theorem 2. *Let* $U(y) \in A(G)$ *satisfy the condition* (11) *and on a smooth surface* S *the inequality*

$$|U(y)| \le \delta,\ 0 < \delta < Me^{-\sigma a} \tag{26}$$

hold on the smooth surface S. *Then the following estimates are true:*

$$|U(x)| \le C(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}},\ \sigma > 1,\ x \in G. \tag{27}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \le C(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}},\ \sigma > 1,\ x \in G,\quad j = 1, .., m. \tag{28}$$

Proof. Let us first estimate the inequality (27). Using the integral formula (10), we have

$$U(x) = \int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y + \int\limits_T N_\sigma(y, x; \lambda)) U(y) ds_y,\ x \in G. \tag{29}$$

We also have

$$|U(x)| \le \left|\int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y\right| + \left|\int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y\right|,\ x \in G. \tag{30}$$

Using the inequality (26), we estimate the first integral in (30).

$$\begin{aligned} \left|\int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y\right| &\le \int\limits_S |N_\sigma(y, x; \lambda)|\, |U(y)|\, ds_y \le \\ &\le \delta \int\limits_S |N_\sigma(y, x; \lambda)|\, ds_y,\ x \in G. \end{aligned} \tag{31}$$

To do this, we estimate the integrals $\int\limits_S |\Phi_\sigma(y,x;\lambda)|\,ds_y$, $\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y$, $(j=1,2,...,m-1)$ and $\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$ on S.

By (16), we have

$$\int\limits_S |\Phi_\sigma(y,x;\lambda)|\,ds_y \le C(x)\sigma^{k+1}e^{\sigma(a-x_m)},\ \sigma>1,\ x\in G. \tag{32}$$

To estimate the second integral, using (16) and (18), we obtain

$$\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y \le C(x)\sigma^{k+1}e^{\sigma(a-x_m)},\ \sigma>1,\ x\in G, \\ j=1,...,m-1. \tag{33}$$

To estimate the integral $\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$, using (16), we obtain

$$\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y \le C(x)\sigma^{k+1}e^{\sigma(a-x_m)},\ \sigma>1,\ x\in G. \tag{34}$$

From (32)-(34), we obtain

$$\left|\int\limits_S N_\sigma(y,x;\lambda)U(y)ds_y\right| \le C(\lambda,x)\sigma^{k+1}\delta\, e^{\sigma(a-x_m)},\ \sigma>1,\ x\in G. \tag{35}$$

The following is known:

$$\left|\int\limits_T N_\sigma(y,x;\lambda)U(y)ds_y\right| \le C(x)\sigma^{k+1}Me^{-\sigma x_m},\ \sigma>1,\ x\in G. \tag{36}$$

Now taking into account (35)-(36), we have

$$|U(x)| \le \frac{C(x)\sigma^{k+1}}{2}(\delta\, e^{\sigma a}+M)e^{-\sigma x_m},\ \sigma>1,\ x\in G. \tag{37}$$

Choosing σ from the equality

$$\sigma = \frac{1}{a}\ln\frac{M}{\delta}, \tag{38}$$

we obtain the estimate (27).

Now let us prove the inequality (28). To do this, we find the partial derivative from the integral formula (10) with respect to the variable x_j, $j=1,...,m-1$:

$$\begin{gathered}\frac{\partial U(x)}{\partial x_j}=\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y+\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y+\\ +\frac{\partial U_\sigma(x)}{\partial x_j}+\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y,\ x\in G,\ j=1,...,m.\end{gathered}\tag{39}$$

Here
$$\frac{\partial U_\sigma(x)}{\partial x_j}=\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y.\tag{40}$$

We have

$$\begin{gathered}\left|\frac{\partial U(x)}{\partial x_j}\right|\leq\left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|+\left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\leq\\ \leq\left|\frac{\partial U_\sigma(x)}{\partial x_j}\right|+\left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|,\ x\in G,\ j=1,...,m.\end{gathered}\tag{41}$$

Using (26), we estimate the first integral in (27):

$$\begin{gathered}\left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\leq\int\limits_S\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right||U(y)|\,ds_y\leq\\ \leq\delta\int\limits_S\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y,\ x\in G,\ j=1,...,m.\end{gathered}\tag{42}$$

Now let's estimate the integrals $\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y$, $(j=1,2,...,m-1)$ and $\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m}\right|ds_y$ on S.

By the equalities (16) and (23), we obtain

$$\begin{gathered}\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y\leq C(x)\sigma^{k+1}e^{\sigma(a-x_m)},\ \sigma>1,\ x\in G,\\ j=1,2,...,m-1.\end{gathered}\tag{43}$$

Now, we estimate the integral $\int\limits_S \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m} \right| ds_y.$

Taking into account the equality (16), we obtain

$$\int\limits_S \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m} \right| ds_y \le C(x)\sigma^{k+1} e^{\sigma(a-x_m)},\ \sigma>1,\ x\in G, \tag{44}$$

From (43)-(44), we obtain

$$\left| \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) \right| \le C(x)\sigma^{k+1}\delta\, e^{\sigma(a-x_m)},\ \sigma>1,\ x\in G,\quad j=1,...,m. \tag{45}$$

The following is known:

$$\left| \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y \right| \le C(x)\sigma^{k+1} M e^{-\sigma x_m},\ \sigma>1,\ x\in G,\quad j=1,...,m. \tag{46}$$

Now taking into account (45)-(46), we have

$$\left| \frac{\partial U(x)}{\partial x_j} \right| \le \frac{C(x)\sigma^{k+1}}{2}(\delta\, e^{\sigma a} + M) e^{-\sigma x_m},\ \sigma>1,\ x\in G,\quad j=1,...,m. \tag{47}$$

Choosing σ from the equality (38), we obtain the estimate (28).
Theorem 2 is proved. ◄

Let $U(y)\in A(G)$ and instead of $U(y)$ on S taken with its approximation $f_\delta(y)$, respectively, with an error $0<\delta<Me^{-\sigma a}$,

$$\max_S |U(y) - f_\delta(y)| \le \delta. \tag{48}$$

We put

$$U_{\sigma(\delta)}(x) = \int\limits_S N_\sigma(y,x;\lambda) f_\delta(y) ds_y,\ x\in G. \tag{49}$$

Theorem 3. *Let* $U(y)\in A(G)$ *satisfy the condition* (11) *on the part of the plane* $y_m = 0$

Then the following estimates are true:

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \leq C(x)\sigma^{k+1}M^{1-\frac{x_m}{a}}\delta^{\frac{x_m}{a}},\ \sigma>1,\ x\in G. \tag{50}$$

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C(x)\sigma^{k+1}M^{1-\frac{x_m}{a}}\delta^{\frac{x_m}{a}},\ \sigma>1,\ x\in G,\quad j=1,...,m. \tag{51}$$

Proof. From the integral formulas (10) and (49), we have

$$U(x)-U_{\sigma(\delta)}(x)=\int\limits_{\partial G}N_\sigma(y,x;\lambda)U(y)ds_y-\int\limits_{S}N_\sigma(y,x;\lambda)f_\delta(y)ds_y=$$

$$=\int\limits_{S}N_\sigma(y,x;\lambda)U(y)ds_y+\int\limits_{T}N_\sigma(y,x;\lambda)U(y)ds_y-\int\limits_{S}N_\sigma(y,x;\lambda)f_\delta(y)ds_y=$$

$$=\int\limits_{S}N_\sigma(y,x;\lambda)\left\{U(y)-f_\delta(y)\right\}ds_y+\int\limits_{T}N_\sigma(y,x;\lambda)U(y)ds_y.$$

and

$$\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}=\int\limits_{\partial G}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y-\int\limits_{S}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}f_\delta(y)ds_y=$$

$$=\int\limits_{S}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y+\int\limits_{T}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y-\int\limits_{S}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}f_\delta(y)ds_y=$$

$$=\int\limits_{S}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\left\{U(y)-f_\delta(y)\right\}ds_y+\int\limits_{T}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y,\ j=1,...,m.$$

Using conditions (11) and (48), we estimate

$$\left|U(x)-U_{\sigma(\delta)}(x)\right|=\left|\int\limits_{S}N_\sigma(y,x;\lambda)\left\{U(y)-f_\delta(y)\right\}ds_y\right|+$$

$$+\left|\int\limits_{T}N_\sigma(y,x;\lambda)U(y)ds_y\right|\leq\int\limits_{S}\left|N_\sigma(y,x;\lambda)\right|\left|\left\{U(y)-f_\delta(y)\right\}\right|ds_y+$$

$$+\int\limits_{T}\left|N_\sigma(y,x;\lambda)\right|\left|U(y)\right|ds_y\leq\delta\int_{S}\left|N_\sigma(y,x;\lambda)\right|ds_y+M\int\limits_{T}\left|N_\sigma(y,x;\lambda)\right|ds_y$$

and

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right|=\left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\left\{U(y)-f_\delta(y)\right\}ds_y\right|+$$

$$+\left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\leq\int\limits_S\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|\left|\left\{U(y)-f_\delta(y)\right\}\right|ds_y+$$

$$+\int\limits_T\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|\left|U(y)\right|ds_y\leq\delta\int\limits_S\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y+$$

$$+M\int\limits_T\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y,\ j=1,...,m.$$

Now, repeating the proofs of Theorems 2 and 3, we obtain

$$\left|U(x)-U_{\sigma(\delta)}(x)\right|\leq\frac{C(x)\sigma^{k+1}}{2}(\delta e^{\sigma a}+M)e^{-\sigma x_m},$$

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{U_{\sigma(\delta)}(x)}{\partial x_j}\right|\leq\frac{C(x)\sigma^{k+1}}{2}(\delta e^{\sigma a}+M)e^{-\sigma x_m},\ j=1,...,m.$$

From here, choosing σ from the equality (38), we obtain the estimates (50) and (51).

Theorem 3 is proved. ◄

Corollary 3. *For every $x\in G$, the following equalities are true:*

$$\lim_{\delta\to 0}U_{\sigma(\delta)}(x)=U(x),\ \lim_{\delta\to 0}\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}=\frac{\partial U(x)}{\partial x_j},\ j=1,...,m.$$

Corollary 4. *If $x\in\overline{G}_\varepsilon$, then the families of functions $\left\{U_{\sigma(\delta)}(x)\right\}$ and $\left\{\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ converge uniformly as $\delta\to 0$, i.e.*

$$U_{\sigma(\delta)}(x)\rightrightarrows U(x),\ \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\rightrightarrows\frac{\partial U(x)}{\partial x_j},\ j=1,...,m.$$

5. Conclusion

The following results have been obtained in this article.

Using the Carleman function, a formula is obtained for the continuation of the solution of linear elliptic system of the first order with constant coefficients in a spatial bounded domain $\mathbb{R}^m$ $(m = 2k+1,\ k \geq 1)$. The resulting formula is an analogue of the classical formula of B. Riemann, W. Voltaire and J. Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem is considered when, instead of the exact data of the Cauchy problem, their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on the part T of the boundary of the domain G. An explicit regularization formula is obtained.

Thus, the functionals $U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ define the regularization of the solution of the problem (1), (2).

References

[1] A.A. Shlapunov, *On the Cauchy problem for the Laplace equation*, Siberian Mathematical Journal, **43(4)**, 1992, 953–963.

[2] A.A. Shlapunov, *Boundary problems for Helmholtz equation and the Cauchy problem for Dirac operators*, Journal of Siberian Federal University. Mathematics & Physics, **4(2)**, 2011, 217–228.

[3] A. Bers, F. John, M. Shekhter, *Partial Differential Equations*, Mir, Moscow, 1966.

[4] A.N. Tikhonov, *On the solution of ill-posed problems and the method of regularization*, Reports of the USSR Academy of Sciences, **151(3)**, 1963, 501–504.

[5] D.A. Juraev, *The construction of the fundamental solution of the Helmholtz equation*, Reports of the Academy of Sciences of the Republic of Uzbekistan, **2**, 2012, 14–17.

[6] D.A. Juraev, *Regularization of the Cauchy problem for systems of equations of elliptic type*, Saarbrucken: LAP Lambert Academic Publishing, Germany, 2014.

[7] D.A. Juraev, *Regularization of the Cauchy problem for systems of elliptic type equations of first order*, Uzbek Mathematical Journal, **2**, 2016, 61–71.

[8] D.A. Juraev, *The Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain*, Siberian Electronic Mathematical Reports, **14**, 2017, 752–764.

[9] D.A. Juraev, *Cauchy problem for matrix factorizations of the Helmholtz equation*, Ukrainian Mathematical Journal, **69(10)**, 2017, 1364–1371.

[10] D.A. Juraev, *On the Cauchy problem for matrix factorizations of the Helmholtz equation in a bounded domain*, Siberian Electronic Mathematical Reports, **15**, 2018, 11–20.

[11] D.A. Juraev, *The Cauchy problem for matrix factorizations of the Helmholtz equation in* $\mathbb{R}^3$, Journal of Universal Mathematics, **1(3)**, 2018, 312–319.

[12] D.A. Juraev, *On the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain in* $\mathbb{R}^2$, Siberian Electronic Mathematical Reports, **15**, 2018, 1865–1877.

[13] D.A. Juraev, *On a regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation*, Advanced Mathematical Models & Applications, **4(1)**, 2019, 86–96.

[14] D.A. Juraev, *On the Cauchy problem for matrix factorizations of the Helmholtz equation*, Journal of Universal Mathematics, **2(2)**, 2019, 113–126.

[15] D.A. Juraev, *The solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation*, Advanced Mathematical Models & Applications, **5(2)**, 2020, 205–221.

[16] E.V. Arbuzov, A.L. Bukhgeim, *The Carleman formula for the Helmholtz equation*, Siberian Mathematical Journal, **47(3)**, 2006, 518–526.

[17] G.M. Goluzin, V.M. Krylov, *The generalized Carleman formula and its application to the analytic continuation of functions*, Sbornik: Mathematics, **40(2)**, 1993, 144–149.

[18] I.E. Kabanikhin, Y.S. Gasimov, D.B. Nurseitov, M.A. Shishlenin, B.B. Sholpanbaev, S. Kasenov, *Regularization of the continuation problem for elliptic equation*, Journal of Inverse and Ill-Posed Problems, **21(6)**, 2013, 871–874.

[19] I.E. Niyozov, *On the continuation of the solution of systems of equations of the theory of elasticity*, Uzbek Mathematical Journal, **3**, 2015, 95–105.

[20] I.E. Niyozov, *Regularization of a nonstandard Cauchy problem for a dynamic Lame system*, Izvestiya Vysshikh Uchebnykh Zavedenii, **4**, 2020, 54–63.

[21] J. Hadamard, *The Cauchy problem for linear partial differential equations of hyperbolic type*, Nauka, Moscow, 1978.

[22] L.A. Aizenberg, *Carleman's formulas in complex analysis*, Nauka, Novosibirsk, 1990.

[23] M.M. Lavrent'ev, *On the Cauchy problem for second-order linear elliptic equations*, Reports of the USSR Academy of Sciences, **112(2)**, 1957, 195–197.

[24] M.M. Lavrent'ev, *On some ill-posed problems of mathematical physics*, Nauka, Novosibirsk, 1962.

[25] M.A. Aleksidze, *Fundamental functions in approximate solutions of boundary value problems*, Nauka, Moscow, 1991.

[26] M. Ikehata, *Inverse conductivity problem in the infinite slab*, Inverse Problems, **17**, 2001, 437–454.

[27] M. Ikehata, *Probe method and a Carleman function*, Inverse Problems, **23**, 2007, 1871–1894.

[28] N.N. Tarkhanov, *Stability of the solutions of elliptic systems*, Funct. Anal. Appl., **19(3)**, 1985, 245–247.

[29] N.N. Tarkhanov, *On the Carleman matrix for elliptic systems*, Reports of the USSR Academy of Sciences, **284(2)**, 1985, 294–297.

[30] N.N. Tarkhanov, *The solvability criterion for an ill-posed problem for elliptic systems*, Reports of the USSR Academy of Sciences, **380(3)**, 1989, 531–534.

[31] N.N. Tarkhanov, *The Cauchy problem for solutions of elliptic equations*, Akad. Verl., Berlin, V. 7. 1995.

[32] P.K. Kythe, *Fundamental solutions for differential operators and applications*, Birkhauser, Boston, 1996.

[33] K.A. Berdawood, A. Nachaoui, R.K. Saeed, M. Nachaoui, F. Aboud, *The solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation*, Advanced Mathematical Models & Applications, **5(1)**, 2020, 131–139.

[34] T. Carleman, *Les fonctions quasi analytiques*, Gautier-Villars et Cie., Paris, 1926.

[35] Sh. Yarmukhamedov, *On the Cauchy problem for the Laplace equation*, Reports of the USSR Academy of Sciences, **235(2)**, 1977, 281–283.

[36] Sh. Yarmukhamedov, *On the extension of the solution of the Helmholtz equation*, Reports of the Russian Academy of Sciences, **357(3)**, 1977, 320–323.

[37] Sh. Yarmukhamedov, *The Carleman function and the Cauchy problem for the Laplace equation*, Siberian Mathematical Journal, **45(3)**, 2004, 702–719.

[38] Sh. Yarmukhamedov, *Representation of Harmonic Functions as Potentials and the Cauchy Problem*, Math. Notes, **83(5)**, 2008, 763–778.

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