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Existence of Global Attractors for the Coupled System of Suspension Bridge Equations

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Abstract. In this paper we study the mathematical model of the bridge problem where the roadbed and the tensioning cable have a common point. The correctness of the considered problem is proved and in the linear case the exponential energy decay of the system is shown.
In the case of non-focused non-linear source terms we show the existence of an absorbing set and the asymptotic compactness of the nonlinear semigroup generated by the corresponding dynamic system. By using these results we show that the same nonlinear semigroup has a global minimal attractor.

Key Words and Phrases: coupled suspension bridge equations, semigroup, exponential stability, absorbing set, global attractor.

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1. Introduction

In [1] Lazer and McKenna studied the problem of nonlinear oscillation in a suspension bridge. They presented a one-dimensional mathematical model for the bridge that takes account of the fact that the coupling provided by the stays connecting the main cable to the deck of the road bed is fundamentally nonlinear, that is, they gave rise to some system of semi linear hyperbolic equations (see [2, 3]), where the first equation describes the vibration of the roadbed in the vertical plain and the second equation describes that of the main cable from which the roadbed is suspended by the tie cables. More recently, there has been a growing interest in this area.

For the mathematical model of suspension bridge we refer the readers to [4, 5, 6, 7, 8, 9, 10] and references therein, where the existence and asymptotic behavior of solutions have been studied.

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In some bridge models, the roadbed and the cable have a common point. The mathematical model of these problems has not been studied. In the previous studies, the cases have been considered where the roadbed and cable do not have the common point. In recent studies, the existence of global minimal attractors of dynamic systems created by these problems has been considered [3, 4, 5, 6, 7, 8, 9, 10].

Note that, the existence of global minimal attractor of the dynamic system, created by the bridge problem, has been studied by several authors (see [11], [14, 15, 16, 17, 18], and the references therein).

In this paper, we consider the corresponding mixed problem in the case where the roadbed and tensioning cable have one common point. The existence and uniqueness of the solution is investigated by modelling this problem as the Cauchy problem for an operator coefficient equation in some space. Then, it is shown that the energy function approaches zero exponentially in the case of linear dissipation. Using this, the existence of absorbing set of a corresponding dynamic system, in the case of non-focused non-linear source functions, has been shown and the assertion about asymptotic compactness has been proved. We also show that the corresponding dynamic system possesses a global minimal attractor.

2. Statement of the problem. Existence and uniqueness of the solution

We consider the following mathematical model for the oscillations of the bridge the roadbed of which has one common point with the cable:

$$\begin{cases} u_{tt} + u_{xxxx} + (u-v)_+ = f_1(x,u,v), \\ v_{tt} - v_{xx} - (u-v)_+ = f_2(x,u,v), \end{cases} \tag{1}$$

where $u(x,t)$ is a state function of the roadbed and $v(x,t)$ is that of the main cable.

Here $0 \le x \le l$, $t > 0$, $z_+ = \max\{z, 0\}$. $f_1(\cdot)$ and $f_2(\cdot)$ are real-valued functions defined on $[0,l] \times R^2$.

As the cable and the roadbed are fixed at the endpoints, the following boundary conditions must hold:

$$u(0,t) = u_x(0,t) = u(l,t) = u_x(l,t) = v(0,t) = v(l,t) = 0. \tag{2}$$

Suppose that at some point $\xi \in (0,l)$, the tension cable and the roadbed have a common point, i.e.

$$u(\xi - 0, t) = u(\xi + 0, t) = v(\xi - 0, t) = v(\xi + 0, t). \tag{3}$$

According to the physical meaning, assume that ξ is not the bend point of the roadbed, i.e.

$$u_{xx}(\xi-0,t)=u_{xx}(\xi+0,t)=0. \tag{4}$$

We also assume that the equiliubrium condition is satisfied at the point ξ. Mathematically, this means that the following equality holds:

$$u_{xxx}(\xi-0,t)-u_{xxx}(\xi+0,t)-v_x(\xi-0,t)+v_x(\xi+0,t)=0. \tag{5}$$

We also impose the following initial conditions:

$$u(x,0)=u_0(x),\ \ u'(x,0)=u_1(x), \quad x\in(0,l) \tag{6}$$

$$v(x,0)=v_0(x),\ \ v'(x,0)=v_1(x), \quad x\in(0,l) \tag{7}$$

The mixed problem (1)-(7) represents the mathematical model of the suspension bridge in the case where tensioning cable and roadbed have one common point.

Firstly, we will investigate the existence and uniqueness of solutions to problem (1)-(7).

In the case where aerodynamic forces are linear, the exponential decay of the solution of the corresponding homogeneous problem is investigated.

Then, in the case of non-focused non-linear source functions, the existence of absorbing set of a corresponding dynamic system is investigated.

Finally, the existence of global minimal attractor of the same problem is shown by using the general theory of existence of the attractor in dynamic systems.

3. Existence and uniqueness of the solution

In order to investigate the problem (1)-(7), we introduce the following notations:

$$\begin{aligned}
H^k(a,b) &= \left\{v : v, v', ..., v^{(k)} \in L_2(a,b)\right\}\\
{}_0H^1(0,\xi) &= \left\{v : v\in H^1(0,\xi),\ v(0)=0\right\}\\
{}_{00}H^2(0,\xi) &= \left\{v : v\in H^2(0,\xi),\ v(0)=v'(0)=0\right\}\\
H_0^1(\xi,l) &= \left\{v : v\in H^1(\xi,l),\ v(l)=0\right\}\\
H_{00}^2(\xi,l) &= \left\{v : v\in H^2(\xi,l),\ v(l)=v'(l)=0\right\}
\end{aligned}$$

Let's define the following Hilbert space:

$$\mathcal{H}=\{w : w^*=(u_{11},u_{12},u_{21},u_{22},u_{31},u_{32},u_{41},u_{42}),\ u_{11}\in_{00} H^2(0,\xi),$$

$$u_{12} \in L_2(0,\xi), u_{21} \in H^2_{00}(\xi,l),\ u_{22} \in L_2(\xi,l),\ u_{31} \in_0 H^1(0,\xi),\ u_{32} \in L_2(0,\xi),$$
$$u_{41} \in H^1_0(\xi,l), u_{42} \in L_2(\xi,l),\ u_{11}(\xi) = u_{21}(\xi) = u_{31}(\xi) = u_{41}(\xi)\}.$$

The scalar product in the space $\mathcal{H}$ is defined as follows:

$$< w, z >= \int_0^\xi u_{11xx} z_{11xx} dx + \int_0^\xi u_{12} z_{12} dx + \int_\xi^l u_{21xx} z_{21xx} dx + \int_\xi^l u_{22} z_{22} dx +$$
$$+ \int_0^\xi u_{31x} z_{31x} dx + \int_0^\xi u_{32} z_{32} dx + \int_\xi^l u_{41x} z_{41x} dx + \int_\xi^l u_{42} z_{42} dx;$$

where $w = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42})^* \in \mathcal{H}$ and
$z = (z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}, z_{41}, z_{42})^t \in \mathcal{H}$.

We will denote the norm in the space $\mathcal{H}$ as $\|\cdot\|_{\mathcal{H}}$.

Let's define the linear operator A in the space $\mathcal{H}$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial^2 & 0 \end{bmatrix},$$

$$D(A) = \{w : w = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}), u_{11} \in_{00} H^4(0,\xi),$$
$$u_{12} \in_{00} H^2(0,\xi), u_{21} \in H^4_{00}(\xi,l), u_{22} \in H^2_{00}(\xi,l),$$
$$u_{31} \in_0 H^2(0,\xi), u_{32} \in_0 H^1(0,\xi), u_{41} \in H^2_0(\xi,l), u_{42} \in H^1_0(\xi,l),$$
$$u_{11xxx}(\xi) - u_{21xxx}(\xi) - u_{31x}(\xi) + u_{41x}(\xi) = 0, u_{11xx}(\xi) = u_{21xx}(\xi) = 0\}.$$

The set $\mathcal{H}_1 = D(A)$ is a Banach space with respect to the norm

$$\|w\|_{\mathcal{H}_1} = \|Aw\|_{\mathcal{H}} + \|w\|_{\mathcal{H}}.$$

Let's define the following non-linear operator acting in the space $\mathcal{H}$:
$F(w) = F_1(w) + F_2(w)$, with

$$F_1(w) = (0, -(u_{11} - u_{31})_+, 0, -(u_{21} - u_{41})_+ 0, -(u_{31} - u_{11})_+ 0, -(u_{41} - u_{21})_+)^*$$

$F_2(w) = (0, f_1(x, u_{11}, u_{31}), 0, f_1(x, u_{21}, u_{41}), 0, f_2(x, u_{11}, u_{31}), 0, f_2(x, u_{21}, u_{41}))^*$

The problem (1)-(7) is written in the form

$$w' = Aw + F(w), \tag{8}$$

$$w(0) = w_0, \tag{9}$$

where $w = w(t) = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42})^*,\ w' = \frac{dw(t)}{dt}$, $w_0 = (u_{110}, u_{120}, u_{210}, u_{220}, u_{310}, u_{320}, u_{410}, u_{420})$,

$$u_{11} = u(x,t),\quad u_{12} = u_t(x,t),\quad 0 < x < \xi,$$

$$u_{21} = u(x,t),\quad u_{22} = u_t(x,t),\quad \xi < x < l,$$

$$u_{31} = v(x,t),\quad u_{32} = v_t(x,t),\quad 0 < x < \xi,$$

$$u_{41} = v(x,t),\quad u_{42} = v_t(x,t),\quad \xi < x < l,$$

$$u_{110} = u_0(x),\quad u_{120} = u_1(x),\ 0 < x < \xi,$$

$$u_{210} = u_0(x),\quad u_{220} = u_1(x),\ \xi < x < l,$$

$$u_{310} = v_0(x),\quad u_{320} = v_1(x),\ 0 < x < \xi,$$

$$u_{410} = v_0(x),\quad u_{420} = v_1(x),\ \xi < x < l.$$

Lemma 1. *A is a maximal dissipative operator.*

Lemma 2. *Suppose that the functions $f_1(\cdot), f_2(\cdot)$ are mapping from $[0,l] \times R^2$ to R and satisfy local Lipschitz conditions. Then the nonlinear operator $w \to F(w) : \mathcal{H} \to \mathcal{H}$ satisfies local Lipschitz condition (see [12]).*

By virtue of Lemmas 1 and 2, we have the following result about the existence of local solutions:

Theorem 1. *Suppose that $f_1(\cdot), f_2(\cdot)$ are the functions mapping from $[0,l] \times R^2$ to R and satisfying local Lipschitz conditions. Then for any $w_0 = (u_{110}, u_{120}, u_{210}, u_{220}, u_{310}, u_{320}, u_{410}, u_{420}) \in \mathcal{H}$ there exists $T' > 0$, such that the problem (8), (9) has a unique solution $w(\cdot) = (u_{11}(\cdot), u_{12}(\cdot), u_{21}(\cdot), u_{22}(\cdot), u_{31}(\cdot), u_{32}(\cdot), u_{41}(\cdot), u_{42}(\cdot)) \in C([0,T'];\mathcal{H})$. If $w_0 \in D(A)$, then $w(\cdot) \in C([0,T'];\mathcal{H}_1) \cap C^1([0,T'];\mathcal{H})$.*

In either case, the following alternative is true: If $[0, T_{\max})$ is the maximum interval on which the local solution exists, then

1) $T_{\max} = +\infty$ or

2) $\lim_{t\to T_{\max}} \|w(\cdot)\|_{\mathcal{H}} = +\infty$.

According to this theorem, if a priori estimate

$$\|w(t)\|_{\mathcal{H}} \leq C, \quad 0 \leq t < T_{\max}$$

is true for the solution of the problem (8)-(9), then $T_{\max} = +\infty$, i.e., in this case, solution established by Theorem 1 is a global solution.

Theorem 2. *Let's suppose that all the conditions of Theorem 1 hold. Moreover, assume that there is a function $F(x,u,v)$ defined on $[0,L] \times R^2$ which satisfies the following conditions:*

$$\begin{aligned} F'_u(x,u,v) &= f_1(x,u,v), \\ F'_v(x,u,v) &= f_2(x,u,v), \\ F(x,u,v) &\leq c_1(u^2+v^2)+c_2, \end{aligned}$$

where c_1 and c_2 are some positive constants. Then for any $w_0 \in \mathcal{H}$, the problem (8), (9) *has a unique solution $w(\cdot) \in C([0;+\infty);\mathcal{H})$.*

It follows from Theorem 2 that the problem (8)-(9) creates a dynamical system in the space $\mathcal{H}$. Let's denote a semigroup corresponding to this dynamical system by $W_t = W(t)$.

If $U_t = U(t) = e^{-At}$ is the C_0 semigroup generated by the linear operator A, then

$$W_t(w_0) = U_t w_0 + \int_0^t U_{t-s} F(w(s))ds. \tag{10}$$

Let's introduce some concepts that characterize the asymptotics of a dynamical system corresponding to the semigroup W_t.

Definition 1. *Suppose that $W(t)$ is a nonlinear semigroup generated by the problem* (27), (14)–(19). *A subset B_0 of $\mathcal{H}$ is called an absorbing set in $\mathcal{H}$ if, for any bounded subset B of $\mathcal{H}$, there exists some $t_0 = t(B)$ such that $U(t)B \subset B_0$, for all $t \geq t_0$*

Definition 2. *Suppose that $\{W_t\}_{t\geq 0}$ is a semigroup in some metric space (X,d) and X has a minimal set $\mathfrak{U} \subset X$ invariant for the corresponding dynamical system such that for any bounded set $B \subset X$, the following condition holds:*

$$\lim_{t\to\infty} \sup_{v\in B} \inf_{u\in\mathfrak{U}} d(S(t)v,u) = 0.$$

Then it's said that the semigroup $\{W_t\}_{t\geq 0}$ has a minimal global attractor $\mathfrak{U}$.

4. Influence of linear aerodynamic resistance without external forces

The mathematical model of the influence of linear aerodynamic resistance without external forces has the following form:

$$\begin{cases} u_{tt} + u_{xxxx} + (u-v)_+ + u_t = 0, \\ v_{tt} - v_{xx} - (u-v)_+ + v_t = 0 \end{cases} \tag{11}$$

Let us consider the system (11) with the boundary conditions (2), interaction conditions (3) - (5) and initial conditions (6), (7).

We introduce the following functional corresponding to the solution of the problem (2)-(7), (11)

$$E_0(t) = \frac{1}{2}\int_0^l u_t^2(t,x)dx + \frac{1}{2}\int_0^l u_{xx}^2(t,x)dx + \frac{1}{2}\int_0^l v_t^2(t,x)dx + \frac{1}{2}\int_0^l v_x^2(t,x)dx.$$

Theorem 3. *There exist numbers $M > 0$ and $\alpha > 0$ such that for the energy functional $E_0(t)$ of the problem (2)-(7), (11), the estimate*

$$E_0(t) \le Me^{-\alpha t}E_0(0) \tag{12}$$

is true.

Before proving Theorem 3, consider the following problem:

$$\begin{cases} u_{tt} + u_{xxxx} + (u-v)_+ + u_t = g_1(t,x), \\ v_{tt} - v_{xx} - (u-v)_+ + v_t = g_2(t,x). \end{cases} \tag{13}$$

$$u(0,t) = u_x(0,t) = u(l,t) = u_x(l,t) = v(0,t) = v(l,t) = 0, \tag{14}$$

$$u(\xi-0,t) = u(\xi+0,t) = v(\xi-0,t) = v(\xi+0,t). \tag{15}$$

$$u_{xx}(\xi-0,t) = u_{xx}(\xi+0,t) = 0. \tag{16}$$

$$u_{xxx}(\xi-0,t) - u_{xxx}(\xi+0,t) - v_x(\xi-0,t) + v_x(\xi+0,t) = 0. \tag{17}$$

$$u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\quad x \in (0,l) \tag{18}$$

$$v(x,0) = v_0(x),\ v'(x,0) = v_1(x),\qquad x \in (0,l). \tag{19}$$

Let's define the following functionals other than $E_0(t)$, according to the solution of this problem:

$$E_1(t) = \frac{1}{2}\int_0^l u_t^2(t,x)dx + \frac{1}{2}\int_0^l u_{xx}^2(t,x)dx + \frac{1}{2}\int_0^l v_t^2(t,x)dx +$$

$$+\eta\int_0^l u_t(t,x)u(t,x)dx+\eta\int_0^l v_t(t,x)v(t,x)dx+$$

$$+\frac{1}{2}\int_0^l v_x^2(t,x)dx+\frac{1}{2}\int_0^l |(u(t,x)-v(t,x))_+|^2dx;$$

$$E_2(t)=(1-\eta)\int_0^l u_t^2(t,x)dx+(1-\eta)\int_0^l v_t^2(t,x)dx+$$

$$+\eta\int_0^l u_{xx}^2(t,x)dx+\eta\int_0^l v_x^2(t,x)dx+\eta\int_0^l (u(t,x)-v(t,x))^2dx+$$

$$+\eta\int_0^l u_t(t,x)u(t,x)dx+\eta\int_0^l v_t(t,x)v(t,x)dx.$$

Lemma 3. *If the functions $u\in H^2$ and $v\in H^1$ satisfy the conditions* (14)-(17)*, then there exist constants $c_1>0, c_2>0$, such that*

$$\|v\|_{L_2(0,l)}\le\lambda_1\|v_x\|_{L_2(0,l)}, \tag{20}$$

$$\|u\|_{L_2(0,l)}\le\lambda_2\|u_{xx}\|_{L_2(0,l)}. \tag{21}$$

Lemma 4. *There exists a number $\eta_0>0$ such that for any $0<\eta\le\eta_0$*

$$\mu_1E_1(t)\le E_0(t)\le\mu_1^{-1}E_1(t),\quad t>0, \tag{22}$$

$$\mu_2E_2(t)\le E_0(t)\le\mu_2^{-1}E_2(t),\quad t>0 \tag{23}$$

where $\mu_1>0$ and $\mu_2>0$ are the constants depending on η.

Lemma 5. *For the functionals $E_1(t)$ and $E_2(t)$ of the problem* (13)-(19)*, the following identity is valid:*

$$\frac{d}{dt}E_1(t)+E_2(t)=\int_0^l g_1(t,x)u_t(t,x)dxdt+\int_0^l g_2(t,x)v_t(t,x)dx+$$

$$+\eta\int_0^l g_1(t,x)u(t,x)dxdt+\eta\int_0^l g_2(t,x)v(t,x)dx. \tag{24}$$

According to Lemma 4 and Lemma 5, we get

$$\frac{d}{dt}E_1(t)+\mu_1\mu_2E_1(t)\leq\int\limits_0^l f_1(t,x)u_t(t,x)dxdt+\int\limits_0^l f_2(t,x)v_t(t,x)dx+$$

$$+\eta\int\limits_0^l f_1(t,x)u(t,x)dxdt+\eta\int\limits_0^l f_2(t,x)v(t,x)dx. \tag{25}$$

In particular, if $f_1(t,x)=f_2(t,x)=0$, then we obtain the following inequality:

$$E_1(t)\leq E_1(0)e^{-\mu_1\mu_2 t},\quad t>0. \tag{26}$$

Finally, if we use Lemma 4, we get Theorem 3.

5. The existence of an absorbing set in the case of a non-linear non-focused source

In this section we study the existence of an absorbing set of the dynamic system related to the following system with non-linear non-focused source function under the conditions (14)-(19).

$$\begin{cases} u_{tt}+u_{xxxx}+(u-v)_+ + u_t+|u|^p|v|^{p+2}u=h_1(x), \\ v_{tt}-v_{xx}-(u-v)_+ + v_t+|u|^{p+2}|v|^p v=h_2(x). \end{cases} \tag{27}$$

Here

$$p\geq 0, \tag{28}$$

$$h_1(\cdot),h_2(\cdot)\in L_2(0,l). \tag{29}$$

According to Theorem 3, the problem (27), (14) - (19) is correct in $\mathcal{H}$, when the conditions (28), (29) are satisfied. Let us denote the semigroup generated by this problem by $W(t)$, as in (10).

Theorem 4. *Suppose that conditions* (28), (29) *are satisfied. Then the semigroup* W_t *generated by the problem* (27), (14) - (19) *has a bounded absorbing set in the space* $\mathcal{H}$.

Proof. Since $g_1(t,x)=-|u|^p|v|^{p+2}u+h_1(x), g_2(t.x)=-|u|^{p+2}|v|^p v+h_2(x)$ in (27), we obtain the following inequality to solve the problem (27), (14) - (19) by applying the Holder inequality in (25):

$$\frac{d}{dt}E_1(t)+\mu_1\mu_2E_1(t)\leq-\frac{1}{p+1}\frac{d}{dt}\int\limits_0^l|uv|^{p+1}dx-2\eta\int\limits_0^l|uv|^{p+1}dx+$$

$$+\left(\int_0^l |h_1(x)|^2dx\right)^{1/2}\left(\int_0^l |u_t(t,x)|^2dx\right)^{1/2}+$$

$$+\left(\int_0^l |h_2(x)|^2dx\right)^{1/2}\left(\int_0^l |(v_t(t,x)|^2dx\right)^{1/2}+$$

$$+\eta\left(\int_0^l |h_1(x)|^2dx\right)^{1/2}\left(\int_0^l |u(t,x)|^2dx\right)^{1/2}+$$

$$+\eta\left(\int_0^l |h_2(x)|^2dx\right)^{1/2}\left(\int_0^l |(v(t,x)|^2dx\right)^{1/2}.$$

Applying the Young inequality to each of the terms on the right, we obtain

$$\frac{d}{dt}\left[E_1(t)-\varepsilon\left(\int_0^l |u_t|^2dx+\int_0^l |v_t|^2dx\right)+\frac{1}{p+1}\int_0^l |uv|^{p+1}dx\right]+$$

$$+\mu_1\mu_2E_1(t)+2\eta\int_0^l |uv|^{p+1}dx-\varepsilon\left[\int_0^l |u|^2dx+\int_0^l |v|^2dx\right]\le$$

$$\le\frac{2\eta^2}{\varepsilon}\int_0^l |h_1(x)|^2dx+\frac{2\eta^2}{\varepsilon}\int_0^l |h_2(x)|^2dx. \tag{30}$$

In the inequality (30), we take $0<\varepsilon<\min\{\frac{1}{2c_1},\frac{1}{2c_2}\}$. Then we get the following inequality:

$$\frac{d}{dt}E_{1\varepsilon}(t)+E_{2\varepsilon}(t)\le\gamma, \tag{31}$$

where

$$E_{1\varepsilon}(t)=[1/2-\varepsilon]\int_0^l u_t^2(t,x)dx+[1/2-\varepsilon]\int_0^l u_{xx}^2(t,x)dx+1/2\int_0^l v_t^2(t,x)dx+$$

$$+\eta\int_0^l u_t(t,x)u(t,x)dx+\eta\int_0^l v_t(t,x)v(t,x)dx+$$

$$+1/2\int_0^l v_x^2(t,x)dx+1/2\int_0^l |(u(t,x)-v(t,x)_+|^2dx;$$

$$E_{2\varepsilon}(t)=(1-\eta)\int_0^l u_t^2(t,x)dx+(1-\eta)\int_0^l v_t^2(t,x)dx+$$

$$+[\eta-\varepsilon c_1]\int_0^l u_{xx}^2(t,x)dx+[\eta-\varepsilon c_2]\int_0^l v_x^2(t,x)dx+\eta\int_0^l (u(t,x)-v(t,x))^2dx+$$

$$+\eta\int_0^l u_t(t,x)u(t,x)dx+\eta\int_0^l v_t(t,x)v(t,x)dx.$$

$$\gamma=\frac{2\eta^2}{\varepsilon}\int_0^l |h_1(x)|^2dx+\frac{2\eta^2}{\varepsilon}\int_0^l |h_2(x)|^2dx.$$

◀

Lemma 6. *Suppose that conditions* (28), (29) *are fulfilled. Then there exist numbers* $\mu_3>0$, $\mu_4>0$, *such that*

$$\mu_3 E_{1\varepsilon}(t)\le E_0(t)\le \mu_3^{-1}E_{1\varepsilon}(t),\quad t>0,$$

$$\mu_4 E_{2\varepsilon}(t)\le E_0(t)\le \mu_4^{-1}E_{2\varepsilon}(t),\ t>0.$$

According to Lemma 6 and (31), there exists $c_3>0$ such that

$$E_{2\varepsilon}(t)\ge c_3E_{1\varepsilon}(t).$$

Based on this inequality, from (31) we get

$$\frac{d}{dt}E_{1\varepsilon}(t)+c_3E_{1\varepsilon}(t)\le\gamma.$$

Here, taking into account Lemma 6, we get:

$$E_0(t)\le\frac{\gamma\mu_3}{c_3}+\left(E_0(0)-\frac{\gamma}{c_3\mu_3}\right)e^{-c_3t},\quad t>0. \tag{32}$$

If we take $r=\frac{\gamma\mu_3}{c_3}+1$, it follows from (32) that $B_0=\{w:\|w\|_{\mathcal{H}}\le r+1\}$ is the absorbing set.

6. Asymptotic compactness and existence of a minimal global attractor in the case of a nonlinear non-focused source

Suppose that $B \subset \mathcal{H}$ and $w_0 \in B$. We introduce the notation

$V_t(B) = \{y : y = V_t(w_0), w_0 \in B\}$, where $V_t(w_0) = \int\limits_0^t U_{t-s}G(w(s))ds$,

$$G(w(\cdot)) = \begin{pmatrix} 0 \\ -|u_{11}(\cdot)|^p|u_{31}(\cdot)|^{p+2}u_{11}(\cdot) + h_{11}(\cdot) \\ 0 \\ -|u_{21}(\cdot)|^p|u_{41}(\cdot)|^{p+2}u_{21}(\cdot) + h_{21}(\cdot) \\ 0 \\ -|u_{11}(\cdot)|^{p+2}|u_{31}(\cdot)|^p u_{31}(\cdot) + h_{12}(\cdot) \\ 0 \\ -|u_{21}(\cdot)|^{p+2}|u_{41}(\cdot)|^p u_{41}(\cdot) + h_{41}(\cdot) \end{pmatrix}$$

$$h_{11}(x) = h_1(x), \quad 0 \le x \le \xi; \quad h_{21}(x) = h_1(x), \quad \xi \le x \le l;$$

$$h_{12}(x) = h_2(x), \quad 0 \le x \le \xi; \quad h_{22}(x) = h_2(x), \quad \xi \le x \le l;$$

Theorem 5. *Suppose that conditions* (28), (29) *are satisfied. Then* $\overline{\bigcup\limits_{t\ge 0}\overline{V_t(B)}}$ *is precompact in* $\mathcal{H}$ *for any bounded set* $B \subset \mathcal{H}$.

Proof. For any bounded set $B \subset \mathcal{H}$, there is $t_B > 0$ such that $W_t(B) \subset B_0$ for $t \ge t_B$. It follows from here that if for any $t, w(t)$ is the solution of the problem (27), (14) - (19), then $\|w(t)\|_{\mathcal{H}} \le c(r_0), t \ge 0$. In this case, since $u_{k1t} = u_{k2},\ k = 1,2,3,4$, the following estimates are true:

$$\|u_{k1t}(t,\cdot\|_{L_2(0,\xi)} \le c(r_0), \quad k = 1,3,$$

$$\|u_{k1t}(t,\cdot\|_{L_2(\xi,l)} \le c(r_0), \quad k = 2,4.$$

Using these estimates, we get

$$\|f_{ju}(u_{11}(t,\cdot), u_{31}(t,\cdot))u_{11t}(t,\cdot)\|_{L_2(0,\xi)} \le$$

$$\le \sup_{0\le x\le \xi} f_{ju}(u_{11}(t,x), u_{31}(t,x)) \cdot \|u_{11t}(t,\cdot\|_{L_2(0,\xi)} \le$$

$$\le c(\|u_{11}(t,\cdot)\|_{{}_{00}H^2(0,\xi)}, \|u_{31}(t,\cdot)\|_{{}_0H^1(0,\xi)})\|u_{11t}(t,\cdot\|_{L_2(0,\xi)} \le c_j. \qquad (33)$$

Here, $f_1(u,v) = |u(\cdot)|^p|v(\cdot)|^{p+2}u(\cdot) + h_1(\cdot),\ f_2(u,v) = |u(\cdot)|^{p+2}|v(\cdot)|^p v(\cdot) + h_2(\cdot)$.

Similarly, we have

$$\|f_{jv}(u_{11}(t,\cdot), u_{31}(t,\cdot))u_{31t}(t,\cdot)\|_{L_2(0,\xi)} \le c_j, \quad j = 1, 2, \tag{34}$$

$$\|f_{ju}(u_{21}(t,\cdot), u_{41}(t,\cdot))u_{21t}(t,\cdot)\|_{L_2(\xi,l)} \le c_j, \quad j = 1, 2, \tag{35}$$

$$\|f_{jv}(u_{21}(t,\cdot), u_{41}(t,\cdot))u_{41t}(t,\cdot)\|_{L_2(\xi,l)} \le c_j, \quad j = 1, 2. \tag{36}$$

According to the definition of $F(w)$ and the a priori estimate (33)-(36), we obtain

$$\|(G(w))'\|_{\mathcal{H}} \le c(\|w_0\|_{\mathcal{H}}). \tag{37}$$

It's obvious that the function $y = \int\limits_0^t U_{t-s}F(w(s))ds$ is the solution of the problem

$$y' = Ay + G(w), \tag{38}$$

$$y(0) = 0. \tag{39}$$

If we denote $z = y'$, then

$$z = U_t(w_0) + \int\limits_0^t U_{t-s}(G(w(s)))'ds.$$

From the estimates (22), (37), we derive

$$\|y'\|_{\mathcal{H}} = \|z\|_{\mathcal{H}} \le Me^{-\omega t}\|w_0\|_{\mathcal{H}} + \frac{M}{\omega}(1 - e^{-\omega t})c(\|w_0\|_{\mathcal{H}}) \le c(B), \quad t \ge 0. \tag{40}$$

Considering (40) in (38), we get

$$\|Ay\|_{\mathcal{H}} \le c(B), \quad t \ge 0. \tag{41}$$

Since the space $\mathcal{H}_1$ is compactly embedded in $\mathcal{H}$, it follows from (41) that $\overline{\{y(t), t \ge 0, w\_0 \in B\}} = \overline{\bigcup\limits_{t\ge 0} \overline{V_t(B)}}$ is a compact set. ◄

Now, let's show the existence of the minimal global attractor. The main result of this paper is the following:

Theorem 6. *Suppose that conditions* (28) *and* (29) *are satisfied. Then the semigroup W_t generated by the problem* (27), (14) - (19) *has a global minimum attractor, so that it is connected, invariant in $\mathcal{H}$, and bounded set in $\mathcal{H}_1$.*

Proof. According to Theorem 6, W_t is continuous, point-dissipative and asymptotic compact semigroup. Then, according to the results in the papers [11, 15], W_t has a minimal global attractor $\mathfrak{U} \subset \mathcal{H}$.

Let's show that $\mathfrak{U}$ is bounded in the set $\mathcal{H}_1$. Since $\mathfrak{U}$ is an invariant for the semigroup W_t, for any $y \in \mathfrak{U}$ there is $y_k \in \mathfrak{U}$ and $t_k \to +\infty$, such that $W_{t_k} y_k = y$.

From here we get

$$U_{t_k} y_k + V_{t_k} y_k = y. \tag{42}$$

Based on the inequality $\|U_t\| \leq M e^{-\omega t}$, from the relation (42) we obtain

$$V_{t_k} y_k \to y, \quad k \to \infty, \tag{43}$$

in $\mathcal{H}$.

From the inequality (41) we derive $\|V_{t_k} y_k(t)\|_{\mathcal{H}_1} \leq c$. Thus, there is a $X \in \mathcal{H}_1$ such that

$$V_{t_k} y_k \to X \quad \text{weakly in } \mathcal{H}_1. \tag{44}$$

It follows from (43) and (44) that $y = \chi$ and $\|y(t)\|_{\mathcal{H}_1} \leq c$. In other words, the set $\mathfrak{U}$ is bounded in $\mathcal{H}_1$. ◄

Conclusion: It is clear that $\mathcal{H}_1$ is compactly embedded in $\mathcal{H}$, so $\mathfrak{U}$ is a compact set in $\mathcal{H}$.

7. Proof of Lemmas

Proof of Lemma 1. Let $w = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42})^* \in D(A)$. Then by taking into account the boundary conditions (2), we get the following equality:

$$\langle Aw, w \rangle = \int_0^\xi \partial^2 u_{12} \cdot \partial^2 u_{11} dx - \partial^3 u_{11}(\xi) \cdot u_{12}(\xi) + \partial^2 u_{11}(\xi) \cdot \partial^2 u_{12}(\xi) -$$

$$-\partial^2 u_{21}(\xi) \cdot \partial^2 u_{22}(\xi) - \int_\xi^l \partial^2 u_{21} \cdot \partial^2 u_{22} dx + \int_0^\xi \partial u_{32} \cdot \partial u_{31} dx -$$

$$-\partial u_{41}(\xi) \cdot \partial u_{42}(\xi) - \int_\xi^l \partial u_{41} \cdot \partial u_{42} dx.$$

Taking into account the continuity and interaction conditions (3)-(5), we get

$$< Aw, w >= 0.$$

Thus, we proved that A is a dissipative operator.

Now let's show that A is a maximal operator. For this purpose, we must show that the image of A coincides with $\mathcal{H}$.

Let $h = (h_{11}, h_{12}, h_{21}, h_{22}, h_{31}, h_{32}, h_{41}, h_{42}) \in \mathcal{H}$. Then the equation

$$Aw = h, \tag{45}$$

is equivalent to the following boundary problem:

$$\begin{cases} u_{12} = h_{11} \\ \partial^4 u_{11} = h_{12} \\ u_{22} = h_{21} \\ -\partial^4 u_{21} = h_{22} \\ u_{32} = h_{31} \\ \partial^2 u_{31} = h_{32} \\ u_{42} = h_{41} \\ \partial^2 u_{41} = h_{42} \end{cases} \qquad \begin{cases} u_{11}(0) = \partial u_{11}(0) = u_{12}(0) = u_{12}(0) = 0 \\ u_{21}(l) = \partial u_{21}(l) = u_{22}(l) = u_{22}(l) = 0 \\ u_{31}(0) = u_{32}(0) = 0 \\ u_{41}(l) = \partial u_{42}(l) = 0 \\ u_{11}'''(\xi) - u_{21}'''(\xi) - u_{31}'(\xi) + u_{41}'(\xi) = 0 \\ u_{11}''(\xi) = u_{21}''(\xi) = 0. \end{cases} \tag{46}$$

Let's show that this problem has a solution

$$w = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}) \in D(A).$$

It's clear from the expressions obtained for w that

$$\|Aw\|_{\mathcal{H}} \geq m\|w\|_{\mathcal{H}}. \tag{47}$$

Here $m > 0$ is a positive number which doesn't depend on A.

From the existence of the solution of the problem (46) and estimate (47), it follows that the linearly closed operator A has a bounded inverse. Thus, $\lambda = 0$ belongs to the resolvent set. Since the resolvent set is open and A is dissipative, it follows that any $\lambda > 0$ belongs to the resolvent set [13, 19]. ◄

Proof of Lemma 2. Since $|\theta_{2_+} - \theta_{1_+}| \leq |\theta_2 - \theta_1|$ for any θ_1, θ_2, we get

$$\|F_1(w_2) - F_1(w_1)\|_{\mathcal{H}} \leq \|w_2 - w_1\|_{\mathcal{H}}.$$

Suppose B_r is a ball of radius r in space $\mathcal{H}$ and

$$w^k = (u_{11}^k, u_{12}^k, u_{21}^k, u_{22}^k, u_{31}^k, u_{32}^k, u_{41}^k, u_{42}^k) \in B_r, \; k = 1, 2.$$

Then

$$\|f_1(x, u_{11}^2, u_{31}^2) - f_1(x, u_{11}^1, u_{31}^1)\|_{L_2(0,\xi)}^2 \leq$$

$$\leq \int_0^\xi |c(u_{11}^1, u_{11}^2, u_{11}^1, u_{11}^2)|^2 \left[|u_{11}^2 - u_{11}^1|^2 + |u_{31}^2 - u_{31}^1|^2\right] dx \leq$$

$$\leq \sup_{0\leq x\leq \xi} |c(u_{11}^1, u_{11}^2, u_{11}^1, u_{11}^2)|^2 \int_0^\xi [|u_{11}^2 - u_{11}^1|^2 + |u_{31}^2 - u_{31}^1|^2] dx \leq$$

$$\leq c^2 \left(\|u_{11}^1\|_{{}_0H^2(0,\xi)}, \|u_{11}^2\|_{{}_0H^2(0,\xi)} \|u_{31}^1\|_{{}_0H^1(0,\xi)}, \|u_{31}^2\|_{{}_0H^2(0,\xi)} \right) \times$$

$$\times \int_0^\xi \left[|u_{11}^2 - u_{11}^1|^2 + |u_{31}^2 - u_{31}^1|^2 \right] dx \leq c^2(r) \|w^2 - w^1\|^2. \tag{48}$$

Similarly, we have

$$\|f_2(x, u_{11}^2, u_{31}^2) - f_2(x, u_{11}^1, u_{31}^1)\|_{L_2(0,\xi)}^2 \leq c^2(r) \|w^2 - w^1\|^2, \tag{49}$$

$$\|f_j(x, u_{21}^2, u_{41}^2) - f_j(x, u_{21}^1, u_{41}^1)\|_{L_2(\xi,l)}^2 \leq c^2(r) \|w^2 - w^1\|^2, \quad j = 1, 2. \tag{50}$$

According to (48)-(50), we have

$$\|F(w^2) - F(w^1)\|_{\mathcal{H}} \leq c(r) \|w^2 - w^1\|_{\mathcal{H}}.$$

◀

Proof of Lemma 3. Since $u_{11}(0) = u_{11}'(0) = 0$, we obtain

$$\|u_{11}\|_{L_2(0,\xi)} \leq c_{11} \|u_{11}x\|_{L_2(0,\xi)} \leq \widetilde{c_{11}} \|u_{11xx}\|_{L_2(0,\xi)}. \tag{51}$$

Similarly, since $u_{21}(l) = u_{21}'(l) = 0$, we have

$$\|u_{21}\|_{L_2(\xi,l)} \leq c_{21} \|u_{21x}\|_{L_2(\xi,l)} \leq \widetilde{c_{21}} \|u_{21xx}\|_{L_2(\xi,l)}. \tag{52}$$

In the same way, we get the following inequalities:

$$\|u_{31}\|_{L_2(0,\xi)} \leq c_{31} \|v_x\|_{L_2(0,\xi)},$$

$$\|u_{41}\|_{L_2(\xi,l)} \leq c_{41} \|v_x\|_{L_2(\xi,l)}.$$

It follows from (51) and (52) that

$$\|u\|_{L_2(0,l)} \leq c_4 \|u_{xx}\|_{L_2(0,l)},$$

$$\|v\|_{L_2(0,l)} \leq c_5 \|v_x\|_{L_2(0,l)}.$$

◀

Proof of Lemma 4. Using Holder's and Young's inequalities, from Lemma 3 we obtain

$$\left| \int_0^l u_s(\tau, x) u(\tau, x) dx \right| \leq \varepsilon/2 \int_0^l |u_s(\tau, x)|^2 dx + \frac{\lambda_2}{2\varepsilon} \int_0^l |u_{xx}(\tau, x)|^2 dx. \tag{53}$$

Similarly we get

$$\left|\int_0^l v_s(\tau,x)v(\tau,x)dx\right| \le \frac{\varepsilon}{2}\int_0^l |v_s(\tau,x)|^2dx + \frac{\lambda_1}{2\varepsilon}\int_0^l |v_x(\tau,x)|^2dx. \tag{54}$$

$$\frac{1}{2}\int_0^l |(u(t,x)-v(t,x)_+|^2dx \le \int_0^l |u(t,x)|^2dx + \int_0^l |v(t,x)|^2dx \le$$

$$\le \lambda_2\int_0^l |u_{xx}(t,x)|^2dx + \lambda_1\int_0^l |v_x(t,x)|^2dx. \tag{55}$$

$$E_1(t) \le \max\left\{1+2\varepsilon\eta, 1+(1+\lambda_1)\frac{\eta}{\varepsilon}, 1+(1+\lambda_2)\frac{\eta}{\varepsilon}\right\}E_0(t). \tag{56}$$

η and ε are chosen so that the conditions $\eta < \sqrt{\min\left\{\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right\}}, \varepsilon < \frac{1}{\eta}$ are satisfied.

In this case, from the definition of $E_1(t)$ and $\varepsilon\cdot\eta < 1, \frac{\eta\lambda_1}{\varepsilon} < 1$, we get

$$E_1(t) \ge \min\left\{1-2\varepsilon\eta, 1-\frac{\eta\lambda_1}{\varepsilon}, 1-\frac{\eta\lambda_2}{\varepsilon}\right\}E_0(t). \tag{57}$$

So if we take μ_1 as follows, we obtain the inequality (22).

$$\mu_1 = \min\left\{1-2\varepsilon\eta, 1-\frac{\eta\lambda_1}{\varepsilon}, 1-\frac{\eta\lambda_2}{\varepsilon}, \frac{1}{1+2\varepsilon\eta}, \frac{1}{(1+\lambda_1)\frac{\eta}{\varepsilon}}, \frac{1}{1+(1+\lambda_2)\frac{\eta}{\varepsilon}}\right\}.$$

The inequality (23) can be proved in the same way. ◄

Proof of Lemma 5. Multiplying first equation of the system (13) by $u_t(t,x)$, second equation by $v_t(t,x)$, integrating over the domain and summing up we get:

$$\frac{d}{dt}\left\{\frac{1}{2}\int_0^l |u_t(t,x)|^2dx + \frac{1}{2}\int_0^l |v_t(t,x)|^2dx + \frac{1}{2}\int_0^l |u_{xx}(t,x)|^2dx +\right.$$

$$\left.+\frac{1}{2}\int_0^l |v_x(t,x)|^2dx\right\} + \int_0^l |u_t(t,x)|^2dx + \int_0^l |v_t(t,x)|^2dx =$$

$$= \int_0^l g_1(t,x)\cdot u_t(t,x)dx + \int_0^l g_2(t,x)\cdot v_t(t,x)dx. \tag{58}$$

Multiplying first equation of the system (13) by $u(t,x)$, second equation by $v(t,x)$, integrating over the domain and summing up we get:

$$\frac{d}{dt}\left\{\int_0^l u_t(t,x)\cdot u(t,x)dx+\int_0^l v_t(t,x)\cdot v(t,x)dx\right\}-$$

$$-\left\{\int_0^l |u_t(t,x)|^2dx+\int_0^l |v_t(t,x)|^2dx\right\}+\int_0^l |u_{xx}(t,x)|^2dx+\int_0^l |v_x(t,x)|^2dx+$$

$$+\int_0^l u_t(t,x)\cdot u(t,x)dx+\int_0^l v_t(t,x)\cdot v(t,x)dx=$$

$$=\int_0^l g_1(t,x)\cdot u(t,x)dx+\int_0^l g_2(t,x)\cdot v(t,x)dx. \tag{59}$$

Multiplying both sides of the equality (59) by the parameter η and adding it to the equality (57), leads to the identity (24). ◄

The proof of Lemma 6 is the same as that of Lemma 4.

References

[1] A.C. Lazer, P.J. McKenna, *Large scale oscillatory behavior in loaded asymmetric systems*, Ann. Ins. H. Poincare, Anal. Nonlinear, **4**, 1987, 243–274.

[2] P.J. McKenna, W. Walter, *Traveling waves in a suspension bridge*, SIAM J. Appl.Math., **50**, 1990, 703-715.

[3] Q.F. Ma, S.H. Wang, C.K. Zhong, *Necessary and sufficient conditions for the existence of global attractor for the semigroup and application*, Indiana Univ. Math. J., **51(6)**, 2002, 1541–1559.

[4] Q. Ma, C. Zhong, *Existence of global attractors for the coupled system of suspension bridge equations*, J. Math. Anal. Appl., **308**, 2005, 365-379.

[5] Q.Z. Ma, C.K. Zhong, *Existence of strong solutions and global attractors for the coupled suspension bridge equations*, J. Differ. Equat., **246**, 2009, 3755–3775.

[6] N.U. Ahmed, H. Harbi, *Mathematical analysis of dynamic models of suspension bridges*, SIAM J. Appl.Math., **58**, 1998, 853–874.

[7] Y. An, *Nonlinear perturbations of a coupled system of steady state suspension bridge equations*, Nonlinear Anal., **51**, 2002, 1285-1292.

[8] J.R. Kang, *Asymptotic behavior of the thermoelastic suspension bridge equation with linear memory*, Boundary Value Problems, **206**, 2016, 1-18.

[9] Y. Park, J.R. Kang, *Global attractors for the suspension bridge equations with non-linear damping*, Quart. Appl. Math., **69**, 2011, 465-475.

[10] C.K. Zhong, Q.Z. Ma, C.Y. Sun, *Existence of strong solutions and global attrctors for the suspension bridge equations*, Nonlinear Analysis, **67(2)**, 2007, 442–454.

[11] O.A. Ladyzhenskaya, *Some complements and corrections to my papers on the theory of attractors for abstract semigroups.* (English. Russian original) Zbl 0783.47054J. Sov. Math., **62(3)**, 1992, 2789-2794; translation from Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **182**, 1990, 102-112.

[12] A.B. Aliev, *Solvability "in the large" of the Cauchy problem for quasi-linear equations of hyperbolic type*, Sov. Math. Dokl., **19**, 1978, 563-566.

[13] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1968.

[14] V.K. Kalantarov, E.S. Titi, *Global attractors and determining modes for the 3D Navier-Stokes-Voight equations*, Chin. Ann. Math., **30B(6)**, 2009, 697–714.

[15] A.V. Babin, M.I. Vishik, *Attractors of partial differential evolution equations and estimates of their dimension*, Uspekhi Mat. Nauk, **38(4)**(232), 1983, 133–187; Russian Math. Surveys, **38(4)**, 1983, 151–213.

[16] R. Temam, *Infinite Dimensional Dynamical System in Mechanics and Physics*, second ed., Spring-Verlag, 1997.

[17] V.V. Chepyzhov, M.I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society Colloquium Publications, **49**, Providence, RI, 2002.

[18] A.B. Aliev, A.Kh. Khanmamedov, *A On the existence of a minimal global attractor for the nonlinear wave equation with antidissipation in the domain and with dissipation on a part of the boundary*, Differ. Uravn., **34(3)**, 1998, 326–330 (in Russian).

[19] H. Brezis, *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Elsevier, 1973.

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