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On Double Difference of Composition Operators from a Space Generated by the Cauchy Kernel and a Special Measure

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Abstract. In this paper, compact double difference of composition operators acting from a space generated by the Cauchy kernel and a special measure to analytic Besov spaces is characterized. Moreover, operator norm of these operators acting from Cauchy transforms to analytic Besov spaces is obtained explicitly.

Key Words and Phrases: double difference of composition operators, space generated by the Cauchy kernel and a special measure, Bergman space, Besov space, Dirichlet space.

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1. Front matter

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ the unit circle, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, $H^\infty = \{f \in H(\mathbb{D}) : \|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty\}$ and dA the normalized area measure on $\mathbb{D}$, that is,

$$dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta.$$

The space of functions generated by Cauchy transforms of special measures, denoted by $\mathcal{F}$, is the subspace of $H(\mathbb{D})$ consisting of functions which admits an integral representation defined as

$$f(z) = \int_{\mathbb{T}} \frac{1}{1-\overline{x}z}d\mu(x). \tag{1}$$

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With respect to the norm defined as

$$\|f\|_{\mathcal{F}} = \inf\left\{\|\mu\| : \mu \in \mathcal{M} \text{ and } f(z) = \int_{\mathbb{T}} \frac{1}{1-\overline{x}z} d\mu(x)\right\},$$

$\mathcal{F}$ is a Banach space, where $\mathcal{M}$ is a class consisting of all complex Borel measures on $\mathbb{T}$ and $\|\mu\|$ is the total variation of μ. For more about these spaces and linear operators on them, we refer to [1]-[4], [7]-[10] and [12].

Let dA_α, $\alpha \in (-1,\infty)$ be a probability measure on $\mathbb{D}$ defined as

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z), \quad z \in \mathbb{D}.$$

For $1 \le p < \infty$ and $\alpha > -1$, the weighted Bergman space $\mathcal{A}^p_\alpha$ is a subspace of $H(\mathbb{D})$ and is a Banach space with the norm

$$||f||_{\mathcal{A}^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)\right)^{1/p} < \infty.$$

The analytic Besov space B^p, $(1 < p < \infty)$ is the Banach subspace $H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f'(z)|^p dA_{p-2}(z) < \infty.$$

The norm for a B^p space is defined by

$$\|f\|_{\mathcal{B}^p} := |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p dA_{p-2}(z)\right)^{1/p} < \infty.$$

The space B^2 is the Dirichlet space $\mathcal{D}$ equipped with an equivalent norm. For more about analytic Besov spaces we refer the reader to [13] and [14].

Let $S(\mathbb{D}) = \{\varphi \in H(\mathbb{D}) : \varphi(\mathbb{D}) \subset \mathbb{D}\}$. Then for a $\varphi \in S(\mathbb{D})$, composition operator C_φ is defined as

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbb{D})$ and is extensively studied on different subspaces of $H(\mathbb{D})$. See [6] and [11].

Recently, Choe et. al. [5] proved that $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{A}^p_\alpha \to \mathcal{A}^p_\alpha$ is compact if and only if

$$\lim_{|z|\to 1} (M_{12} + M_{34})(z)(\widehat{M_{13} + M}_{24})(z) = 0,$$

where

$$M_{ij} = \left[\sum_{k\in\{i,j\}} \frac{1-|z|}{1-|\varphi_k(z)|}\right]\rho(\varphi_i(z), \varphi_j(z))$$

and $\rho(\zeta, z)$ is the pseudo-hyperbolic distance between ζ and z in $\mathbb{D}$ given by

$$\rho(\zeta, z) = \left|\frac{\zeta - z}{1 - \bar{\zeta} z}\right|.$$

Motivated by their results, we completely characterize compact double difference of composition operators from the space of functions generated by Cauchy transforms of special measures to Besov spaces. Let $\mathbb{N}_4 = \{1, 2, 3, 4\}$ and $\varphi_i \in S(\mathbb{D})$ for $i \in \mathbb{N}_4$. Throughout this paper, constants are positive and not necessarily the same at each occurrence and are denoted by C.

2. Primary document

In this section, double difference of composition operators from Cauchy transforms to Besov spaces are characterized.

Theorem 1. *Let $1 < p < \infty$ and $\varphi_i \in S(\mathbb{D})$, $i \in \mathbb{N}_4$. Then $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{F} \to \mathcal{B}^p$ is bounded if and only if the family of functions*

$$\left\{ \sum_{i \in \mathbb{N}_4} \eta_i \frac{\varphi_i'}{(1 - \overline{x}\varphi_i)^2} : x \in \mathbb{T} \right\} \subset \mathcal{A}^p_{p-2}$$

is norm bounded, where

$$\eta_i = \begin{cases} 1 & \text{if} \quad i = 1 \text{ or } 4; \\ -1 & \text{if} \quad i = 2 \text{ or } 3, \end{cases}$$

that is,

$$N^p = \sup_{x \in \mathbb{T}} \int_{\mathbb{D}} \left| \sum_{i \in \mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1 - \overline{x}\varphi_i(z))^2} \right|^p dA_{p-2}(z) \leq C < \infty. \tag{2}$$

for some $C > 0$. Furthermore, the following equality holds:

$$\|(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4})\|_{\mathcal{F} \to \mathcal{B}^p} = \sup_{x \in \mathbb{T}} \left| \sum_{i \in \mathbb{N}_4} \frac{\eta_i}{1 - \overline{x}\varphi_i(0)} \right| + N. \tag{3}$$

Proof. First suppose that condition in (2) holds. For each $f \in \mathcal{F}$, there is a measure μ in $\mathcal{M}$ such that (1) and the equality $\|\mu\| = \|f\|_{\mathcal{F}}$ hold. Taking derivative of (1) with respect to z, replacing z in the equation so obtained by $\varphi_i,\ i \in \mathbb{N}_4$, respectively, we obtain

$$f'(\varphi_i(z)) = \int_{\mathbb{T}} \frac{\overline{x}}{(1-\overline{x}\varphi_i(z))^2} d\mu(x), \qquad i = \mathbb{N}_4. \tag{4}$$

Multiplying the equations in (4) by $\eta_i\varphi_i'(z)$, respectively, adding the equations so obtained, using a well known inequality, we obtain

$$\Big|\sum_{i\in\mathbb{N}_4} \eta_i f'(\varphi_i(z))\varphi_i'(z)\Big|^p \le \|\mu\|^p \int_{\mathbb{T}} \Big|\sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p \frac{d|\mu|(x)}{\|\mu\|}. \tag{5}$$

Integrating (5) with respect to $dA_{p-2}(z)$, then using Fubini's theorem, by (1) we have

$$\begin{aligned} &\int_{\mathbb{D}}\Big|\sum_{i\in\mathbb{N}_4} \eta_i f'(\varphi_i(z))\varphi_i'(z)\Big|^p dA_{p-2}(z) \\ &\quad\le \|\mu\|^{p-1} \int_{\mathbb{T}} \Big[\int_{\mathbb{D}} \Big|\sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p dA_{p-2}(z)\Big] d|\mu|(x) \qquad (6) \\ &\quad\le N^p\|\mu\|^{p-1}\int_{\mathbb{T}} d|\mu|(x) = N^p\|\mu\|^p = N^p\|f\|_{\mathcal{F}}^p. \qquad (7) \end{aligned}$$

Using (7) and the fact that

$$\Big|\Big(\Big(\sum_{i\in\mathbb{N}_4}\eta_i C_{\varphi_i}\Big)f\Big)(0)\Big| \le \sup_{x\in\mathbb{T}}\Big|\sum_{i\in\mathbb{N}_4}\frac{\eta_i}{1-\overline{x}\varphi_i(0)}\Big|\|f\|_{\mathcal{F}},$$

we have

$$\Big\|\Big(\sum_{i\in\mathbb{N}_4}\eta_i C_{\varphi_i}\Big)f\Big\|_{\mathcal{B}^p} \le \Big\{\sup_{x\in\mathbb{T}}\Big|\sum_{i\in\mathbb{N}_4}\frac{\eta_i}{1-\overline{x}\varphi_i(0)}\Big| + N\Big\}\|f\|_{\mathcal{F}}.$$

Thus $(C_{\varphi_1}-C_{\varphi_2})-(C_{\varphi_3}-C_{\varphi_4}) : \mathcal{F}\to\mathcal{B}^p$ is bounded and

$$\|(C_{\varphi_1}-C_{\varphi_2})-(C_{\varphi_3}-C_{\varphi_4})\|_{\mathcal{F}\to\mathcal{B}^p} \le \sup_{x\in\mathbb{T}}\Big|\sum_{i\in\mathbb{N}_4}\frac{\eta_i}{1-\overline{x}\varphi_i(0)}\Big| + N. \tag{8}$$

Conversely, suppose that $(C_{\varphi_1}-C_{\varphi_2})-(C_{\varphi_3}-C_{\varphi_4})$ maps $\mathcal{F}$ boundedly into $\mathcal{B}^p$. For each $x\in\mathbb{T}$, consider the family $\{f_x : x\in\mathbb{T}$, where $f_x(z) = 1/(1-\overline{x}z)$. Then for each $x\in\mathbb{T}$, we have $\|f_x\|_{\mathcal{F}} = 1$. Since $(C_{\varphi_1}-C_{\varphi_2})-(C_{\varphi_3}-C_{\varphi_4})$ maps $\mathcal{F}$

boundedly into $\mathcal{B}^p$, by the closed graph theorem we have $((C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}))f \in \mathcal{B}^p$ for every $f \in \mathcal{F}$. In particular,

$$\Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f_x \in \mathcal{B}^p,$$

and so we have

$$\sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'}{(1-\overline{x}\varphi_i)^2} \in \mathcal{A}^p_{p-2}$$

for every $x \in \mathbb{T}$. Moreover,

$$\begin{aligned}
&\sup_{x\in\mathbb{T}} \Big| \sum_{i\in\mathbb{N}_4} \frac{\eta_i}{1-\overline{x}\varphi_i(0)} \Big| + N \\
&\le \sup_{x\in\mathbb{T}} \Bigg(\Big| \sum_{i\in\mathbb{N}_4} \frac{\eta_i}{1-\overline{x}\varphi_i(0)} \Big| + \Big(\int_{\mathbb{D}} \Big| \sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2} \Big|^p dA_{p-2}(z) \Big)^{1/p} \Bigg) \\
&= \sup_{x\in\mathbb{T}} \Big\| \Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i} \Big) \Big(\frac{1}{1-\overline{x}z} \Big) \Big\|_{\mathcal{B}^p} \\
&\le \|(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4})\|_{\mathcal{F}\to\mathcal{B}^p} \sup_{x\in\mathbb{T}} \Big\| \frac{1}{1-\overline{x}z} \Big\|_{\mathcal{F}} \\
&= \|(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4})\|_{\mathcal{F}\to\mathcal{B}^p}. \qquad (9)
\end{aligned}$$

Therefore, the inequality in (2) holds. Again, by combining (8) and (9), we get the validity of (3). ◄

The proof of next lemma is standard. See Proposition 3.11 of [6] for detailed arguments.

Lemma 1. *Let $1 < p < \infty$ and $\varphi_i \in S(\mathbb{D})$, $i \in \mathbb{N}_4$ such that each $C_{\varphi_i} : \mathcal{F} \to \mathcal{B}^p$, $i \in \mathbb{N}_4$ is bounded. Then $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{F} \to \mathcal{B}^p$ is compact if and only if for any norm bounded sequence $\{f_j\}$ in $\mathcal{F}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\lim_{j\to\infty} ||((C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}))f_j||_{\mathcal{B}^p} \to 0$.*

Theorem 2. *Let $1 < p < \infty$ and $\varphi_i \in S(\mathbb{D})$, $i \in \mathbb{N}_4$ such that each $C_{\varphi_i} : \mathcal{F} \to \mathcal{B}^p$, $i \in \mathbb{N}_4$ is bounded and $(C_{\varphi_1} - C_{\varphi_2}), (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{F} \to \mathcal{B}^p$ are not compact. Then the following statements are equivalent:*

(1) $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{F} \to \mathcal{B}^p$ is compact.

(2) $\varphi_i \in S(\mathbb{D})$, $i \in \mathbb{N}_4$ are such that

$$\lim_{r\to 1} \sup_{x\in\mathbb{T}} \int_{\min\{|\varphi_i(z)|: i\in\mathbb{N}_4\} > r} \Big| \sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2} \Big|^p dA_{p-2}(z) = 0.$$

(3) For each of $x \in \mathbb{T}$, the integral transform

$$\int_{\mathbb{D}} \left| \sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2} \right|^p dA_{p-2}(z)$$

is a continuous function of x.

(4) For any given $\varepsilon > 0$ and any given $E \subset \mathbb{D}$, there is a $\delta > 0$ such that if $A(E) < \delta$,, then $\nu_x(E) < \varepsilon$ for all $x \in \mathbb{T}$, where the measure ν_x is defined by

$$\nu_x(E) = \int_{E} \left| \sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2} \right|^p dA_{p-2}(z),$$

for each $x \in \mathbb{T}$.

Proof. (1) $\Leftrightarrow$ (2). Suppose that (2) holds. Since $C_{\varphi_i} : \mathcal{F} \to \mathcal{B}^p$, $i \in \mathbb{N}_4$ are bounded, there is a constant $C > 0$ such that $||C_{\varphi_i} f||_{\mathcal{B}^p} \leq C||f||_{\mathcal{F}}$ for all $i \in \mathbb{N}_4$. By considering the identity function $I_0(z) = z$ in $\mathcal{F}$, we can easily show that $\varphi_i \in \mathcal{B}^p$ for each $i \in \mathbb{N}_4$. Thus we can choose $r \in (0,1)$ such that

$$\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r} |\varphi_i'(z)|^2 dA_{p-2}(z) < \varepsilon \tag{10}$$

for each $i \in \mathbb{N}_4$ and for each $\epsilon > 0$. Let f be a function of unit norm in $\mathcal{F}$. For each $t \in (0,1)$, let $f_t(z) = f(tz)$, $z \in \mathbb{D}$. Then $f_t \in \mathcal{F}$ and $\sup_{t\in(0,1)} \|f_t\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$. Moreover, on compact subsets of $\mathbb{D}$, $f_t \to f$ uniformly as $t \to 1$. So the compactness of $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{F} \to \mathcal{B}^p$ asserts that

$$\lim_{t\to1} \left\| \Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f_t - \Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f \right\|_{\mathcal{B}^p} \to 0.$$

Thus there is some $t \in (0,1)$ such that

$$\int_{\mathbb{D}} \left| \left(\Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f_t\right)'(z) - \left(\Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f\right)'(z) \right|^p dA_{p-2}(z) < \varepsilon. \tag{11}$$

Using inequalities (10) in (11), we have

$$\begin{aligned} &\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r} \left| \left(\Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f\right)'(z) \right|^p dA_{p-2}(z) \\ &\leq C \int_{\mathbb{D}} \left| \left(\Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f_t\right)'(z) - \left(\Big(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i}\Big) f\right)'(z) \right|^p dA_{p-2}(z) \end{aligned}$$

$$+ C \int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r} \left| \left(\left(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i} \right) f_t \right)'(z) \right|^p dA_{p-2}(z)$$
$$\leq \varepsilon C(1 + \|f_t'\|_\infty^p).$$

Therefore for every f of unit norm in $\mathcal{F}$, there is a $\gamma_0(f, \varepsilon)$ in $(0,1)$ such that for $r \in (\gamma_0, 1)$, we have

$$\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r} \left| \left(\left(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i} \right) f \right)'(z) \right|^p dA_{p-2}(z) < \varepsilon. \tag{12}$$

Since $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{F} \to \mathcal{B}^p$ is compact, for every f of unit norm in $\mathcal{F}$ and every $\varepsilon > 0$, there is some k and functions f_j, $j \in \mathbb{N}_k$ in the unit ball of $\mathcal{F}$, and some $j \in \mathbb{N}_k$ such that

$$\int_{\mathbb{D}} \left| \left(\left(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i} \right) f \right)'(z) - \left(\left(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i} \right) f_j \right)'(z) \right|^p dA_{p-2}(z) < \varepsilon. \tag{13}$$

From (12), there is some $\gamma \in (0,1)$ such that for $r \in (\delta, 1)$ and $j \in \mathbb{N}_k$, we have

$$\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r} \left| \left(\left(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i} \right) f_j \right)'(z) \right|^p dA_{p-2}(z) < \varepsilon. \tag{14}$$

Combining (13) and (14), we see that for every f in the unit ball of $\mathcal{F}$, there is $r \in (\gamma, 1)$ such that

$$\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r} \left| \left(\left(\sum_{i\in\mathbb{N}_4} \eta_i C_{\varphi_i} \right) f \right)'(z) \right|^p dA_{p-2}(z) < \varepsilon C. \tag{15}$$

Replacing f in (15) by f_x defined in (1), we obtain

$$\sup_{x\in\mathbb{T}} \int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r} \left| \sum_{i=1}^4 \eta_i \frac{\varphi_i'(z)}{(1 - \overline{x}\varphi_i(z))^2} \right|^p dA_{p-2}(z) < \varepsilon C$$

for $r \in (\delta, 1)$. Hence (2) holds.
Again assume that (2) holds. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence as in Lemma 1. Then we need to show that

$$\|((C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}))f_j\|_{\mathcal{B}^p} \to 0 \text{ as } j \to \infty.$$

There exists $\mu_j \in \mathcal{M}$ with $\|\mu_j\| = \|f_j\|_{\mathcal{F}}$ and

$$f_j(z) = \int_{\mathbb{T}} \frac{d\mu_j(x)}{1 - \overline{x}z}. \tag{16}$$

Proceeding as in Theorem 1, we can easily show that

$$\Big|\sum_{i\in\mathbb{N}_4}\eta_i f_j'(\varphi_i(z))\varphi_i'(z)\Big|^p \le \|\mu\|^{p-1}\int_{\mathbb{T}}\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p d|\mu_j|(x). \tag{17}$$

By the condition in (4), for every $\varepsilon>0$, there is some $r\in(0,1)$ such that

$$\sup_{x\in\mathbb{T}}\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r}\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p dA_{p-2}(z)<\varepsilon. \tag{18}$$

Now $(C_{\varphi_1}-C_{\varphi_2}),(C_{\varphi_3}-C_{\varphi_4}):\mathcal{F}\to\mathcal{B}^p$ are not compact, so $\|\varphi_i\|_\infty=1$ for all $i\in\mathbb{N}_4$. Therefore,

$$\begin{aligned}&\|((C_{\varphi_1}-C_{\varphi_2})-(C_{\varphi_3}-C_{\varphi_4}))f_j\|_{\mathcal{B}^p}^p\\ &\le C\bigg(\Big|\sum_{i\in\mathbb{N}_4}\eta_i f_j(\varphi_i(0))\Big|+\int_{\max\{|\varphi_i(z)|:i\in\mathbb{N}_4\}\le r}\Big|\sum_{i\in\mathbb{N}_4}\eta_i f_j'(\varphi_i(z))\varphi_i'(z)\Big|^p dA_{p-2}(z)\\ &\quad+\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r}\Big|\sum_{i\in\mathbb{N}_4}\eta_i f_j'(\varphi_i(z))\varphi_i'(z)\Big|^p dA_{p-2}(z)\bigg).\end{aligned}$$

Using (17), (18), Fubini's theorem and the fact that $\varphi_i\in\mathcal{B}^p$, $i\in\mathbb{N}_4$, and $\sup_{|w|\le r}|f_j'(w)|^p\to 0$ as $j\to\infty$, we have

$$\begin{aligned}&\|((C_{\varphi_1}-C_{\varphi_2})-(C_{\varphi_3}-C_{\varphi_4}))f_j\|_{\mathcal{B}^p}^p\\ &\le C\bigg(\sum_{i\in\mathbb{N}_4}|f_j(\varphi_i(0))|+\sup_{\max\{|\varphi_i(z)|:i\in\mathbb{N}_4\}\le r}|f_j'(\varphi_i(z))|^p\\ &\qquad\times\int_{\max\{|\varphi_i(z)|:i\in\mathbb{N}_4\}\le r}\sum_{i\in\mathbb{N}_4}|\varphi_i'(z)|^p dA_{p-2}(z)\\ &\quad+C\int_{\mathbb{T}}\int_{\min\{|\varphi_i(z)|:i\in\mathbb{N}_4\}>r}\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p dA_{p-2}(z)d|\mu_j|(x)\bigg)\\ &< C\Big(C+\int_{\mathbb{T}}d|\mu_j|(x)\Big)\varepsilon< C\varepsilon\end{aligned}$$

for $j\ge j_0$. Hence (1) holds.

(1) $\Rightarrow$ (3). Let us suppose that $x_j\in\mathbb{T}$ with $x_j\to x$ as $j\to\infty$, and let f_{x_j} be defined as in (1). Then f_{x_j} is of unit norm in $\mathcal{F}$ and $f_{x_j}\to f_x$ uniformly on compact subsets of $\mathbb{D}$. Since $(C_{\varphi_1}-C_{\varphi_2})-(C_{\varphi_3}-C_{\varphi_4}):\mathcal{F}\to\mathcal{B}^p$ is compact. By Lemma 1, we have

$$\Big\|\Big(\sum_{i\in\mathbb{N}_4}\eta_i C_{\varphi_i}\Big)f_{x_j}-\Big(\sum_{i\in\mathbb{N}_4}\eta_i C_{\varphi_i}\Big)f_x\Big\|_{\mathcal{B}^p}\to 0\quad\text{as}\quad j\to\infty.$$

Since $C_{\varphi_i}:\mathcal{F}\to\mathcal{B}^p$, $i\in\mathbb{N}_4$ are bounded, there is a constant $C>0$ such that $||C_{\varphi_i}f_x||_{\mathcal{B}^p}\le C||f_x||_{\mathcal{F}}=C$ for all $i\in\mathbb{N}_4$ and for all $x\in\mathbb{T}$. Therefore, we have

$$\begin{aligned}
&\int_{\mathbb{D}}\bigg|\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x_j}\varphi_i(z))^2}\Big|^p-\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p\bigg|dA_{p-2}(z)\\
&\quad=\int_{\mathbb{D}}\bigg|\Big|\sum_{i\in\mathbb{N}_4}\eta_i f_{x_j}'(\varphi_i(z))\varphi_i'(z)\Big|^p-\Big|\sum_{i\in\mathbb{N}_4}\eta_i f_x'(\varphi_i(z))\varphi_i'(z)\Big|^p\bigg|dA_{p-2}(z)\\
&\quad\le C\bigg(\int_{\mathbb{D}}\Big|\sum_{i\in\mathbb{N}_4}\eta_i f_{x_j}'(\varphi_i(z))\varphi_i'(z)-\sum_{i\in\mathbb{N}_4}\eta_i f_x'(\varphi_i(z))\varphi_i'(z)\Big|^p dA_{p-2}(z)\bigg)^{1/p}\\
&\quad=C\bigg\|\Big(\sum_{i\in\mathbb{N}_4}\eta_i C_{\varphi_i}\Big)f_{x_j}-\Big(\sum_{i\in\mathbb{N}_4}\eta_i C_{\varphi_i}\Big)f_x\bigg\|_{\mathcal{B}^p}\to 0 \text{ as } j\to\infty.
\end{aligned}$$

Thus

$$\int_{\mathbb{D}}\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x_j}\varphi_i(z))^2}\Big|^p dA_{p-2}(z)\ \to\ \int_{\mathbb{D}}\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p dA_{p-2}(z)$$

as $j\to\infty$. Hence for each of $x\in\mathbb{T}$, the integral transform

$$\int_{\mathbb{D}}\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p dA_{p-2}(z)$$

is continuous on $\mathbb{T}$.

(3) $\Rightarrow$ (4). If possible, assume that (4) is not true. Then there are sequences $\{x_k\}\in\mathbb{T}$ and $\{E_k\}\in\mathbb{D}$ such that $x_k\to x$, $A(E_k)\to 0$ as $k\to\infty$, and $\nu_{x_k}(E_k)\ge C$ for all $k\in\mathbb{N}$. Now

$$\begin{aligned}
&|\nu_{x_k}(E_k)-\nu_x(E_k)|\\
&\quad\le\int_{E_k}\bigg|\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x_k}\varphi_i(z))^2}\Big|^p-\Big|\sum_{i=1}^{4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p\bigg|dA_{p-2}(z).
\end{aligned}$$

Thus

$$\begin{aligned}
\nu_{x_k}(E_k)\le\int_{E_k}\bigg|\Big|&\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x_k}\varphi_i(z))^2}\Big|^p\\
&-\Big|\sum_{i\in\mathbb{N}_4}\eta_i\frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2}\Big|^p\bigg|dA_{p-2}(z)+\nu_x(E_k)
\end{aligned}$$

$$\leq C \int_{\mathbb{E}_k} \left| \sum_{i=1}^{4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x_k}\varphi_i(z))^2} - \sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2} \right|^p dA_{p-2}(z) + \nu_x(E_k). \tag{19}$$

Boundedness of $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) : \mathcal{F} \to \mathcal{B}^p$ asserts that $\nu_x(E_k) \to 0$ and so we arrive at a contradiction as then $\nu_{x_k}(E_k) \to 0$ as $k \to \infty$. Therefore, (3) $\Rightarrow$ (4) holds.
(4) $\Rightarrow$ (1). Let $\epsilon > 0$ be given. Then

$$\begin{aligned}
&\int_{\mathbb{D}} \left| \sum_{i\in\mathbb{N}_4} \eta_i f_k'(\varphi_i(z))\varphi_i'(z) \right|^p dA_{p-2}(z) \\
&\quad \leq \|\mu_k\| \int_{\mathbb{T}} \int_{\mathbb{D}} \left| \sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2} \right|^p dA_{p-2}(z) d|\mu_k|(x).
\end{aligned}$$

Choose a compact subset K of $\mathbb{D}$ such that $A(\mathbb{D}\setminus K) < \delta$. Then

$$\begin{aligned}
&\int_{\mathbb{D}\setminus K} \left| \sum_{i\in\mathbb{N}_4} \eta_i f_k'(\varphi_i(z))\varphi_i'(z) \right|^p dA_{p-2}(z) \\
&\quad \leq \|\mu_k\|^{p-1} \int_{\mathbb{T}} \int_{\mathbb{D}\setminus K} \left| \sum_{i\in\mathbb{N}_4} \eta_i \frac{\varphi_i'(z)}{(1-\overline{x}\varphi_i(z))^2} \right|^p dA_{p-2}(z) d|\mu_k|(x) \\
&\quad \leq \epsilon \|\mu_k\|^{p-1} \int_{\mathbb{T}} d|\mu_k|(x) = \epsilon ||f_k||_{\mathcal{F}}^p < \epsilon.
\end{aligned} \tag{20}$$

On K, there is some k_0 such that $|f_k'(\varphi_i(z))|^2 < \epsilon$ for all $i \in \mathbb{N}_4$ and for $k \geq k_0$. Thus for $k \geq k_0$, we have

$$\begin{aligned}
&\int_{K} \left| \sum_{i\in\mathbb{N}_4} \eta_i f_k'(\varphi_i(z))\varphi_i'(z) \right|^p dA_{p-2}(z) \\
&\quad \leq C \int_{K} \sum_{i\in\mathbb{N}_4} |f_k'(\varphi_i(z))|^p |\varphi_i'(z)|^p dA_{p-2}(z) \\
&\quad \leq \epsilon C \int_{K} \sum_{i\in\mathbb{N}_4} |\varphi_i'(z)|^p dA_{p-2}(z) < \epsilon C \sum_{i=1}^{4} \|\varphi_i\|_{\mathcal{B}^p}^p.
\end{aligned} \tag{21}$$

Therefore, by (20), (21) and the fact that $\varphi_i \in \mathcal{B}^p$ for all $i \in \mathbb{N}_4$, we have $||((C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}))f_k||_{\mathcal{B}^p} \to 0$ as $k \to \infty$. This completes the proof of (3) $\Rightarrow$ (1). ◄

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