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Existence and Ulam Stability Results for Fractional Differential Equations with Mixed Nonlocal Conditions

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Abstract. In this paper, we use the fixed point theory to obtain the existence and uniqueness of solutions for fractional differential equations with mixed nonlocal conditions. Also, we show the Ulam stability of the solutions. Finally, an example is given to illustrate this work.

Key Words and Phrases: fractional differential equations, existence, uniqueness, Ulam stability, fixed point theorems.

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1. Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non integer order. Fractional differential equations with and without delay arise in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillators, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[22], [24]–[31] and the references therein. The study of Ulam stability for fractional differential equations was initiated by Wang et al. [29]. An overview on the development of theory of the Ulam-Hyers and the Ulam-Hyers-Rassias

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stability for fractional differential equations can be found in [29, 30] and the references therein. Subsequently, many authors discussed various Ulam-Hyers stability problems for different kinds of fractional integral and fractional differential equations by using different techniques, see [7, 15, 18, 29, 30] and the references therein.

Recently, Asawasamrit in [7] investigated the existence and Ulam stability analysis of the following fractional differential equation with mixed nonlocal conditions:

$$\begin{cases} {}^cD^{\alpha}u(t) = f\left(t, u\left(t\right)\right), \ t \in \left[0, T\right], \\ \sum\limits_{i=1}^{m} \gamma_i u(\eta_i) + \sum\limits_{j=1}^{n} \lambda_j {}^cD^{\beta_j} u(\zeta_j) + \sum\limits_{r=1}^{k} \sigma_j I_{0+}^{\delta_r} u(\phi_r) = A, \end{cases}$$

where ${}^cD^{\alpha}$ and ${}^cD^{\beta_j}$ denote the Caputo fractional derivatives of orders α and β_j, respectively, $0 < \beta_j < \alpha \leq 1$, for $j = 1, 2, ..., n$, the notation $I_{0+}^{\delta_r}$ is the right Riemann-Liouville fractional integral of order $\delta_r > 0$, for $r = 1, 2, ..., k$, the given constants $\gamma_i, \lambda_j, \sigma_j, A \in \mathbb{R}$, the points $\eta_i, \zeta_j, \theta_r \in \left[0, T\right]$, $i = 1, 2, ..., m$ and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In [13], the authors used the Krasnoselskii fixed point theorem to prove the existence of solutions to the following problem involving both left Riemann-Liouville and right Caputo fractional derivatives:

$$\begin{cases} {}^cD_{1^-}^{\beta}\left(D_{0+}^{\alpha}u(t)\right) + f\left(t, u\left(t\right)\right) = 0, \ t \in \left(0, 1\right), \\ u\left(0\right) = u'\left(0\right) = u\left(1\right) = 0, \end{cases}$$

where ${}^cD_{1^-}^{\beta}$ and D_{0+}^{α} denote the right Caputo fractional derivative of order $\beta \in (0, 1)$ and the left Riemann-Liouville fractional derivative of order $\alpha \in (1, 2)$, respectively, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In [14], the following boundary value problem involving both left Riemann-Liouville and right Riemann-Liouville fractional derivatives was studied:

$$\begin{cases} D_{1^-}^{\beta}\left(D_{0+}^{\alpha}u(t)\right) + f\left(t, u\left(t\right)\right) = 0, \ t \in \left(0, 1\right), \\ D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0, \\ u\left(0\right) = u'\left(0\right) = 0, \end{cases}$$

where $D_{1^-}^{\beta}$ and D_{0+}^{α} denote the right Riemann-Liouville derivative of order $\beta \in (1, 2)$ and the left Riemann-Liouville fractional derivative of order $\alpha \in (1, 2)$, respectively, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Motivated by the above works, in this paper, we investigated the existence and Ulam stability analysis of the following fractional differential equation with mixed nonlocal conditions:

$$\begin{cases} {}^cD_{1^-}^{\beta}\left({}^cD_{0^+}^{\alpha}u(t)\right) = f\left(t, u\left(t\right)\right),\ t \in [0,1]\,, \\ \sum\limits_{i=1}^{m}\gamma_i u(\eta_i) + \sum\limits_{r=1}^{n}\sigma_r I_{0^+}^{\delta_r}u(\theta_r) = A, \\ {}^cD_{0^+}^{\alpha}u(0) = {}^c D_{0^+}^{\alpha}u(1) = 0, \end{cases} \tag{1}$$

where ${}^cD_{1^-}^{\beta}$ and ${}^cD_{0^+}^{\alpha}$ denote, the left and the right Caputo fractional derivatives of orders β and α, respectively, $0 < \alpha \leq 1$, $1 < \beta \leq 2$, the notation $I_{0^+}^{\delta_r}$ is the right Riemann-Liouville fractional integral of order $\delta_r > 0$, for $r = 1, 2, ..., n$, the given constants $\gamma_i, \sigma_r, A \in \mathbb{R}$, the points $\eta_i, \theta_r \in [0,1]\,,\ i = 1, 2, ..., m$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. To show the existence and uniqueness of solutions, we transform (1) into an integral equation and then use the Banach and Schauder fixed point theorems. Further, we obtain Ulam-Hyers and Ulam-Hyers-Rassias stability results for (1). Finally, we provide an example to illustrate our results.

The rest of this paper is organized as follows. Some definitions from fractional calculus theory are recalled in Section 2. In Section 3, we will prove the existence and uniqueness of solutions for Problem (1). In Section 4, we discuss the Ulam stability results. Finally, an example is given in Section 5 to illustrate the usefulness of our main results.

2. Preliminaries

In this section, we recall some basic definitions and necessary lemmas related to fractional calculus and fixed point theorems that will be used throughout this paper. Let $C\left([0,1]\,,\mathbb{R}\right)$ denote the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$.

Definition 1 ([16])**.** *The left and right Riemann-Liouville fractional integrals of order* $\alpha \in \mathbb{R}$, $(\alpha > 0)$ *of a function* $f : [a,b] \to \mathbb{R}$ *are defined by*

$$\begin{aligned} I_{a^+}^{\alpha}f\left(t\right) &= \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(s\right)ds, \\ I_{b^-}^{\alpha}f\left(t\right) &= \int_t^b \frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(s\right)ds, \end{aligned}$$

respectively, where Γ *denotes the Gamma function.*

Definition 2 ([16])**.** *Let* $n-1 < \alpha < n$*. The left and right Caputo fractional derivatives of order* $\alpha \in \mathbb{R}$, $(\alpha > 0)$ *of a function* $f : [a,b] \to \mathbb{R}$ *such that* $f \in$

$C^n([a,b],\mathbb{R})$ *are given by*

$$\begin{aligned} {}^cD^{\alpha}_{a^+}x(t) &= \left(I^{n-\alpha}_{a^+}f^{(n)}\right)(t), \\ {}^cD^{\alpha}_{b^-}x(t) &= (-1)^n\left(I^{n-\alpha}_{b^-}f^{(n)}\right)(t), \end{aligned}$$

respectively, where $n=[\alpha]+1$, $[\alpha]$ *denotes the integer part of the real number* α.

Lemma 1 ([16])**.** *Let* $\alpha\in\mathbb{R}$ *with* $\alpha>0$ *and let* $n=[\alpha]+1$. *If* $f\in C^n([a,b],\mathbb{R})$, *then*

1) $I^{\alpha}_{a^+}\,{}^cD^{\alpha}_{a^+}f(t)=f(t)-\sum\limits_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k$.

2) $I^{\alpha}_{b^-}\,{}^cD^{\alpha}_{b^-}f(t)=f(t)-\sum\limits_{k=0}^{n-1}\frac{(-1)^{(k)}f^{(k)}(b)}{k!}(b-t)^k$.

In particular, when $0<\alpha<1$, $I^{\alpha}_{0^+}\,{}^cD^{\alpha}_{0^+}f(t)=f(t)-f(0)$ *and* $I^{\alpha}_{1^-}\,{}^cD^{\alpha}_{1^-}f(t)=f(t)-f(1)$.

To define Ulam's stability, we consider the following fractional differential equation:

$$ {}^cD^{\alpha}u(t)=f(t,u(t)),\ 0<\alpha\le 1,\ t\in[0,1]. \tag{2}$$

Definition 3 ([22])**.** *The equation (2) is said to be Ulam-Hyres (UH) stable if there exists a real number* $k>0$ *such that for each* $\epsilon>0$ *and for each* $y\in C^1([0,1],\mathbb{R})$ *solution of the inequality*

$$\left|{}^cD^{\alpha}y(t)-f(t,y(t))\right|\le\epsilon,\ t\in[0,1], \tag{3}$$

there exists a solution $u\in C^1([0,1],\mathbb{R})$ *of the equation (2) with*

$$|y(t)-u(t)|\le k\epsilon,\ t\in[0,1].$$

Definition 4 ([22])**.** *Assume that* $y\in C^1([0,1],\mathbb{R})$ *satisfies the inequality (3) and* $u\in C^1([0,1],\mathbb{R})$ *is a solution of the equation (2). If there is a function* $\phi_f\in C(\mathbb{R}^+,\mathbb{R}^+)$ *with* $\phi_f(0)=0$ *satisfying*

$$|y(t)-u(t)|\le\phi_f(\epsilon),\ t\in[0,1],$$

then the equation (2) is said to be generalized Ulam-Hyres (GUH) stable.

Definition 5 ([22])**.** *The equation (2) is said to be Ulam-Hyres-Rassias (UHR) stable with respect to* $\phi_f\in C([0,1],\mathbb{R}^+)$ *if there exists a real number* $k>0$ *such that for each* $\epsilon>0$ *and for each* $y\in C^1([0,1],\mathbb{R})$ *solution of the inequality*

$$\left|{}^cD^{\alpha}y(t)-f(t,y(t))\right|\le\epsilon\phi_f(t),\ t\in[0,1], \tag{4}$$

there exists a solution $u\in C^1([0,1],\mathbb{R})$ *of the equation (2) with*

$$|y(t)-u(t)|\le k\phi_f(t)\epsilon,\ t\in[0,1].$$

Definition 6 ([22]). *Assume that* $y \in C^1([0,1],\mathbb{R})$ *satisfies the inequality in (4) and* $u \in C^1([0,1],\mathbb{R})$ *is a solution of the equation (2). If there exists a non zero positive constant* $k_{\phi,f}$ *such that*

$$|y(t) - u(t)| \leq k_{\phi,f}\phi_f(t),\ t \in [0,1],$$

then the equation (2) is said to be generalized Ulam-Hyres-Rassias (GUHR) stable.

Remark 1. *If there is a function* $\Psi \in C([0,1],\mathbb{R})$ *(dependent on y), such that*
1) $|\Psi(t)| \leq \epsilon$, *for all* $t \in [0,1]$,
2) ${}^cD^\alpha y(t) = f(t, y(t)) + \Psi(t)$, $t \in [0,1]$,
then the function $y \in C^1([0,1],\mathbb{R})$ *is a solution of the inequality (3).*

Theorem 1 (Banach's fixed point theorem [23]). *Let* Ω *be a non-empty closed convex subset of a Banach space* $(S, \|.\|)$. *Then any contraction mapping* P *of* Ω *into itself has a unique fixed point.*

Theorem 2 (Schauder's fixed point theorem [23]). *Let* Ω *be a nonempty closed bounded convex subset of a Banach space* $(S, \|.\|)$ *and* $P : \Omega \to \Omega$ *be a continuous and compact operator. Then* P *has a fixed point in* Ω.

3. Existence and uniqueness

Let $E = C([0,1],\mathbb{R})$ be a Banach space of all continuous functions $x : [0,1] \to \mathbb{R}$ endowed with the norm $\|x\| = \sup_{t\in[0,1]}\{|x(t)|\}$.

The following lemma concerns a linear variant of Problem (1).

Lemma 2. *Suppose* $y \in C([0,1],\mathbb{R})$ *and*

$$\omega = \sum_{i=1}^{m}\gamma_i + \sum_{r=1}^{n}\frac{\sigma_r\theta_r^{\delta_r}}{\Gamma(\delta_r+1)} \neq 0. \tag{5}$$

Then, the unique solution of the linear problem

$$\begin{cases} {}^cD_{1^-}^{\beta}\left({}^cD_{0^+}^{\alpha}u(t)\right) = y(t),\ t \in [0,1], \\ \sum\limits_{i=1}^{m}\gamma_i u(\eta_i) + \sum\limits_{r=1}^{n}\sigma_r I_{0^+}^{\delta_r}u(\theta_r) = A, \\ {}^cD_{0^+}^{\alpha}u(0) = {}^cD_{0^+}^{\alpha}u(1) = 0, \end{cases} \tag{6}$$

is given by the integral equation

$$u(t) = I_{0^+}^{\alpha}I_{1^-}^{\beta}y(t) + \left(I_{0^+}^{\alpha}(1-t)\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}y(s)ds\right)$$

$$+\frac{1}{\omega}\left(A-\sum_{i=1}^{m}\gamma_i I_{0^+}^{\alpha}I_{1^-}^{\beta}y(\eta_i)-\left(\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)\,ds\right.\right.$$
$$\left.+\sum_{r=1}^{n}\sigma_r\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}(1-s)\,ds\right)$$
$$\left.\times\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}y(s)ds\right)-\sum_{r=1}^{n}\sigma_r I_{0^+}^{\delta_r+\alpha}I_{1^-}^{\beta}y(\theta_r)\right). \tag{7}$$

Proof. Applying the fractional integral operator $I_{1^-}^{\beta}$ to the first equation of (1), we get

$${}^{c}D_{0^+}^{\alpha}u(t)=I_{1^-}^{\beta}y(t)+c_0+c_1(1-t). \tag{8}$$

According to second condition in (1), it yields

$$c_0=0,\ c_1=\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}y(s)ds. \tag{9}$$

Substituting c_0 and c_1 by their values in (8), we obtain

$${}^{c}D_{0^+}^{\alpha}u(t)=I_{1^-}^{\beta}y(t)+(1-t)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}y(s)ds\right). \tag{10}$$

Now we apply the fractional integral $I_{0^+}^{\alpha}$ to the equation (10) to get

$$u(t)=I_{0^+}^{\alpha}I_{1^-}^{\beta}y(t)+\left(I_{0^+}^{\alpha}(1-t)\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}y(s)ds\right)+c_2. \tag{11}$$

From the first condition in (1), we deduce

$$c_2=\frac{1}{\omega}\left(A-\sum_{i=1}^{m}\gamma_i I_{0^+}^{\alpha}I_{1^-}^{\beta}y(\eta_i)-\sum_{r=1}^{n}\sigma_r I_{0^+}^{\delta_r+\alpha}I_{1^-}^{\beta}y(\theta_r)\right.$$
$$-\left(\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)\,ds\right.$$
$$\left.\left.+\sum_{r=1}^{n}\sigma_r\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}(1-s)\,ds\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}y(s)ds\right)\right).$$

Substituting c_2 in (11), we get the integral equation (7). The converse can be proven by direct computation. The proof is completed. ◄

Moreover, for computational convenience we put

$$L^{*}=\frac{1}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}\left(2\sum_{i=1}^{m}\left|\gamma_{i}\right|+2\sum_{r=1}^{k}\left|\sigma_{r}\right|\right). \tag{12}$$

Using Lemma 2 we define the operator $P:E\rightarrow E$ by

$$\begin{aligned}
\left(Pu\right)\left(t\right) &= \int_{0}^{t}\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\int_{s}^{1}\frac{\left(x-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}f\left(x,u(x)\right)dxds \\
&+ \left(I_{0^{+}}^{\alpha}\left(1-t\right)\right)\left(\frac{1}{\Gamma\left(\beta\right)}\int_{0}^{1}s^{\beta-1}f\left(s,u(s)\right)ds\right) \\
&+ \frac{1}{\omega}\left(A-\sum_{i=1}^{m}\gamma_{i}\int_{0}^{\eta_{i}}\frac{\left(\eta_{i}-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\int_{s}^{1}\frac{\left(x-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}f\left(x,u(x)\right)dxds\right. \\
&- \left(\sum_{i=1}^{m}\gamma_{i}\int_{0}^{\eta_{i}}\frac{\left(\eta_{i}-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left(1-s\right)ds\right. \\
&+ \left.\sum_{r=1}^{n}\sigma_{r}\int_{0}^{\theta_{r}}\frac{\left(\theta_{r}-s\right)^{\delta_{r}+\alpha-1}}{\Gamma\left(\delta_{r}+\alpha\right)}\left(1-s\right)ds\right) \\
&\times \left(\frac{1}{\Gamma\left(\beta\right)}\int_{0}^{1}s^{\beta-1}f\left(s,u(s)\right)ds\right) \\
&- \left.\sum_{r=1}^{n}\sigma_{r}\int_{0}^{\theta_{r}}\frac{\left(\theta_{r}-s\right)^{\delta_{r}+\alpha-1}}{\Gamma\left(\delta_{r}+\alpha\right)}\int_{s}^{1}\frac{\left(x-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}f\left(x,u(x)\right)dxds\right).
\end{aligned} \tag{13}$$

Theorem 3. *Let $f:[0,1]\times\mathbb{R}\rightarrow\mathbb{R}$ be a continuous function and (5) hold. Suppose that*

(H1) $f(t,u)$ is a Lipschitz continuous function (in u), i.e. there exists a constant $k_f>0$ such that

$$\left|f(t,u)-f(t,v)\right|\leq k_{f}\left|u-v\right|,\ \textit{for all } t\in\left[0,1\right],\ u,v\in\mathbb{R}.$$

If

$$\frac{2k_{f}}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}+\frac{k_{f}}{\left|\omega\right|}L^{*}<1,$$

then Problem (1) has a unique solution on $[0,1]$.

Proof. Let us define a positive number M as follows:

$$M\geq\frac{\left|A\right|+N\left(\frac{2\left|\omega\right|}{\Gamma(\alpha+1)\Gamma(\beta+1)}+L^{*}\right)}{\left|\omega\right|-k_{f}\left(\frac{2\left|\omega\right|}{\Gamma(\alpha+1)\Gamma(\beta+1)}+L^{*}\right)},$$

$$N=\sup\left\{ \left|f(t,0)\right| ,\ t\in[0,1]\right\} ,$$

and show that $PB_M \subset B_M$, where

$$B_M=\{u\in E:\|u\|\leq M\}$$

is a closed ball, and we note that

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)}dxds \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}, \tag{14}$$

and

$$\frac{1}{\Gamma(\delta_r+\alpha+1)}\leq\frac{1}{\Gamma(\alpha+1)}.$$

In (14) we have used the fact that $(1-s)^{\beta}\leq 1$ for $1<\beta\leq 2$. Using the above argument, we have

$$\begin{aligned}
|(Pu)(t)| \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)}\left(\left|f(x,u(x))-f(x,0)\right|+\left|f(x,0)\right|\right)dxds \\
& +\left(I_{0^+}^{\alpha}(1-t)\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}\left(\left|f(s,u(s))-f(s,0)\right|+\left|f(s,0)\right|\right)ds\right) \\
& +\frac{1}{|\omega|}\left(|A|+\sum_{i=1}^{m}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\int_s^1\frac{(x-s)^{\beta-1}}{\Gamma(\beta)}\right. \\
& \times\left(\left|f(x,u(x))-f(x,0)\right|+\left|f(x,0)\right|\right)dxds \\
& +\sum_{r=1}^{n}|\sigma_r|\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}\int_s^1\frac{(x-s)^{\beta-1}}{\Gamma(\beta)} \\
& \times\left(\left|f(x,u(x))-f(x,0)\right|+\left|f(x,0)\right|\right)dxds \\
& +\left(\sum_{i=1}^{m}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)\,ds\right. \\
& \left.+\sum_{r=1}^{n}|\sigma_r|\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}(1-s)\,ds\right) \\
& \left.\times\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}\left(\left|f(s,u(s))-f(s,0)\right|+\left|f(s,0)\right|\right)ds\right)\right) \\
\leq & \frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)}(k_f M+N)+\frac{|A|}{|\omega|} \\
& +\frac{(k_f M+N)}{|\omega|}\left(\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left(2\sum_{i=1}^{m}|\gamma_i|+2\sum_{r=1}^{n}|\sigma_r|\right)\right)
\end{aligned}$$

$$= \frac{|A|}{|\omega|} + \frac{(k_f M + N)}{|\omega|} \left(\frac{2|\omega|}{\Gamma(\alpha+1)\Gamma(\beta+1)} + L^* \right) \leq M,$$

which, by taking the norm on $[0,1]$, yields $\|Pu\| \leq M$. This shows that $PB_M \subset B_M$. In order to show that P is a contraction, we put $u, v \in B_M$. Then we obtain

$$\begin{aligned}
& |(Pu)(t) - (Pv)(t)| \\
& \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} (|f(x,u(x)) - f(x,v(x))|) \, dx ds \\
& + (I_{0^+}^{\alpha}(1-t)) \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} (|f(s,u(s)) - f(s,v(s))|) \, ds \right) \\
& + \frac{1}{|\omega|} \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} (|f(x,u(x)) - f(x,v(x))|) \, dx ds \right. \\
& + \sum_{r=1}^n |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r - s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} (|f(x,u(x)) - f(x,v(x))|) \, dx ds \\
& + \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} (1-s) \, ds \right. \\
& \left. + \sum_{r=1}^n |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r - s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} (1-s) \, ds \right) \\
& \left. \times \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} (|f(s,u(s)) - f(s,v(s))|) \, ds \right) \right) \\
& \leq \frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)} \|u-v\| + \frac{k_f \|u-v\|}{|\omega|} \\
& \times \left(\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left(2\sum_{i=1}^m |\gamma_i| + 2\sum_{r=1}^n |\sigma_r| \right) \right) \\
& = \left(\frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{k_f}{|\omega|} L^* \right) \|u-v\|.
\end{aligned}$$

Therefore

$$\|Pu - Pv\| \leq \left(\frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{k_f}{|\omega|} L^* \right) \|u-v\|,$$

which implies that P is a contraction, since

$$\frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{k_f}{|\omega|} L^* < 1.$$

By Banach contraction mapping the operator P defined in (13), has the unique fixed point, which implies that Problem (1) has a unique solution on $[0,1]$. The proof is completed. ◄

Theorem 4. *Let f be a continuous function on $[0,1]\times\mathbb{R}$ and (5) hold. Assume that*

(H2) $|f(t,u(t))| \le \Phi(t)$, $\forall (t,u)\in[0,1]\times\mathbb{R}$, $\Phi\in L^1([0,1],\mathbb{R}^+)$.

Then Problem (1) has at least one solution on $[0,1]$.

Proof. We consider the non-empty closed bounded convex subset $B_M = \{u\in E : \|u\|\le M\}$ of E where M is chosen as follows:

$$M \ge \frac{|A|}{|\omega|} + \Phi^*\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}\right).$$

The continuity of f implies the continuity of the operator P. Now, we need to show that the operator P is compact by applying the well known Arzela-Ascoli theorem. So we will show that $PB_M\subset B_M$ and PB_M is a uniformly bounded and equicontinuous set. Let $\Phi^* = \sup\{\Phi(t) : t\in[0,1]\}$. For $u\in B_M$, it follows that

$$\begin{aligned}
|(Pu)(t)| \le{}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)}|f(x,u(x))|\,dxds \\
&+ \left(I_{0^+}^{\alpha}(1-t)\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}|f(s,u(s))|\,ds\right) \\
&+ \frac{1}{|\omega|}\left(|A| + \sum_{i=1}^{m}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\int_s^1\frac{(x-s)^{\beta-1}}{\Gamma(\beta)}|f(x,u(x))|\,dxds\right. \\
&+ \sum_{r=1}^{n}|\sigma_r|\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}\int_s^1\frac{(x-s)^{\beta-1}}{\Gamma(\beta)}|f(x,u(x))|\,dxds \\
&+ \left(\sum_{i=1}^{m}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)\,ds\right. \\
&\left.+ \sum_{r=1}^{n}|\sigma_r|\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}(1-s)\,ds\right) \\
&\left.\times\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}|f(s,u(s))|\,ds\right)\right) \\
\le{}& \frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Phi^* + \frac{|A|}{|\omega|}
\end{aligned}$$

$$+\frac{\Phi^*}{|\omega|}\left(\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left(2\sum_{i=1}^{m}|\gamma_i|+2\sum_{r=1}^{n}|\sigma_r|\right)\right)$$
$$=\frac{|A|}{|\omega|}+\Phi^*\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{L^*}{|\omega|}\right),$$

and consequently

$$\|Pu\|\leq\frac{|A|}{|\omega|}+\Phi^*\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{L^*}{|\omega|}\right)\leq M,$$

which implies that $PB_M\subset B_M$ and the set PB_M is uniformly bounded. Next, we are going to prove that PB_M is an equicontinuous set. For $t_1,t_2\in[0,1]$ such that $t_1<t_2$ and for $u\in B_M$, we obtain

$$\begin{aligned}&|(Pu)(t_2)-(Pu)(t_1)|\\&\leq\frac{\Phi^*}{\Gamma(\beta+1)}\int_0^{t_1}\left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)ds\\&+\int_0^{t_1}\left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)(1-s)ds\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}|f(s,u(s))|ds\right)\\&+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)ds\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}|f(s,u(s))|ds\right)\\&+\frac{\Phi^*}{\Gamma(\beta+1)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\&\leq\frac{\Phi^*}{\Gamma(\beta+1)}\left(\frac{4(t_2-t_1)^{\alpha}+2(t_1^{\alpha}-t_2^{\alpha})}{\Gamma(\alpha+1)}\right).\end{aligned}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero and the convergence is independent of u in B_M, which means PB_M is equicontinuous. The Arzela-Ascoli theorem implies that P is compact. Hence, by the Schauder fixed point theorem, the operator P has at least one fixed point $u\in B_M$. Therefore, Problem (1) has at least one solution on $[0,1]$. ◄

4. Ulam stability

Lemma 3. *If y satisfies the inequality*

$$\left|{}^cD_{1^-}^{\beta}\left({}^cD_{0^+}^{\alpha}y(t)\right)-f(t,y(t))\right|\leq\epsilon,\ t\in[0,1],\tag{15}$$

then, for $\epsilon \in (0,1]$, y is a solution of the inequality

$$|y(t) - (Py)(t)| \leq \epsilon \left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|} \right), \tag{16}$$

where L^* is defined in (12).

Proof. From Remark 1 and Lemma 2, we have

$$\begin{aligned}
y(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} (f(x,y(x)) + \psi(x))\, dxds \\
& + \left(I_{0^+}^{\alpha}(1-t)\right) \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} (f(s,y(s)) + \psi(s))\, ds \right) \\
& + \frac{1}{\omega} \left(A - \sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} (f(x,y(x)) + \psi(x))\, dxds \right. \\
& - \left(\sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} (1-s)\, ds + \sum_{r=1}^n \sigma_r \int_0^{\theta_r} \frac{(\theta_r - s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} (1-s)\, ds \right) \\
& \times \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} (f(s,y(s)) + \psi(s))\, ds \right) \\
& \left. - \sum_{r=1}^n \sigma_r \int_0^{\theta_r} \frac{(\theta_r - s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} (f(x,y(x)) + \psi(x))\, dxds \right).
\end{aligned}$$

Then, by Remark 1, we obtain

$$\begin{aligned}
& |y(t) - (Py)(t)| \\
& \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds \\
& + \left(I_{0^+}^{\alpha}(1-t)\right) \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} |\psi(s)|\, ds \right) \\
& + \frac{1}{\omega} \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds \right. \\
& + \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} (1-s)\, ds \right. \\
& \left. + \sum_{r=1}^n |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r - s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} (1-s)\, ds \right) \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} |\psi(s)|\, ds \right) \\
& \left. + \sum_{r=1}^n |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r - s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds \right)
\end{aligned}$$

$$\leq \epsilon\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{L^*}{|\omega|}\right).$$

This completes the proof. ◀

Theorem 5. *If the conditions (5), (H1) and (H2) are fulfilled and $d \in (0,1)$ holds, where*

$$d=\frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{k_f}{|\omega|}L^*,$$

then Problem (1) is UH stable.

Proof. Suppose y is a solution of the inequality (15) and let u be the unique solution of Problem (1). We consider

$$\begin{aligned}
&|y(t)-u(t)| \\
&=\Bigg|y(t)-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(x,u(x))\,dxds \\
&-\left(I_{0^+}^{\alpha}(1-t)\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1} f(s,u(s))\,ds\right) \\
&-\frac{1}{\omega}\Bigg(A-\sum_{i=1}^m \gamma_i \int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\int_s^1\frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(x,u(x))\,dxds \\
&-\Bigg(\sum_{i=1}^m \gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)\,ds \\
&+\sum_{r=1}^n \sigma_r\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}(1-s)\,ds\Bigg)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}f(s,u(s))\,ds\right) \\
&-\sum_{r=1}^n \sigma_r\int_0^{\theta_r}\frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}\int_s^1\frac{(x-s)^{\beta-1}}{\Gamma(\beta)}f(x,u(x))\,dxds\Bigg)\Bigg| \\
&\leq |y(t)-(Py)(t)|+|(Py)(t)-(Pu)(t)| \\
&\leq \epsilon\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{L^*}{|\omega|}\right) \\
&+\left(\frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{k_f}{|\omega|}L^*\right)\|y-u\|,
\end{aligned}$$

from which we obtain

$$|y(t)-u(t)| \leq \frac{\epsilon\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{L^*}{|\omega|}\right)}{1-d}.$$

By setting

$$k = \frac{\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}}{1-d},$$

we obtain

$$|y(t) - u(t)| \leq k\epsilon. \tag{17}$$

Therefore, Problem (1) is UH stable. The proof is completed. ◄

Remark 2. *By setting $\phi_f(\epsilon) = k\epsilon$ in (17), we obtain $\phi_f(0) = 0$ and then Problem (1) is GUH stable.*

Lemma 4. *Let y be a solution of the inequality*

$$\left| {}^cD^{\beta}_{1^-}\left({}^cD^{\alpha}_{0^+}y(t)\right) - f\left(t, y\left(t\right)\right)\right| \leq \epsilon\phi_f(t),\ t \in [0,1], \tag{18}$$

and assume that (H3)

$$\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds \\
&+ \left(I^{\alpha}_{0^+}(1-t)\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}|\psi(s)|\, ds\right) \\
&+ \frac{1}{\omega}\left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds\right. \\
&+ \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)\, ds\right. \\
&+ \left.\sum_{r=1}^n |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}(1-s)\, ds\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}|\psi(s)|\, ds\right) \\
&+ \left.\sum_{r=1}^n |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds\right) \\
&\leq \epsilon\phi_f(t)\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}\right).
\end{aligned}$$

Then y satisfies the inequality

$$|y(t) - (Py)(t)| \leq \epsilon\phi_f(t)\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}\right). \tag{19}$$

Proof. From Remark 1 and Lemma 2, we have

$$
\begin{aligned}
y(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} \left(f(x,y(x))+\psi(x)\right) dxds \\
&+ \left(I_{0^+}^{\alpha}(1-t)\right) \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} \left(f(s,y(s))+\psi(s)\right) ds\right) \\
&+ \frac{1}{\omega} \left(A - \sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} \left(f(x,y(x))+\psi(x)\right) dxds\right. \\
&- \left(\sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)} (1-s)\, ds\right. \\
&+ \left.\sum_{r=1}^{k} \sigma_j \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} (1-s)\, ds\right) \\
&\times \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} \left(f(s,y(s))+\psi(s)\right) ds\right) \\
&- \left.\sum_{r=1}^{k} \sigma_j \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} \left(f(x,y(x))+\psi(x)\right) dxds\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|y(t)-(Py)(t)| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds \\
&+ \left(I_{0^+}^{\alpha}(1-t)\right) \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} |\psi(s)|\, ds\right) \\
&+ \frac{1}{\omega} \left(\sum_{i=1}^{m} |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds\right. \\
&+ \left(\sum_{i=1}^{m} |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)} (1-s)\, ds + \sum_{r=1}^{n} |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} (1-s)\, ds\right) \\
&\times \left(\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} |\psi(s)|\, ds\right) \\
&+ \left.\sum_{r=1}^{n} |\sigma_r| \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} |\psi(x)|\, dxds\right) \\
&\leq \epsilon \phi_f(t) \left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}\right),
\end{aligned}
$$

which yields the inequality (19). ◀

Theorem 6. *If the assumptions (5), (H1), (H2), (H3) and* $d \in (0,1)$ *are satisfied, then Problem (1) is UHR stable.*

Proof. Let y be a solution of the inequality (18) and let u be the unique solution of Problem (1). Next we consider

$$
\begin{aligned}
&|y(t)-u(t)| \\
&= \Bigg| y(t) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(x,u(x))\,dxds \\
&- \left(I_{0^+}^{\alpha}(1-t)\right)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1} f(s,u(s))\,ds\right) \\
&- \frac{1}{\omega}\Bigg(A - \sum_{i=1}^{m}\gamma_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(x,u(x))\,dxds \\
&- \Bigg(\sum_{i=1}^{m}\gamma_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}(1-s)\,ds \\
&+ \sum_{r=1}^{k}\sigma_j \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)}(1-s)\,ds\Bigg)\left(\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1} f(s,u(s))\,ds\right) \\
&- \sum_{r=1}^{k}\sigma_j \int_0^{\theta_r} \frac{(\theta_r-s)^{\delta_r+\alpha-1}}{\Gamma(\delta_r+\alpha)} \int_s^1 \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(x,u(x))\,dxds\Bigg)\Bigg| \\
&\le |y(t)-(Py)(t)| + |(Py)(t)-(Pu)(t)| \\
&\le \epsilon\phi_f(t)\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}\right) \\
&+ \left(\frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{k_f}{|\omega|}L^*\right)\|y-u\|,
\end{aligned}
$$

from which we have

$$|y(t)-u(t)| \le \frac{\epsilon\phi_f(t)\left(\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}\right)}{1-d}.$$

By taking a constant

$$k_{\phi,f} = \frac{\frac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{L^*}{|\omega|}}{1-d},$$

we obtain

$$|y(t)-u(t)| \le k_{\phi,f}\epsilon\phi_f(t). \tag{20}$$

Therefore, Problem (1) is UHR stable. The proof is completed. ◀

Remark 3. *By putting $\epsilon = 1$ in (20), we deduce that Problem (1) is GUHR stable.*

5. Example

Consider the following problem

$$\begin{cases} {}^cD_{1^-}^{\frac{3}{2}}\left({}^cD_{0^+}^{\frac{1}{2}}u(t)\right) = \frac{1}{4}t\sin\left(u(t)\right),\ t\in[0,1], \\ 0.25u(1) + 0.5u(0.5) + 0.75I_{0^+}^{0.6}u(1) = 1, \\ {}^cD_{0^+}^{\frac{1}{2}}u(0) = {}^cD_{0^+}^{\frac{1}{2}}u(1) = 0. \end{cases} \tag{21}$$

Here $\alpha = 0.5$, $\beta = 1.5$, $\gamma_1 = 0.25$, $\gamma_2 = 0.5$, $\eta_1 = 1$, $\gamma_1 = 0.5$, $\sigma_1 = 0.75$, $\theta_1 = 1$, $\delta_1 = 0.75$, $A = 1$ and $f(t,u) = \frac{1}{4}t\sin(u)$. From given information, we find that $\omega = 1.50$ and $L^* = 2.5464$. Since

$$f(t,u) = \frac{1}{4}t\sin\left(u(t)\right),$$

we obtain that $|f(t,u) - f(t,v)| \leq \frac{1}{4}|u-v|$. Then we get $k_f = \frac{1}{4}$, $|f(t,u)| \leq \frac{1}{4}t = \Phi(t)$. In view of Theorem 3, $\frac{2k_f}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{k_f}{|\omega|}L^* = 0.8488 < 1$. Hence Problem (21) has the unique solution.

In view of Theorem 5, the conditions (5), (H1), (H2) and $d = 0.8488$ are satisfied. Therefore Problem (21) is UH stable with $k = 22.4553$ and hence GUH stable. By setting $\phi_f(t) = 0.4230t^{0.6} + 0.5770$, we make all conditions of Theorem 6 satisfied. Therefore Problem (21) is UHR stable and also GUHR stable.

References

[1] S. Abbas, *Existence of solutions to fractional order ordinary and delay differential equations and applications*, Electronic Journal of Differential Equations, **2011(9)**, 2011, 1–11.

[2] R.P. Agarwal, Y. Zhou, Y. He, *Existence of fractional functional differential equations*, Computers and Mathematics with Applications, **59**, 2010, 1095–1100.

[3] B. Ahmad, S.K. Ntouyas, *Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations*, Electronic Journal of Differential Equations, **2017(36)**, 2017, 1–11.

[4] A. Ardjouni, *Positive solutions for nonlinear Hadamard fractional differential equations with integral boundary conditions*, AIMS Mathematics, **4(4)**, 2019, 1101–1113.

[5] A. Ardjouni, A. Djoudi, *Approximating solutions of nonlinear hybrid Caputo fractional integro-differential equations via Dhage iteration principle*, Ural Mathematical Journal, **5(1)**, 2019, 3–12.

[6] A. Ardjouni, A. Djoudi, *Positive solutions for first-order nonlinear Caputo-Hadamard fractional relaxation differential equations*, Kragujevac Journal of Mathematics, **45(6)**, 2021, 897–908.

[7] S. Asawasamrit, W. Nithiarayaphaks, S.K. Ntouyas, J. Tariboon, *Existence and stability analysis for fractional differential equations with mixed nonlocal conditions*, Mathematics, **7(117)**, 2019, 1–11.

[8] H. Boulares, A. Ardjouni, Y. Laskri, *Positive solutions for nonlinear fractional differential equations*, Positivity, **21**, 2017, 1201–1212.

[9] H. Boulares, A. Ardjouni, Y. Laskri, *Stability in delay nonlinear fractional differential equations*, Rend. Circ. Mat. Palermo, **65**, 2016, 243–253.

[10] A. Chidouh, A. Guezane-Lakoud, R. Bebbouchi, *Positive solutions of the fractional relaxation equation using lower and upper solutions*, Vietnam J. Math., **44(4)**, 2016, 739–748.

[11] F. Ge, C. Kou, *Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations*, Applied Mathematics and Computation, **257**, 2015, 308–316.

[12] F. Ge, C. Kou, *Asymptotic stability of solutions of nonlinear fractional differential equations of order* $1 < \alpha < 2$, Journal of Shanghai Normal University, **44(3)**, 2015, 284–290.

[13] A. Guezane Lakoud, R. Khaldi, A. Kilicman, *Existence of solutions for a mixed fractional boundary value problem*, Advances in Difference Equations, **2017(164)**, (2017), 1–9.

[14] A. Guezane-Lakoud, S. Ramdane, *Existence of solutions for a system of mixed fractional differential equations*, Journal of Taibah University for Science, **12(4)**, 2018, 421–426.

[15] M. Benchohra, E.L. Jamal, *On stability for nonlinear implicit fractional differential equations*, Le Matematiche, **70(2)**, 2015, 49–61.

[16] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B. V., Amsterdam, 2006.

[17] C. Kou, H. Zhou, Y. Yan, *Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis*, Nonlinear Anal., **74**, 2011, 5975–5986.

[18] K.D. Kucche, S.T. Sutar, *On existence and stability results for nonlinear fractional delay differential equations*, Bol. Soc. Paran. Mat., **36(4)**, 2018, 55–75.

[19] V. Lakshmikantham, A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., **69**, 2008, 2677–2682.

[20] N. Li, C. Wang, *New existence results of positive solution for a class of nonlinear fractional differential equations*, Acta Math. Sci., **33**, 2013, 847–854.

[21] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[22] I.A. Rus, *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math., **26**, 2010, 103–107.

[23] D.R. Smart, *Fixed point theorems*, Cambridge Tracts in Mathematics, no. 66, Cambridge University Press, London–New York, 1974.

[24] X. Tang, *Existence of solutions of four-point boundary value problems for fractional differential equations at resonance*, J. Appl. Math. Comput., **51(1–2)**, 2016, 145–160.

[25] X. Tang, X. Wang, Z. Wang, P. Ouyang, *The existence of solutions for mixed fractional resonant boundary value problem with p(t)-Laplacian operator*, J. Appl. Math. Comput., **61(1–2)**, 2019, 559–572.

[26] X. Tang, C. Yan, Q. Liu, *Existence of solutions of two-point boundary value problems for fractional p-Laplace differential equations at resonance*, J. Appl. Math. Comput., **41(1–2)**, 2013, 119–131.

[27] S. Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl., **252**, 2000, 804–812.

[28] F. Wang, *Existence and uniqueness of solutions for a nonlinear fractional differential equation*, J. Appl. Math. Comput., **39(1–2)**, 2012, 53–67.

[29] J. Wang, L. Lv, Y. Zhou, *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, Electron. J. Qual. Theory Differ. Equ., **2011(63)**, 2011, 1–10.

[30] J. Wang, L. Lv, Y. Zhou, *New concepts and results in stability of fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17**, 2012, 2530–2538.

[31] M. Xu, S. Sun, *Positivity for integral boundary value problems of fractional differential equations with two nonlinear terms*, J. Appl. Math. Comput., **59(1–2)**, 2019, 271–283.

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