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Some Properties of Fuzzy Frame Operator

B. Daraby*, F. Delzendeh, A. Rahimi

Abstract. In this paper, some results about fuzzy frames on fuzzy Hilbert spaces in the sense of Bag and Samanta are proved. We investigate approximation for the inverse fuzzy frame operator.

Key Words and Phrases: fuzzy norm, fuzzy inner product space, fuzzy Hilbert space, fuzzy Riesz bases.

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1. Introduction

The idea of fuzzy norms on a linear space was first introduced by Katsaras [18] in 1984. Later on, many authors, such as Felbin [17], Cheng, Mordeson [8], Bag and Samanta [2] etc. gave different definitions of fuzzy normed linear spaces. R. Biswas [7] and A. M. El-Abye and H. M. El-Hamouly [16] tried to give a meaningful definition of fuzzy inner product space and associated fuzzy norm function restricted to the real linear space only. P. Mazumder and S. K. Samanta introduced the definition of fuzzy inner product space in terms of Bag and Samanta fuzzy norm [2]. Resently, Daraby and et al. [11] studied some properties of fuzzy Hilbert spaces and they showed that all results in classical Hilbert spaces are immediate consequences of the corresponding results for Felbin-fuzzy Hilbert spaces. Moreover, by an example, they showed that the spectrum of the category of Felbin-fuzzy Hilbert spaces is broader than the category of classical Hilbert spaces [12].

One of the important concepts in the study of vector spaces is a basis, which allows every vector to be uniquely represented as a linear combination of the basis

*Corresponding author.

elements. The main feature of a basis $\{x_k\}$ in a Hilbert space H is that every $x \in H$ can be represented as a linear combination of the elements x_k in the form:

$$x = \sum_{k=1}^{\infty} c_k(x) x_k. \tag{1}$$

The coefficients $c_k(x)$ are unique. However, the linear independence property for a basis -which implies the uniqueness of coefficients- is restrictive in applications; sometimes it is impossible to find vectors which both fulfill the basis requirements and also satisfy external conditions demanded by applied problems. For such purposes, a more flexible types of spanning sets is needed. Frames provide these alternatives. Frames are used in signal and image processing, non-harmonic Fourier series, data compression, and sampling theory. Frames were introduced by Duffin and Schaeffer [15] for Hilbert space in 1952 to study the non-harmonic Fourier. After some decades, Daubechies, Grossmann and Meyer reintroduced frames with extensive studies, in 1986 [13, 14]. For a more complete treatment of frame theory we recommend the excellent book of Christensen [9]. Today, frame theory has ever increasing applications for problems in both pure and applied mathematics, physics, engineering, computer science, etc.

Many physical systems are inherently nonlinear functions and must be described by non-linear models. But some systems have structural uncertainty, and it is not possible to provide an accurate mathematical model for them. Therefore, for these systems, the conventional control models can not be used. For example to solving this problems, B. T. Bilalov and etc. investigated the intuitionists and its the basicity properties system in fuzzy metric spaces [4, 5, 6]. So, we need to use a new concept, namely, fuzzy frames theory and fuzzy wavelets. Fuzzy frame and fuzzy wavelet are based on frame theory, wavelet theory and fuzzy concepts. For approximation functions, control and identification of nonlinear systems see [3, 21] . It not only retains the frame and wavelet properties, but also has some advantages such as simple structure for approximation and good interoperability approximation of non-linear functions.

In this paper, some results about fuzzy frames on fuzzy Hilbert spaces in the sense of Bag and Samanta are proved. We define dual fuzzy frames and fuzzy Riesz bases and establish some fundamental results via dual fuzzy frames and fuzzy Riesz bases. We also investigate the relationship between fuzzy frames and fuzzy Riesz bases.

2. Some preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 1. *[2]. Let U be a linear space over the field F. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if for all $x, u \in U$ and $c \in F$, the following conditions are satisfied:*

(N1) $\forall t \in \mathbb{R}$ *with* $t \leq 0$, $N(x,t) = 0$;

(N2) $(\forall t \in \mathbb{R}, t > 0, N(x,t) = 1)$ *iff* $x = \underline{0}$;

(N3) $\forall t \in \mathbb{R}, t > 0, N(cx,t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$;

(N4) $\forall s, t \in \mathbb{R}, x, u \in U, N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$;

(N5) $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$.

The pair (U, N) will be referred to as a fuzzy normed linear space.

Theorem 1. *[2] Let (U, N) be a fuzzy normed linear space. Assume further that,*

(N6) $\forall t > 0, N(x,t) > 0 \Rightarrow x = \underline{0}$.

Define $\|x\|_\alpha = \bigwedge\{t > 0 : N(x,t) \geq \alpha\}$, $\alpha \in (0,1)$. Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U and they are called α- norms on U corresponding to the fuzzy norm N on U.

Definition 2. *[1] Let (U, N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if there exists $x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$, denoted by $\lim x_n$.*

Proposition 1. *[8] Let (U, N) be a fuzzy normed linear space satisfying (N_6) and $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ converges to x iff $x_n \to x$ w.r.t. $\|.\|_\alpha$, for all $\alpha \in (0,1)$.*

Definition 3. *[1] Let (U, N) be a fuzzy normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be α-convergent in U if there exists $x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) > \alpha$, for all $t > 0$ and x is called the limit of $\{x_n\}$.*

Proposition 2. *[20] Let (U, N) be a fuzzy normed linear space satisfying (N_6). If $\{x_n\}$ is an α-convergent sequence in (U, N), then $\|x_n - x\|_\alpha \to 0$ as $n \to \infty$.*

Definition 4. *[19] Let U be a linear space over the field $\mathbb{C}$ of complex numbers. Let $\mu : U \times U \times \mathbb{C} \longrightarrow I = [0,1]$ be a mapping such that the following hold:*

(FIP1) *for* $s, t \in \mathbb{C}, \mu(x+y, z, |t|+|s|) \geq \min\{\mu(x,z,|t|), \mu(y,z,|s|)\}$;

(FIP2) *for* $s, t \in \mathbb{C}, \mu(x,y,|st|) \leq \min\left\{\mu\left(x,x,|s|^2\right), \mu\left(y,y,|t|^2\right)\right\}$;

(FIP3) *for* $t \in \mathbb{C}, \mu(x,y,t) = \mu\left(y,x,\bar{t}\right)$*;*

(FIP4) $\mu(\alpha x,y,t) = \mu(x,y,\frac{t}{|\alpha|}), \alpha\,(\neq 0) \in \mathbb{C}, t \in \mathbb{C}$*;*

(FIP5) $\mu(x,x,t) = 0, \forall t \in \mathbb{C}\backslash\mathbb{R}^+$*;*

(FIP6) $(\mu(x,x,t) = 1, \forall t > 0)$ *iff* $x = \underline{0}$*;*

(FIP7) $\mu(x,x,.) : \mathbb{R} \longrightarrow I$ *is a monotonic non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} \mu(\alpha x,x,t) = 1$.

We call μ *a fuzzy inner product function on* U *and* (U,μ) *a fuzzy inner product space (FIP space).*

Theorem 2. *[19] Let* U *be a linear space over* $\mathbb{C}$ *and* μ *be a FIP on* U*. Then*

$$N(x,t) = \begin{cases} \mu(x,x,t^2) & \text{if } t \in \mathbb{R}, t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

is a fuzzy norm on U*. Now if* μ *satisfies the following conditions:*

(FIP8) $\left(\mu\left(x,x,t^2\right) > 0, \forall t > 0\right) \Rightarrow x = \underline{0}$ *and*

(FIP9) for all $x,y \in U$ *and* $p,q \in \mathbb{R}$,

$$\mu\left(x+y,x+y,2q^2\right) \bigwedge \mu\left(x-y,x-y,2p^2\right) \geq \mu\left(x,x,p^2\right) \bigwedge \mu\left(y,y,q^2\right),$$

then $\|x\|_\alpha = \bigwedge\{t > 0 : N(x,t) \geq \alpha\}, \alpha \in (0,1)$ *is an ordinary norm satisfying parallelogram law. By using polarization identity, we can get ordinary inner product, called the* α*-inner product, as follows:*

$$\langle x,y\rangle_\alpha = \frac{1}{4}\left(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2\right) + \frac{1}{4}i\left(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2\right), \forall \alpha \in (0,1).$$

Definition 5. *[19] Let* (U,μ) *be a FIP space satisfying* $(FIP8)$*. The linear space* U *is said to be level complete if for any* $\alpha \in (0,1)$*, every Cauchy sequence converges w.r.t.* $\|.\|_\alpha$ *(the* α*-norm generated by the fuzzy norm* N *which is induced by fuzzy inner product* μ *).*

Definition 6. *[1] Let* $T : (U,N_1) \longrightarrow (V,N_2)$ *be a fuzzy linear operator where* (U,N_1) *and* (V,N_2) *are fuzzy normed linear spaces. The mapping* T *is said to be strongly fuzzy bounded on* U *if and only if there exists a positive real number* M *such that*

$$N_2(T(x), s) \geq N_1(x, \tfrac{s}{M}), \quad \forall x \in U, \forall s \in \mathbb{R}.$$

Definition 7. *[1] Let $T : (U, N_1) \longrightarrow (V, N_2)$ be a fuzzy linear operator where (U, N_1) and (V, N_2) are fuzzy normed linear spaces. The mapping T is said to be uniformly bounded if there exists $M > 0$ such that*

$$\|Tx\|_\alpha^2 \leq M\|x\|_\alpha^1 \quad \forall \alpha \in (0, 1)$$

where $\|.\|_\alpha^1$ and $\|.\|_\alpha^2$ are α-norms on N_1 and N_2, respectively.

Remark 1. *Let us denote the set of all uniformly bounded linear operators from a fuzzy normed linear space (U, N_1) to (V, N_2) by $B(U, V)$.*

Theorem 3. *[1] Let $T : (U, N_1) \longrightarrow (V, N_2)$ be a fuzzy linear operator where (U, N_1) and (V, N_2) are fuzzy normed linear spaces satisfying (N_6). Then T is strongly fuzzy bounded if and only if it is uniformly bounded with respect to α-norms of N_1 to N_2.*

Definition 8. *[1] Let (U, N_1) and (V, N_2) be two fuzzy normed linear spaces satisfying (N_6). For $T \in B(U, V)$, let*

$$\|T\|_\beta^{'} = \bigvee\nolimits_{x \in U, x \neq \underline{0}} \frac{\|Tx\|_\beta^2}{\|x\|_\beta^1}, \quad \beta \in (0, 1),$$

and

$$\|T\|_\alpha = \bigvee\nolimits_{\beta \leq \alpha} \|T\|_\beta^{'}, \quad \alpha \in (0, 1).$$

Then $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms in $B(U, V)$.

Definition 9. *[19] Let (U, μ) be a FIP space. The linear space U is said to be a fuzzy Hilbert space, if it is level complete.*

Definition 10. *[19] Let $\alpha \in (0, 1)$ and (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. Now, if $x, y \in U$ are such that $\langle x, y\rangle_\alpha = 0$, then we say that x, y are α-fuzzy orthogonal to each other, denoted by $x \perp_\alpha y$. Let M be a subset of U and $x \in U$. Now if $\langle x, y\rangle_\alpha = 0$ for all $y \in M$, then we say that x is α-fuzzy orthogonal to M, denoted by $x \perp_\alpha M$. The set of all α-fuzzy orthogonal elements to M is called an α-fuzzy orthogonal set.*

Definition 11. *[19] Let (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. Now if $x, y \in U$ are such that $\langle x, y\rangle_\alpha = 0$ for all $\alpha \in (0, 1)$, then we say that x, y are fuzzy orthogonal to each other, denoted by $x \perp_\alpha y$.*

Thus $x \perp y$ if and only if $x \perp_\alpha y$ for all $(0, 1)$. The set of all fuzzy orthogonal elements to each other is called a fuzzy orthogonal set.

Definition 12. *[20] Let (U,μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$ and $\alpha \in (0,1)$. An α-fuzzy orthogonal set M in U is said to be α-fuzzy orthonormal if the elements have α-norm $1, \alpha \in (0,1)$, that is for all $x, y \in M$,*

$$\langle x, y\rangle_\alpha = \begin{cases} 1 & , \quad x = y \\ 0 & , \quad x \neq y, \end{cases}$$

where $\langle .,. \rangle_\alpha$ is induced by inner product μ.

Definition 13. *[20] Let (U,μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. A fuzzy orthonormal set M in U is said to be fuzzy orthonormal if the elements have α-norm 1 for all $\alpha \in (0,1)$, that is for all $x, y \in M$*

$$\langle x, y\rangle_\alpha = \begin{cases} 1 & , \quad x = y \\ 0 & , \quad x \neq y, \end{cases}$$

where $\langle .,. \rangle_\alpha$ is induced by inner product μ.

Proposition 3. *[20] An α-fuzzy orthonormal set and a fuzzy orthonormal set in a FIP space are linearly independent.*

Definition 14. *[10] Let (U,μ) and (V,μ) be two fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$. Let T be a strongly fuzzy bounded linear operator from U to V. If there exists an operator T^* from V to U such that for all $\alpha \in (0,1)$*

$$\langle Tx, y\rangle_\alpha = \langle x, T^*y\rangle_\alpha, \forall x \in U, y \in V,$$

then the operator T^* is called fuzzy adjoint of operator T.

In the following example, we give that the fuzzy inner product, results the classic inner product.

Example 1. *[10] Let $(U, \langle .,. \rangle)$ be a real inner product space. Define a function $\mu : U \times U \times \mathbb{C} \to [0,1]$ by*

$$\mu(x,y,t) = \begin{cases} \dfrac{|t|}{|t| + \|x\|\|y\|} & \text{if } t > \|x\|\|y\|, \\ 0 & \text{if } t \le \|x\|\|y\|, \\ 0 & \text{if } t \in \mathbb{C} \setminus \mathbb{R}^+. \end{cases}$$

Then μ is a fuzzy inner product function on U and (U,μ) is a fuzzy inner product space.

Hence, we conclude that every classic inner product induces the fuzzy inner product. Next we will show that $(FIP8)$ also holds. So, we have fuzzy norm in the sense of Bag and Samanta.

(FIP8) $\mu(x,x,t^2) > 0, \forall t > 0 \Rightarrow t > \|x\|^2 \quad \forall t > 0 \Rightarrow x = \underline{0}$.

$$\begin{aligned}
\|x\|_\alpha &= \bigwedge \left\{t : \mu(x,x,t^2) \geq \alpha\right\} \\
&= \bigwedge \left\{t : \frac{|t|^2}{|t|^2 + \|x\|^2} \geq \alpha\right\} \\
&= \sqrt{\frac{\alpha}{1-\alpha}}\|x\|.
\end{aligned}$$

It is clear that $(FIP9)$ *holds. By using polarization identity, the* α*-inner product follows from classic inner product:*

$$\begin{aligned}
\|x-y\|_\alpha^2 + \|x+y\|_\alpha^2 &= \frac{\alpha}{1-\alpha}\|x-y\|^2 + \frac{\alpha}{1-\alpha}\|x+y\|^2 \\
&= \frac{\alpha}{1-\alpha}(\|x-y\|^2 + \|x+y\|^2) \\
&= \frac{\alpha}{1-\alpha}(2\|x\|^2 + 2\|y\|^2) \\
&= 2(\|x\|_\alpha^2 + \|y\|_\alpha^2).
\end{aligned}$$

So, we have

$$\|x-y\|_\alpha^2 + \|x+y\|_\alpha^2 = \frac{\alpha}{1-\alpha}(2\|x\|^2 + 2\|y\|^2) = 2(\|x\|_\alpha^2 + \|y\|_\alpha^2).$$

It follows that

$$\begin{aligned}
\langle x,y\rangle_\alpha &= \frac{1}{4}(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2) + \frac{i}{4}(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2) \\
&= \frac{\alpha}{4(1-\alpha)}(\|x+y\|^2 - \|x-y\|^2) + \frac{\alpha i}{4(1-\alpha)}(\|x+iy\|^2 - \|x-iy\|^2) \\
&= \frac{\alpha}{1-\alpha}\langle x,y\rangle.
\end{aligned}$$

Example (1) shows that the fuzzy inner product implies the classic inner product.

Lemma 1. *[10] Let* (U,μ) *be a fuzzy Hilbert space satisfying* $(FIP8)$ *and* $(FIP9)$*. If* $\alpha \in (0,1)$ *and* $x,y,z \in U$*, where* $\langle x,y\rangle_\alpha = \langle x,z\rangle_\alpha$*, then* $x = z$*.*

Theorem 4. *[10] Let* (U,μ) *be a fuzzy Hilbert space satisfying* $(FIP8)$ *and* $(FIP9)$*. Let* T *be a fuzzy linear operator on* (U,μ)*. Then* T^* *is also a linear operator on* (U,μ) *and following properties hold:*

i) $(T^*)^* = T$;

ii) $(T_1 + T_2)^* = T_1^* + T_2^*$;

iii) $(\lambda T)^* = \overline{\lambda} T^*, \quad \forall \lambda \in \mathbb{C}$;

iv) $(ST)^* = T^* S^*$.

Definition 15. *[10] Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$. A countable family of elements $\{x_k\}_{k=1}^{\infty}$ in U is a fuzzy frame for U if there exist constants $A, B > 0$ such that for all $x \in U$ and $\alpha \in (0,1)$:*

$$A\|x\|_\alpha^2 \leq \sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2 \leq B\|x\|_\alpha^2. \tag{2}$$

The numbers A and B are called fuzzy frame bounds. Fuzzy frame bounds are not unique. The optimal lower fuzzy frame bound is the supremum over all lower fuzzy frame bounds, and the optimal upper fuzzy frame bound is the infimum over all upper fuzzy frame bounds. Note that the optimal fuzzy frame bounds are actually fuzzy frame bounds. If $\|x_k\|_\alpha = 1$, the fuzzy frame is normalized. A fuzzy frame $\{x_k\}_{k=1}^{\infty}$ is tight if $A = B$ and in case $A = B = 1$, we call it **Parseval fuzzy frame**. In case the upper inequality in (2) is satisfied, $\{x_k\}_{k=1}^{\infty}$ is called **fuzzy Bessel sequence**. It follows from the definition that if $\{x_k\}_{k=1}^{\infty}$ is a fuzzy frame for (U, μ), then $\overline{span}\,\{x_k\}_{k=1}^{\infty} = U$.

Consider now a vector space U equipped with a fuzzy frame $\{x_k\}_{k=1}^{\infty}$, and define a linear mapping

$$T : (l^2(\mathbb{N}), \mu) \longrightarrow (U, \mu), \quad T\{\beta_k\}_{k=1}^{\infty} = \textstyle\sum_{k=1}^{\infty} \beta_k x_k$$

T is usually called the pre-fuzzy frame operator or the **fuzzy synthesis operator**. The adjoint operator is given by

$$T^* : (U, \mu) \longrightarrow (l^2(\mathbb{N}), \mu), \quad T^* x = \{\langle x, x_k \rangle_\alpha\}_{k=1}^{\infty},$$

and it is called the **fuzzy analysis operator**. Composing T with its adjoint T^*, we obtain the fuzzy frame operator

$$S : (U, \mu) \longrightarrow (U, \mu), \quad Sx = TT^* x = \sum_{k=1}^{\infty} \langle x, x_k \rangle_\alpha x_k.$$

Note that in terms of the fuzzy frame operator, we have

$$\langle Sx, x \rangle_\alpha = \sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2, \quad \forall x \in U, \forall \alpha \in (0,1).$$

3. Some properties of fuzzy inner product spaces

Theorem 5. *Let $\{x_k\}_{k=1}^{\infty}$ be a fuzzy frame in fuzzy Hilbert space (U, μ) satisfying $(FIP8)$ and $(FIP9)$ with fuzzy frame operator S. Then the following holds:*

i) S is invertible and self-adjoint.

ii) Every $x \in U$ can be represented as

$$x = \sum_{k=1}^{\infty} \langle x, S^{-1}x_k \rangle_\alpha x_k, \quad x \in U, \tag{3}$$

and

$$x = \sum_{k=1}^{\infty} \langle x, x_k \rangle_\alpha S^{-1}x_k, \quad x \in U. \tag{4}$$

Both series converge (w.r.t. $\|.\|_\alpha$; $\alpha \in (0,1)$, where $\|.\|_\alpha$ are the α-norms of N induced by U) iff $\sum_{k=1}^{\infty} |\langle x, S^{-1}x_k \rangle_\alpha|^2$ converges.

Proof.
i) Since $S^* = (TT^*)^* = TT^* = S$, the operator S is self-adjoint. For invertibility of S, firstly, we show that S is one to one. By definition, one has to show that if $Sx = 0$, then $x = \underline{0}$.

If for all $x \in U$, $Sx = 0$, then $0 = \langle Sx, x \rangle_\alpha = \sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2$.

$$A\|x\|_\alpha^2 \leq \textstyle\sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2 = 0$$

$$A\|x\|_\alpha^2 = 0 \Rightarrow \|x\|_\alpha^2 = 0 \Rightarrow x = \underline{0}.$$

So S is injective and actually implies that S^* is surjective and $S = S^*$. Thus S is surjective , but let us give a direct proof. The fuzzy frame condition implies that $\overline{span}\,\{x_k\}_{k=1}^{\infty} = U$. So the fuzzy synthesis operator T is surjective. Given $x \in U$ we can therefore find $y \in l^2(\mathbb{N})$ such that $Ty = x$. We can choose $y \in N_T^{\perp} = R_{T^*}$, so it follows that $R_S = R_{TT^*} = U$. This shows that S is invertible.
ii) Every $x \in U$ has the representation

$$x = SS^{-1}x = TT^*S^{-1}x = \textstyle\sum_{k=1}^{\infty} \langle S^{-1}x, x_k \rangle_\alpha x_k.$$

As S is self-adjoint, we arrive at

$$x = \textstyle\sum_{k=1}^{\infty} \langle x, S^{-1}x_k \rangle_\alpha x_k.$$

Suppose $\Phi_n = \sum_{k=1}^{n} \langle x, S^{-1}x_k \rangle_\alpha x_k$ and $\varphi_n = \sum_{k=1}^{n} |\langle x, S^{-1}x_k \rangle_\alpha|^2$. Then for all $\alpha \in (0,1)$ and $n > m$

$$\|\Phi_n - \Phi_m\|_\alpha^2 = \left\langle \sum_{k=1}^{n} \langle x, S^{-1}x_k\rangle_\alpha x_k, \sum_{k=1}^{m} \langle x, S^{-1}x_k\rangle_\alpha x_k \right\rangle_\alpha,$$

i.e.

$$\|\Phi_n - \Phi_m\|_\alpha^2 = |\langle x, S^{-1}x_{m+1}\rangle_\alpha|^2 + |\langle x, S^{-1}x_{m+2}\rangle_\alpha|^2 + \cdots + |\langle x, S^{-1}x_n\rangle_\alpha|^2.$$

Then

$$\|\Phi_n - \Phi_m\|_\alpha^2 = \varphi_n - \varphi_m, \quad \forall \alpha \in (0,1).$$

Hence Φ_n is Cauchy w.r.t. $\|.\|_\alpha$, for all $\alpha \in (0,1)$ iff φ_n is Cauchy in $\mathbb{R}$. Hence Φ_n is Cauchy iff φ_n is Cauchy in $\mathbb{R}$.

The expansion (4) is proved similarly, using $x = S^{-1}Sx$. ◄

Theorem 5 shows that all information about a given vector $x \in U$ is contained in the sequence $\{\langle x, S^{-1}x_k\rangle_\alpha\}_{k=1}^\infty$. The numbers $\langle x, S^{-1}x_k\rangle_\alpha$ are called fuzzy frame coefficients.

Note that because $S : (U, \mu) \longrightarrow (U, \mu)$ is bijective, the sequence $\left\{S^{-1}x_k\right\}_{k=1}^\infty$ is also a fuzzy frame; it is called the canonical dual fuzzy frame of $\{x_k\}_{k=1}^\infty$.

Lemma 2. *Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$. If $\{x_k\}_{k=1}^\infty$ is a tight fuzzy frame with fuzzy bound A, then the canonical dual fuzzy frame is $\left\{A^{-1}x_k\right\}_{k=1}^\infty$, and*

$$x = \frac{1}{A} \sum_{k=1}^{\infty} \langle x, x_k\rangle_\alpha x_k, \quad \forall x \in U, \alpha \in (0,1).$$

Proof. If $\{x_k\}_{k=1}^\infty$ is a tight fuzzy frame with a fuzzy frame bound A and a fuzzy frame operator S, by Definition 15 for all $x \in U$ and $\alpha \in (0,1)$ we have

$$\langle Sx, x\rangle_\alpha = \sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 = A\|x\|_\alpha^2 = \langle Ax, x\rangle_\alpha.$$

This implies that $S = AI$. Thus, S^{-1} is equal to A^{-1}, and the assertion follows from Theorem 5. ◄

Example 2. *Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$, $\alpha \in (0,1)$ and $\{e_k\}_{k=1}^2$ be a fuzzy orthonormal sequence in U. Suppose*

$$x_1 = e_1, \quad x_2 = e_1 - e_2, \quad x_3 = e_1 + e_2.$$

Then $\{x_k\}_{k=1}^3$ is a fuzzy frame for U. Using the definition of the fuzzy frame operator

$$Sx = \sum_{k=1}^{3} \langle x, x_k\rangle_\alpha x_k,$$

noting that

$$\langle x, y\rangle_\alpha = \frac{\alpha}{1-\alpha}\langle x, y\rangle,$$

we obtain

$$\begin{aligned} Se_1 &= \langle e_1, x_1\rangle_\alpha x_1 + \langle e_1, x_2\rangle_\alpha x_2 + \langle e_1, x_3\rangle_\alpha x_3 \\ &= \frac{\alpha}{1-\alpha}\left(\langle e_1, x_1\rangle x_1 + \langle e_1, x_2\rangle x_2 + \langle e_1, x_3\rangle x_3\right) \\ &= \frac{\alpha}{1-\alpha}\left(e_1 + e_1 - e_2 + e_1 + e_2\right) \\ &= \frac{3\alpha}{1-\alpha} e_1, \end{aligned}$$

and

$$\begin{aligned} Se_2 &= \langle e_2, x_1\rangle_\alpha x_1 + \langle e_2, x_2\rangle_\alpha x_2 + \langle e_2, x_3\rangle_\alpha x_3 \\ &= \frac{\alpha}{1-\alpha}\left(\langle e_2, x_1\rangle x_1 + \langle e_2, x_2\rangle x_2 + \langle e_2, x_3\rangle x_3\right) \\ &= \frac{2\alpha}{1-\alpha} e_2. \end{aligned}$$

Thus

$$S^{-1}e_1 = \frac{1-\alpha}{3\alpha} e_1, \quad S^{-1}e_2 = \frac{1-\alpha}{2\alpha} e_2.$$

By linearity, the canonical dual fuzzy frame is

$$\begin{aligned} \left\{S^{-1}x_k\right\}_{k=1}^{3} &= \left\{S^{-1}x_1, S^{-1}x_2, S^{-1}x_3\right\} \\ &= \left\{S^{-1}e_1, S^{-1}e_1 - S^{-1}e_2, S^{-1}e_1 + S^{-1}e_2\right\} \\ &= \left\{\frac{1-\alpha}{3\alpha}e_1, \frac{1-\alpha}{3\alpha}e_1 - \frac{1-\alpha}{2\alpha}e_2, \frac{1-\alpha}{3\alpha}e_1 + \frac{1-\alpha}{2\alpha}e_2\right\}. \end{aligned}$$

Via Theorem 5, the representation of $x \in U$ in terms of fuzzy frame is given by

$$\begin{aligned} x &= \sum_{k=1}^{3}\langle x, S^{-1}x_k\rangle_\alpha x_k \\ &= \frac{1-\alpha}{3\alpha}\langle x, e_1\rangle_\alpha e_1 + \langle x, \frac{1-\alpha}{3\alpha}e_1 - \frac{1-\alpha}{2\alpha}e_2\rangle_\alpha (e_1 - e_2) \\ &+ \langle x, \frac{1-\alpha}{3\alpha}e_1 + \frac{1-\alpha}{2\alpha}e_2\rangle_\alpha (e_1 + e_2) \\ &= 3\left(\frac{1-\alpha}{3\alpha}\right)\langle x, e_1\rangle_\alpha e_1 + 2\left(\frac{1-\alpha}{2\alpha}\right)\langle x, e_2\rangle_\alpha e_2 \\ &= \frac{1-\alpha}{\alpha}\langle x, e_1\rangle_\alpha e_1 + \frac{1-\alpha}{\alpha}\langle x, e_2\rangle_\alpha e_2 \\ &= \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2. \end{aligned}$$

We finish this section by an example of canonical dual fuzzy frame and non-canonical dual fuzzy frame.

Example 3. *Let* (U,μ) *be a fuzzy Hilbert spaces satisfying* $(FIP8)$ *and* $(FIP9)$ *and let* $\{e_k\}_{k=1}^{\infty}$ *be a fuzzy orthonormal sequence in* U*. Consider fuzzy frame*

$$\{x_k\}_{k=1}^{\infty}=\{e_1,e_1,e_2,e_3,e_4,\cdots\}$$

with fuzzy bounds $A=\dfrac{\alpha}{1-\alpha}$*,* $B=\dfrac{2\alpha}{1-\alpha}$*. The canonical dual fuzzy frame is given by*

$$\{S^{-1}x_k\}_{k=1}^{\infty}=\left\{\frac{1+\alpha}{2\alpha}e_1,\frac{1+\alpha}{2\alpha}e_1,e_2,e_3,e_4,\cdots\right\}.$$

As examples of non-canonical dual fuzzy frames, we have

$$\{y_k\}_{k=1}^{\infty}=\left\{0,\frac{1-\alpha}{\alpha}e_1,e_2,e_3,e_4,\cdots\right\}$$

and

$$\{y_k\}_{k=1}^{\infty}=\left\{\frac{1-\alpha}{3\alpha}e_1,\frac{2-2\alpha}{3\alpha}e_1,e_2,e_3,e_4,\cdots\right\}.$$

4. Approximation of the inverse fuzzy frame operator

Consider a fuzzy frame $\{x_k\}_{k=1}^{\infty}$ and the associated fuzzy frame operator for $\alpha\in(0,1)$

$$S:U\longrightarrow U,\quad Sx=\textstyle\sum_{k=1}^{\infty}\langle x,x_k\rangle_{\alpha}x_k.$$

One of our main results is the fuzzy frame decomposition 4, which for $\alpha\in(0,1)$ and $x\in U$ states that

$$x=\sum_{k=1}^{\infty}\langle x,S^{-1}x_k\rangle_{\alpha}x_k. \tag{5}$$

In practice it can be very difficult to apply the fuzzy frame decomposition directly. The reason is that U usually is an infinite-dimensional fuzzy Hilbert space, which makes it hard to invert the fuzzy frame operator. In case we can not find S^{-1} explicitly, we need to approximate S^{-1}.
Given a fuzzy frame $\{x_k\}_{k=1}^{\infty}$ with fuzzy frame operator S, it is natural to try to approximate S^{-1} using finite subsets of $\{x_k\}_{k=1}^{\infty}$. Given $n\in\mathbb{N}$ the family $\{x_k\}_{k=1}^{n}$

is a fuzzy frame for $U_n := span\,\{x_k\}_{k=1}^n$. Denote its fuzzy frame operator for $\alpha \in (0,1)$ by

$$S_n : U_n \longrightarrow U_n, \quad S_n x = \sum_{k=1}^{n} \langle x, x_k \rangle_\alpha x_k. \tag{6}$$

Note that U_n is finite-dimensional; thus, at least in principle we can find S^{-1} using fuzzy linear algebra. Our first question is whether S_n^{-1} approximates for $\alpha \in (0,1)$, S^{-1} in the sense that

$$\langle x, S_n^{-1} x_k \rangle_\alpha \longrightarrow \langle x, S^{-1} x_k \rangle_\alpha \quad as \quad n \to \infty, \quad \forall x \in U, k \in \mathbb{N}. \tag{7}$$

Theorem 6. *Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$ and let $\{x_k\}_{k=1}^\infty$ be a fuzzy frame. Then (7) holds for $\alpha \in (0,1)$ if and only if*

$$\forall j \in \mathbb{N}, \exists c_j \in \mathbb{R} : \|S_n^{-1} x_j\|_\alpha \le c_j, n \ge j. \tag{8}$$

Proof. First, suppose that (8) is satisfied. Fix $j \in \mathbb{N}$, and define

$$\Phi_n := S_n^{-1} x_j - S^{-1} x_j, \quad n \ge j.$$

We need to prove that for all $x \in U$ and $\alpha \in (0,1)$, $\langle x, \Phi_n \rangle_\alpha$ is α-convergent to zero as $n \to \infty$. Observe that for $\alpha \in (0,1)$

$$Sx = \sum_{k=1}^{\infty} \langle x, x_k \rangle_\alpha x_k = S_n x + \sum_{k=n+1}^{\infty} \langle x, x_k \rangle_\alpha x_k. \tag{9}$$

We will use this to obtain an alternative formula for Φ_n. First, since

$$S\Phi_n := SS_n^{-1} x_j - x_j, \quad n \ge j,$$

an application of (9) on $S_n^{-1} x_j$ for $\alpha \in (0,1)$ yields

$$\begin{aligned} S\Phi_n &= S_n S_n^{-1} x_j + \sum_{k=n+1}^{\infty} \langle S_n^{-1} x_j, x_k \rangle_\alpha x_k - x_j \\ &= \sum_{k=n+1}^{\infty} \langle S_n^{-1} x_j, x_k \rangle_\alpha x_k. \end{aligned}$$

It follows that for $\alpha \in (0,1)$

$$\Phi_n = \textstyle\sum_{k=n+1}^{\infty} \langle S_n^{-1} x_j, x_k \rangle_\alpha S^{-1} x_k, \quad n \ge j.$$

Therefore for $x \in U$ and $\alpha \in (0,1)$ we have

$$\begin{aligned} |\langle x, \Phi_n \rangle_\alpha|^2 &= |\sum_{k=n+1}^{\infty} \langle x_k, S_n^{-1} x_j \rangle_\alpha \langle x, S^{-1} x_k \rangle_\alpha|^2 \\ &\leq \sum_{k=n+1}^{\infty} |\langle x_k, S_n^{-1} x_j \rangle_\alpha|^2 \sum_{k=n+1}^{\infty} |\langle x, S^{-1} x_k \rangle_\alpha|^2 \\ &\leq B \|S_n^{-1} x_j\|_\alpha^2 \sum_{k=n+1}^{\infty} |\langle x, S^{-1} x_k \rangle_\alpha|^2 \\ &\leq B c_j^2 \sum_{k=n+1}^{\infty} |\langle x, S^{-1} x_k \rangle_\alpha|^2. \end{aligned}$$

Since $\{x_k\}_{k=1}^{\infty}$ is a fuzzy frame for $\alpha \in (0,1)$, the series $\sum_{k=n+1}^{\infty} |\langle x, S^{-1} x_k \rangle_\alpha|^2$ is α-convergent to zero as $n \to \infty$. Therefore our estimate proves that $\langle x, \Phi_n \rangle_\alpha$ is α-convergent to zero as $n \to \infty$ as required. On the other hand, if we assume that (7) is satisfied, we can fix an arbitrary $j \in \mathbb{N}$ and consider the functionals

$$A_n : U \longrightarrow \mathbb{C}, \quad A_n x = \langle x, S_n^{-1} x_j \rangle_\alpha, \quad n \geq j, \alpha \in (0,1).$$

Each A_n is bounded, and by (7) the family of operators $\{A_n\}_{n \geq j}$ is α-convergent. Thus, the family of α-norms $\{\|A_n\|_\alpha\}_{n \geq j}$ is fuzzy bounded. ◀

Remark 2. *If the results of Theorem 6 are equivalent for each $\alpha \in (0,1)$, then the same result hold in the classical form.*

Example 4. *Let (U, μ) be a fuzzy Hilbert spaces satisfying $(FIP8)$ and $(FIP9)$, $\alpha \in (0,1)$ and $\{e_k\}_{k=1}^{\infty}$ be a fuzzy orthonormal basis for U. We define*

$$x_1 = e_1, \quad x_k = e_{k-1} + \frac{1}{k} e_k, \quad k \geq 2.$$

We know that $\{x_k\}_{k=2}^{\infty}$ is a fuzzy frame with fuzzy bounds $\frac{\alpha}{4(1-\alpha)}$, $\frac{9\alpha}{4(1-\alpha)}$; so $\{x_k\}_{k=1}^{\infty}$ is a fuzzy frame. For $n \in \mathbb{N}$ we want to find $x := S_n^{-1} x_1$, i.e., to solve the equation

$$\textstyle\sum_{k=1}^{n} \langle x, x_k \rangle_\alpha x_k = x_1, \quad x \in U_n, \alpha \in (0,1).$$

The equation for $\alpha \in (0,1)$ can be written as

$$\sum_{k=1}^{n-1} \left(\frac{\alpha}{k(1-\alpha)} \langle x, x_k \rangle + \langle x, x_{k+1} \rangle \right) e_k + \frac{\alpha}{n(1-\alpha)} \langle x, x_n \rangle e_n = e_1. \tag{10}$$

It follows that

$$\langle x, x_n\rangle = 0, \quad \langle x, x_k\rangle = -k\langle x, x_{k-1}\rangle, \quad k = 2, 3, \cdots, n-1.$$

So $\langle x, x_k\rangle = 0$ *for all* $k = 2, 3, \cdots, n$. *Again by 10 we have* $\langle x, x_1\rangle = \frac{\alpha}{1-\alpha}$*; expressing that last two conclusions in terms of* $\{e_k\}_{k=1}^{\infty}$ *we have*

$$\langle x, e_1\rangle_\alpha = \frac{\alpha}{1-\alpha}\langle x, e_1\rangle = \frac{\alpha}{1-\alpha}, \quad \langle x, e_2\rangle_\alpha = -2\langle x, e_1\rangle_\alpha = \frac{-2\alpha}{1-\alpha},$$

and in general

$$\langle x, e_k\rangle_\alpha = -k\langle x, e_{k-1}\rangle_\alpha = (-1)^{k-1}\frac{k!\alpha}{1-\alpha}, \quad \alpha \in (0,1), k = 2, \cdots, n.$$

Since $x \in U_n = span\{e_k\}_{k=1}^{n}$, *this implies that*

$$S_n^{-1}x_1 = \textstyle\sum_{k=1}^{n}\langle x, e_k\rangle_\alpha e_k = \sum_{k=1}^{n}(-1)^{k-1}\frac{k!\alpha}{1-\alpha}e_k.$$

5. Conclusion

In this paper, we consider fuzzy inner product space introduced by Bag and Samanta. α- fuzzy orthonormal set, complete fuzzy orthonormal set etc. have been introduced. We also investigate approximation for the inverse fuzzy frame operator. In fuzzy Hilbert space, finding and estimating S^{-1} is better than in the classic Hilbert space. We think that these results will be helpful for the researchers to develop fuzzy functional analysis especially for frame theory.

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B. Daraby
Department of Mathematics, University of Maragheh, P. O. Box 55181-83111, Maragheh, Iran,
E-mail: bdaraby@maragheh.ac.ir

F. Delzendeh
Department of Mathematics, University of Maragheh, Maragheh, Iran,
E-mail: fatenehdelzende@yahoo.com

A. Rahimi
Department of Mathematics, University of Maragheh, P. O. Box 55181-83111, Maragheh, Iran,
E-mail: rahimi@maragheh.ac.ir

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