Azerbaijan Journal of Mathematics
V. 11, No 2, 2021, July
ISSN 2218-6816

The Discontinuous Nonlinear Dirichlet Boundary Value Problem with p-Laplacian

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Abstract. This paper is dedicated to the class of locally bounded weakly upper semi-continuous set-valued operators of generalized (S_+) type in the weighted Sobolev space. The aim of this work is to transform this Dirichlet boundary value problem related to the p-Laplacian with discontinuous nonlinearity into a new one governed by a Hammerstein equation. Then, we establish the existence of weak solutions of the state problem by using the topological degree theory.

Key Words and Phrases: weighted Sobolev spaces, Hardy inequality, topological degree, set-valued operator, p-elliptic problems.

2010 Mathematics Subject Classifications: 35K45, 35K61, 35K65

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$with Lipschitz boundary if $N \geq 2$ and let p be a real number such that $2 < p < \infty$ and $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions on Ω, i.e. each $w_i(x)$ is a measurable a.e. positive on Ω. Let $W_0^{1,p}(\Omega, w)$ be the weighted Sobolev space associated with the vector w.

The p-Laplacian problem appear naturally in a number of fields such as physics, climatological model [10], bimaterial problems in which there are two environments with different resistance constants [20], image processing [17, 29]. It is also applied to polymer rheology, regular variation in thermodynamics, fitting of experimental data, blood flow phenomena, aerodynamics, electro analytical chemistry, electro-dynamics of complex medium, viscoelasticity, electrical circuits, biology, control theory, Bode analysis of feedback amplifiers, capacitor theory, non-Newtonian fluids, electrorheological fluids, the flow of a fluid through a porous medium [13, 21, 24, 27]. Furthermore, in [23], a theoretical point of view of the p-Laplacian game is studied.

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Motivated by the works mentioned above, in this paper we prove existence of weak solutions to the following discontinuous nonlinearity boundary value problem for a p-Laplacian equation

$$\begin{cases} -\Delta_p u + u \in -\left[\underline{\psi}(x,u), \overline{\psi}(x,u)\right] & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}\big(a(x,\nabla u)\big) = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$. We shall suppose that the following degenerate ellipticity condition is satisfied for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$:

$$a(x,\xi)\cdot\xi \geq \varrho \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{2}$$

such that ϱ is a positive constant. The function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a possibly discontinuous function in the sense that

$$\underline{\psi}(x,s) = \liminf_{\eta\to s} \psi(x,\eta) = \lim_{\delta\to 0^+} \inf_{|\eta-s|<\delta} \psi(x,\eta),$$

$$\overline{\psi}(x,s) = \limsup_{\eta\to s} \psi(x,\eta) = \lim_{\delta\to 0^+} \sup_{|\eta-s|<\delta} \psi(x,\eta).$$

Suppose that $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a real-valued function such that

(H_1) $\overline{\psi}$ and $\underline{\psi}$ are super-positionally measurable, that is, $\overline{\psi}(\cdot,u(\cdot))$ and $\underline{\psi}(\cdot,u(\cdot))$ are measurable on Ω for any measurable function $u : \Omega \to \mathbb{R}$;

(H_2) ψ satisfies the growth condition:

$$|\psi(x,s)| \leq b(x) + c\,\sigma(x)\,|s|^{p/p'},$$

for almost all $x \in \Omega$ and all $s \in \mathbb{R}$, where $b \in L^{p'}(\Omega,\sigma^*)$, and c is a positive constant.

In the case $\left[\underline{\psi}(x,u), \overline{\psi}(x,u)\right] = \{1\}$, the p-Laplacian problem (1) models elastic-plastic torsion problems that arise in solid mechanics [14]. In particular, the case $p = 2$ indicates the perfectly elastic torsion model. In addition, the limit case $p \to \infty$ refers to the perfectly plastic torsion model [14].

In this paper, we propose a topological degree theory developed by Kim in [16] for a class of locally bounded weakly upper semi-continuous set-valued operators of generalized (S_+) type in real reflexive separable Banach spaces, based on the Berkovits-Tienari degree [5]. The topological degree theory was first introduced by Leray-Schauder [18] in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Browder [6] constructed a topological degree for operators of class (S_+) in reflexive Banach spaces by the Galerkin method, see also [22, 25, 26]. Among many examples, we refer the reader to the classical works [9, 28] for more details.

On the other hand, S. Liu [19] by using Morse theory, has established the existence of weak solutions to the equation $-\triangle_p u = f(x, u)$ with Dirichlet boundary conditions. It should be mentioned that the results in this paper generalise the p-Laplacian results obtained in [2, 16] to the strongly nonlinear case using the topological degree.

The layout of the paper is as follows. In the next section, we give some preliminaries and the definition of weighted Sobolev spaces and we recall some classes of mappings of generalized (S_+) type and the topological degree. In the third section, we discuss the p-Laplace operator and technical lemmas. Finally we show some existence results of weak solutions of problem (1).

2. Preliminaries

In order to discuss problem (1), we need some theories on topological degree and on spaces $W^{1,p}(\Omega, w)$ which we call weighted Lebesgue−Sobolev spaces. Firstly we state some classes of mappings and topological degree, secondly we give basic properties of spaces $W^{1,p}(\Omega, w)$ which will be used later.

2.1. Classes of mappings and topological degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle \cdot, \cdot \rangle$ between X^* and X in this order, and given a nonempty subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. The symbol $\to$ $(\rightharpoonup)$ stands for strong (weak) convergence.

Definition 1. *Let* Y *be another real Banach space. A set-valued operator* F *:* $\Omega \subset X \to 2^{\mathrm{Y}}$ *is said to be*

1. *bounded, if F maps bounded sets into bounded sets;*
2. *locally bounded, if for each $u \in \Omega$ there exists a neighborhood* U *of u such that the set* $F(\mathrm{U}) = \bigcup_{u\in \mathrm{U}} Fu$ *is bounded;*
3. *upper semicontinuous (u.s.c.), if the set* $F^{-1}(A) = \{u \in \Omega | Fu \cap A \neq \emptyset\}$ *is closed in X for each closed set A in* Y;
4. *weakly upper semicontinuous (w.u.s.c.), if $F^{-1}(A)$ is closed in X for each weakly closed set A in* Y;

5. *compact, if it is upper semicontinuous and the image of any bounded set is relatively compact.*

Definition 2. *A set-valued operator $F : \Omega \subset X \to 2^{X^*}\backslash\emptyset$ is said to be*

1. *of class (S_+), if for any sequence (u_n) in Ω and any sequence (h_n) in X^* with $h_n \in Fu_n$ such that $u_n \rightharpoonup u$ in X and*

$$\limsup_{n\to\infty}\langle h_n, u_n - u\rangle \le 0,$$

we have $u_n \to u$ in X;

2. *quasimonotone, if for any sequence (u_n) in Ω and any sequence (w_n) in X^* with $w_n \in Fu_n$ such that $u_n \rightharpoonup u$ in X, we have*

$$\liminf_{n\to\infty}\langle w_n, u_n - u\rangle \ge 0.$$

Definition 3. *Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. A set-valued operator $F : \Omega \subset X \to 2^X\backslash\emptyset$ is said to be:*

1. *of class $(S_+)_T$, if for any sequence (u_n) in Ω and any sequence (h_n) in X with $h_n \in Fu_n$ such that $u_n \rightharpoonup u$ in X, $Tu_n \rightharpoonup y$ in X^* and*

$$\limsup_{n\to\infty}\langle h_n, Tu_n - y\rangle \le 0,$$

we have $u_n \to u$ in X;

2. *T-quasimonotone, written $F \in (QM)_T$, if for any sequence (u_n) in Ω and any sequence (h_n) in X with $h_n \in Fu_n$ such that $u_n \rightharpoonup u$ in X, $Tu_n \rightharpoonup y$ in X^*, we have*

$$\liminf_{n\to\infty}\langle h_n, Tu_n - y\rangle \ge 0.$$

Remark 1. *([16]) Notice that if $F : \Omega \subset X \to 2^X\backslash\emptyset$ is locally bounded and of class $(S_+)_T$, where Ω is closed in X and $T : \Omega \to X^*$ is bounded and continuous, then F is T-quasimonotone. Moreover, the collection of operators of class $(S_+)_T$ is stable under $(QM)_T$-perturbations.*

Let $\mathcal{O}$ be the collection of all bounded open sets in X. For any $\Omega \subset X$, we consider the following classes of operators:

$\mathcal{F}_1(\Omega) := \{F : \Omega \to X^* | F$ is bounded, demicontinuous and satifies condition $(S_+)\}$,

$\mathcal{F}_{T,B}(\Omega) := \{F : \Omega \to 2^X | F$ is bounded, w.u.s.c. and satifies condition $(S_+)_T\}$,

$\mathcal{F}_T(\Omega) := \{F : \Omega \to 2^X | F$ is locally bounded, w.u.s.c. and satifies condition $(S_+)_T\}$,
$\mathcal{F}_B(X) := \{F \in \mathcal{F}_{T,B}(\overline{G}) |\ G \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{G}})\}$.
$\mathcal{F}(X) := \{F \in \mathcal{F}_T(\overline{G}) |\ G \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{G}})\}$.

Throughout the paper $T \in \mathcal{F}_1(\overline{G})$ is called an essential inner map to F.

Lemma 1. *([16, Lemma 1.4]) Let $T \in \mathcal{F}_1(\overline{G})$ be continuous and $S : D_S \subset X^* \to 2^X$ be locally bounded and weakly upper semicontinuous such that $T(\overline{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

1. If *S is quasimonotone, then $I + ST \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*
2. *If S is of class (S_+), then $ST \in \mathcal{F}_T(\overline{G})$.*

Definition 4. *([16]) For a bounded operator $T : \overline{G} \subset X \to X^*$, a homotopy $H : [0, 1] \times \overline{G} \to 2^X$ is said to be of class $(S_+)_T$, if for any sequence (t_k, u_k) in $[0, 1] \times \overline{G}$ and any sequence (a_k) in X with $a_k \in H(t_k, u_k)$ such that*

$$u_k \rightharpoonup u\ \in X,\ t_k \to t \in [0, 1],\ Tu_k \rightharpoonup y \quad \text{in } X^* \text{ and } \limsup_{k\to\infty} \langle a_k, Tu_k - y\rangle \le 0,$$

we have $u_k \to u$ in X.

The following result says that every affine homotopy with a common essential inner map T is of class $(S_+)_T$.

Lemma 2. *([16]) Let G be a bounded open subset of a real reflexive Banach space X and let $T : \overline{G} \to X^*$ be bounded and continuous. If F, S are bounded and of class $(S_+)_T$, then an affine homotopy $H : [0, 1] \times \overline{G} \to 2^X$ defined by*

$$H(t, u) := (1 - t)Fu + tSu, \quad \text{for } (t, u) \in [0, 1] \times \overline{G},$$

is of class $(S_+)_T$.

Now, we introduce the topological degree for the class $\mathcal{F}(X)$ (for more details see [16]).

Theorem 1. *There exists a unique degree function*

$$d : \big\{(F,\ G,\ g) | G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ g \notin F(\partial G)\big\} \longrightarrow \mathbb{Z}$$

that satisfies the following properties:

1. *(*Normalization*) For any $g \in G$, we have $d(I,\ G,\ g) = 1$.*

2. *(*Additivity*) Let $F \in \mathcal{F}_T(\overline{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $g \notin F(\overline{G} \backslash (G_1 \cup G_2))$, then we have*

$$d(F,\ G,\ g) = d(F,\ G_1,\ g) + d(F,\ G_2,\ g).$$

3. *(*Homotopy invariance*) If $H : [0,1] \times \overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and g: $[0,1] \to X$ is a continuous path in X such that $g(t) \notin H(t,\ \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,\cdot),\ G,\ g(t))$ is constant for all $t \in [0,1]$.*

4. *(*Existence*) if $d(F,\ G,\ g) \neq 0$, then the equation $g \in Fu$ has a solution in G.*

2.2. The weighted Sobolev space

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, p be a real number such that $1 < p < \infty$ and $\omega = \{\omega_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $\omega_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose for any $0 \leq i \leq N$ in all our considerations that

$$w_i \in L^1_{\text{loc}}(\Omega), \tag{3}$$
$$w_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega). \tag{4}$$

The weighted Sobolev space, denoted by $W^{1,p}(\Omega, w)$, is defined as follows:

$$W^{1,p}(\Omega,\ w) = \left\{\ u \in L^p(\Omega, w_0) \quad \text{and} \quad \partial_i u \in L^p(\Omega, w_i), \quad i = 1, ..., N \right\},$$

Note that the derivatives $\partial_i u := \dfrac{\partial u}{\partial x_i}$ are understood in the sense of distributions. This is a Banach space under the norm

$$\|u\|_{1,p,w} = \left[\int_\Omega |u(x)|^p w_0(x)\, dx + \sum_{i=1}^N \int_\Omega |\partial_i u(x)|^p\, w_i(x)\, dx\right]^{1/p}. \tag{5}$$

The condition (3) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega, w)$ and consequently, we can introduce the subspace

$$X = W_0^{1,p}(\Omega, w)$$

of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (5). Moreover, condition (4) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.
We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0,\dots,N\}$ and where p' is the conjugate of p, i.e. $p' = \frac{p}{p-1}$ (for more details we refer to [1, 3, 4]).

Let us define the norm on X equivalent to the norm (5) by

$$|||u|||_X = \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u(x)|^p\, w_i(x)\, dx\Big)^{1/p}. \tag{6}$$

We can find a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^* := \sigma^{1-q'} \in L^1(\Omega) \quad \text{and} \quad \sigma^{-2/(p-2)} \in L^1(\Omega) \tag{7}$$

with $q' = \frac{q}{q-1}$, such that the Hardy inequality

$$\Big(\int_\Omega |u(x)|^q \sigma\, dx\Big)^{1/q} \le c\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u(x)|^p w_i(x)\, dx\Big)^{1/p} \tag{8}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, otherwise the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega,\sigma), \tag{9}$$

expressed by the inequality (8) is compact. Note that $(X, |||\cdot|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 2. Suppose that $w_0(x) \equiv 1$ and the integrability condition holds, i.e. there exists $\nu \in]\frac{N}{p}, +\infty[\cap]\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega), \ \text{ for all } \quad i = 1.....N. \tag{10}$$

Note that the assumptions (10) is stronger than (4). *Then,*

$$|||u||| = \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p\, w_i(x)\, dx\Big)^{1/p} \tag{11}$$

is a norm defined on $W_0^{1,p}(\Omega,w)$ and it is equivalent to (5) and the imbedding

$$W_0^{1,p}(\Omega,w) \hookrightarrow L^q(\Omega), \tag{12}$$

is compact for all $1 \le q \le p_1^*$ if $p\nu < N(\nu+1)$ and for all $q \ge 1$ if $p\nu \ge N(\nu+1)$ where $p_1 = p\nu/\nu + 1$ *and* p_1^* *is the Sobolev conjugate of* p_1 *(see [11, pp.30-31]).*

3. Properties of p-Laplace operator and Technical Lemmas

In this section, we give the definition of a weak solution for problem (1), and we discuss the p-Laplace operator $\triangle_p u := div(|\nabla u|^{p-2}\nabla u)$.
Let us consider the following functional:

$$Lu = \int_\Omega \frac{1}{p}|\nabla u|^p dx, \quad u \in X := W_0^{1,p}(\Omega, w).$$

In view of [8], we have $K \in C^1(X, \mathbb{R})$ and the p-Laplace operator is the derivative operator of L in the weak sense. We denote $K = L' : X \to X^*$. Then

$$\langle Ku, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx \qquad \forall v, \ u \in X.$$

Lemma 3. *i) $K : X \to X^*$ is a continuous, bounded and strictly monotone operator;*

ii) K is a mapping of type (S_+);

iii) $K : X \to X^*$ is a homeomorphism.

Proof. i) It is obvious that K is continuous and bounded. For all $\xi, \eta \in \mathbb{R}^N$, we obtain the following inequality (see [15]) from which we can get the strict monotonicity of L:

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq (\frac{1}{2})^p|\xi - \eta|^p, \qquad p \geq 2. \tag{13}$$

ii) From (i), if $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Ku_n - Ku \, , \, u_n - u\rangle \leq 0$, then

$$\lim_{n\to\infty}\langle Ku_n \, , \, u_n - u\rangle = \lim_{n\to\infty}\langle Ku_n - Ku \, , \, u_n - u\rangle = 0.$$

In view of (13), ∇u_n converges in measure to ∇u in Ω, so we get a subsequence denoted again by ∇u_n satisfying $\nabla u_n(x) \to \nabla u(x)$, a.e. $x \in \Omega$.
Since $u_n \rightharpoonup u$ in $X = W_0^{1,p}(\Omega, w)$, then $(u_n)_n$ is bounded. Therefore the sequence $(|\nabla u_n|^{p-2}\nabla u_n)_n$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, \ w_i^*)$ and $|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u$ a.e. in Ω. According to Lemma 2.1 in [3] we have

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in} \quad \prod_{i=1}^N L^{p'}(\Omega, \ w_i^*) \quad \text{and a.e. in } \Omega.$$

We set $\overline{y}_n = |\nabla u_n|^p$ and $\overline{y} = |\nabla u|^p$. As in [12, Lemma 5], we can write

$$\overline{y}_n \to \overline{y} \quad \text{in } \ L^1(\Omega).$$

By (2) we have

$$\gamma \sum_{i=1}^{N} w_i |\partial_i u_n|^p \le |\nabla u_n|^p.$$

Let $z_n = \displaystyle\sum_{i=1}^{N} w_i |\partial_i u_n|^p, \ z = \sum_{i=1}^{N} w_i |\partial_i u|^p, y_n = \frac{\overline{y}_n}{\gamma}$ and $y = \dfrac{\overline{y}}{\gamma}$. Then, by Fatou's theorem we obtain

$$\int_\Omega 2y dx \le \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z| dx,$$

i.e. $0 \le -\displaystyle\limsup_{n\to\infty} \int_\Omega |z_n - z| dx$. Then

$$0 \le \liminf_{n\to\infty} \int_\Omega |z_n - z| dx \le \limsup_{n\to\infty} \int_\Omega |z_n - z| dx \le 0,$$

this implies,

$$\nabla u_n \to \nabla u \quad \text{in} \quad \prod_{i=1}^{N} L^p(\Omega, \ w_i).$$

Hence $u_n \to u$ in $W_0^{1,p}(\Omega, w)$, i.e. K is of type (S_+).

iii) By the strict monotonicity, K is an injection. Since

$$\lim_{|||u|||\to\infty} \frac{\langle K\, u, \, u\rangle}{|||u|||} = \lim_{|||u|||\to\infty} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{|||u|||} = \infty,$$

K is coercive, thus K is a surjection in view of Minty−Browder theorem (see [28, Theorem 26A]) Hence K has an inverse mapping $K^{-1} : X^* \to X$. Therefore, the continuity of K^{-1} is sufficient to ensure K to be a homeomorphism.
If $f_n, \ f \in X^*, \ f_n \to f$, let $u_n = K^{-1} f_n, \ u = K^{-1} f$. Then $K\, u_n = f_n, \ K\, u = f$. So $(u_n)_n$ is bounded in X. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$. Since $f_n \to f$, then

$$\lim_{n\to\infty} \langle K u_n - K u_0, \ u_n - u_0\rangle = \lim_{n\to\infty} \langle f_n, u_n - u_0\rangle = 0. \tag{14}$$

Since K is of type (S_+), $u_n \to u_0$, we conclude that $u_n \to u$, so K^{-1} is continuous.
◄

Proposition 1. *([7], Proposition 1) For each fixed $x \in \Omega$, the functions $\overline{\psi}(x,s)$ and $\underline{\psi}(x,s)$ are u.s.c functions on $\mathbb{R}^N$.*

Lemma 4. *Let $W_0^{1,p}(\Omega,\omega)$ be the Sobolev space. Then the following statements hold:*

(a) The operator $A : W_0^{1,p}(\Omega,\omega) \to W^{-1,p'}(\Omega,\omega^*)$ defined by

$$\langle Au, v\rangle = -\int_\Omega uv dx \quad \textit{for} \quad u, v \in W_0^{1,p}(\Omega,\omega)$$

is compact.

(b) Under assumptions (H_1) and (H_2), the set-valued operator $N : W_0^{1,p}(\Omega,\omega) \to 2^{W^{-1,p'}(\Omega,\omega^*)}$ defined by

$$Nu = \Big\{ z \in W^{-1,p'}(\Omega,\omega^*) \mid \exists\, h \in L^{p'}(\Omega,\sigma^*); \\ \underline{\psi}(x,u(x)) \le h(x) \le \overline{\psi}(x,u(x)) \ \ \textit{a.e.} \ \ x\in\Omega \\ \textit{and} \ \ \langle z, v\rangle = \int_\Omega hv dx, \quad \forall v \in W_0^{1,p}(\Omega,\omega) \Big\}$$

is bounded, u.s.c and compact.

Proof. (a) Since $p \ge 2$, we have $p' \le 2 \le p$. Then the embedding $i : L^p(\Omega,\sigma) \to L^{p'}(\Omega,\sigma^*)$ is continuous.
Since the embedding $I : W_0^{1,p}(\Omega,\omega) \to L^p(\Omega,\sigma)$ is compact, it is known that the adjoint operator $I^* : L^{p'}(\Omega,\sigma^*) \to W^{-1,p'}(\Omega,\omega^*)$ is also compact. Therefore, $A = I^*\, o\, i\, o\, I$ is compact.

(b) Let $\phi : L^p(\Omega,\sigma) \to 2^{L^{p'}(\Omega,\sigma^*)}$ be the set-valued operator given by

$$\phi u = \Big\{ h \in L^{p'}(\Omega,\sigma^*)|;\ \underline{\psi}(x,u(x)) \le h(x) \le \overline{\psi}(x,u(x)) \quad \text{a.e.} \quad x\in\Omega \Big\}.$$

For each $u \in W_0^{1,p}(\Omega,\sigma)$, by using the growth condition (H_2) we obtain

$$\max\{\ |\underline{\psi}(x,s)|\,;\ |\overline{\psi}(x,s)|\ \} \le b(x) + c\,\sigma(x)\,|s|^{p/p'}.$$

According to the Hardy inequality, it follows that

$$\begin{aligned}\|\overline{\psi}(x,u(x))\|^{p'}_{p',\sigma^*} &\le \int_\Omega |\overline{\psi}(x,u(x))|^{p'}\sigma^* dx\\ &\le 2^{p'}\Big(\int_\Omega |b(x)|^{p'}\sigma^*(x)dx + c^{p'}\int_\Omega |u(x)|^p\sigma^{p'}(x)\sigma^*(x)dx\Big)\\ &\le 2^{p'}\Big(\int_\Omega |b(x)|^{p'}\sigma^*(x)dx + c^{p'}\int_\Omega |u(x)|^p\sigma(x)dx\Big)\\ &\le 2^{p'}\Big(\int_\Omega |b(x)|^{p'}\sigma^*(x)dx + C'\sum_{i=1}^N\int_\Omega |\partial_i u|^p w_i(x)dx\Big).\end{aligned}$$

A similar inequality holds for $\underline{\psi}(x,s)$, so ϕ is bounded on $W_0^{1,p}(\Omega,w)$.
Now we are going to show that ϕ is u.s.c of ϕ, i.e.,

$$\forall\varepsilon>0,\ \exists\delta>0,\quad \|u-u_0\|_{p,\sigma}<\delta \ \Rightarrow\ \phi u\subset \phi u_0+B_\varepsilon,$$

where B_ε is the ε-ball in $L^{p'}(\Omega,\sigma^*)$.
To this end, given $u_0\in L^p(\Omega,\sigma)$, we consider the point sets

$$E_{m,\varepsilon}=\bigcap_{t\in\mathbb{R}} G_t,$$

with

$$\begin{aligned}G_t=\Big\{x\in\Omega;\ |t-u_0(x)|<\frac{1}{m}\ \Rightarrow\ &[\underline{\psi}(x,t),\overline{\psi}(x,t)]\\ &\subset \Big]\underline{\psi}(x,u_0(x)-\frac{\varepsilon}{R},\overline{\psi}(x,u_0(x))+\frac{\varepsilon}{R}\Big[\ \Big\},\end{aligned}$$

m being an integer and R being a constant to be determined.
It is obvious that

$$E_{1,\varepsilon}\subset E_{2,\varepsilon}\subset\cdots$$

By virtue of Proposition 1,

$$\bigcup_{m=1}^{\infty} E_{m,\varepsilon}=\Omega,$$

thus there is an integer m_0 such that

$$m(E_{m_0,\varepsilon})>m(\Omega)-\frac{\varepsilon}{R}. \tag{15}$$

But for all $\varepsilon>0$, there is $\eta=\eta(\varepsilon)>0$, such that $m(T)<\eta$ implies

$$2^{p'}\int_T 2|b(x)|^{p'}\sigma^*(x)+c'(2^p+1)|u_0(x)|^p\sigma(x)dx<\Big(\frac{\varepsilon}{3}\Big)^{p'}, \tag{16}$$

due to $b \in L^{p'}(\Omega, \sigma^*)$ and $u_0 \in L^p(\Omega, \sigma)$.
Now, let

$$0 < \delta < \min\Big\{\frac{1}{m_0}\Big(\frac{\eta}{2}\Big)^{1/p}, \frac{1}{2}\Big(\frac{\varepsilon}{6C}\Big)^{p'/p}\Big\}, \tag{17}$$

$$R > \max\Big\{\frac{2\varepsilon}{\eta}, 3\, C_\sigma^{1/p'}\Big\}. \tag{18}$$

Suppose that $\|u - u_0\|_{p,\sigma} < \delta$ and consider the set $E = \{x \in \Omega \setminus |u(x) - u_0(x)| \geq \frac{1}{m_0}\}$. We have

$$m(E) < (m_0\delta)^p < \frac{\eta}{2} \tag{19}$$

If $x \in E_{m_0,\varepsilon} \backslash E$, then, for each $h \in \phi u$,

$$|u(x) - u_0(x)| < \frac{1}{m_0}$$

and

$$h(x) \in \Big]\underline{\psi}(x, u_0(x)) - \frac{\varepsilon}{R}, \overline{\psi}(x, u_0(x)) + \frac{\varepsilon}{R}\Big[.$$

Let

$$\begin{aligned} G^+ &= \Big\{x \in \Omega; \quad h(x) > \overline{\psi}(x, u_0(x))\Big\}, \\ G^- &= \Big\{x \in \Omega; \quad h(x) < \underline{\psi}(x, u_0(x))\Big\}, \\ G^0 &= \Big\{x \in \Omega; \quad h(x) \in \big[\underline{\psi}(x, u_0(x)), \overline{\psi}(x, u_0(x))\big]\Big\} \end{aligned}$$

and

$$y(x) = \begin{cases} \overline{\psi}(x, u_0(x)), & \text{for} \quad x \in G^+; \\ h(x)\ , & \text{for} \quad x \in G^0; \\ \underline{\psi}(x, u_0(x)), & \text{for} \quad x \in G^-. \end{cases}$$

Then $y \in \phi u_0$ and

$$|y(x) - w(x)| < \frac{\varepsilon}{R} \quad \text{for all} \quad x \in E_{m_0,\varepsilon} \backslash E. \tag{20}$$

Thanks to (18) and (20), we have

$$\int_{E_{m_0,\varepsilon} \backslash E} |y(x) - h(x)|^{p'} \sigma^*(x) dx < \Big(\frac{\varepsilon}{R}\Big)^{p'} C_\sigma < \Big(\frac{\varepsilon}{3}\Big)^{p'}. \tag{21}$$

Let V be the coset in Ω of $E_{m_0,\varepsilon} \backslash E$. Then $V = (\Omega \backslash E_{m_0,\varepsilon}) \cup (E_{m_0,\varepsilon} \cap E)$ and

$$m(V) \leq m(\Omega \backslash E_{m_0,\varepsilon}) + m(E_{m_0,\varepsilon} \cap E) < \frac{\varepsilon}{R} + m(E) < \eta,$$

in view of (15), (18) and (19) . Combining (H_2) and (16) with (17), we have

$$\begin{aligned}
\int_V |y(x)-h(x)|^{p'}\sigma^*(x)dx &\le \int_V |y(x)|^{p'}\sigma^*(x)+|h(x)|^{p'}\sigma^*(x)dx \\
&\le 2^{p'}\Big(\int_V |b(x)|^{p'}\sigma^*(x)+c^{p'}|u_0(x)|^p\sigma^{p'}(x)\sigma^*(x) \\
&\quad +|b(x)|^{p'}\sigma^*(x)+c^{p'}|u(x)|^p\sigma^{p'}(x)\sigma^*(x)dx\Big) \\
&\le 2^{p'}\Big(\int_V 2|b(x)|^{p'}\sigma^*(x)+c^{p'}(2^p+1)|u_0(x)|^p\sigma(x) \\
&\quad +2^p\,c^{p'}|u(x)-u_0(x)|^p\sigma(x)dx\Big) \qquad (22)\\
&\le 2^{p'}\int_V 2|b(x)|^{p'}\sigma^*(x)+c^{p'}(2^p+1)|u_0(x)|^p\sigma(x)dx \\
&\quad +2^{p+p'}\,c^{p'}\int_V |u(x)-u_0(x)|^p\sigma(x)dx \\
&\le \Big(\frac{\varepsilon}{3}\Big)^{p'}+2^{p+p'}\,c^{p'}\delta^p \le 2\Big(\frac{\varepsilon}{3}\Big)^{p'} \le \varepsilon^{p'}.
\end{aligned}$$

Combining (21) with (22), we see that $\|y-h\|_{p',\sigma^*}<\varepsilon$.
Therefore ϕ is u.s.c.
Hence $I^*o\phi oI$ is obviously bounded, u.s.c and compact. ◄

4. Notions of solutions and existence results

In this section, we give the definition of a weak solution for problem (1) and we present existence results for the strongly nonlinear problem (1) based on the degree theory in Section 2.

Definition 5. *We say that $u\in W_0^{1,p}(\Omega,w)$ is a weak solution of the problem* (1)*, if there exists a point $z\in Nu$ such that*

$$\int_\Omega |\nabla|^{p-2}\nabla u\cdot\nabla v dx+\int_\Omega uvdx+\langle z,v\rangle=0 \quad \textit{for all} \quad v\in W_0^{1,p}(\Omega,w).$$

Theorem 2. *Assume* (2), (7), (H_1) *and* (H_2). *There exists a weak solution u in $W_0^{1,p}(\Omega,w)$ of the elliptic problem related to the $p-$Laplacian* (1).

Proof. Let $K,A : W_0^{1,p}(\Omega,w)\to W^{-1,p'}(\Omega,w^*)$ and N: $W_0^{1,p}(\Omega,w)\to 2^{W^{-1,p'}(\Omega,w^*)}$ be defined in Lemmas 3 and 4, respectively. Then $u\in W_0^{1,p}(\Omega,w)$ is a weak solution of (1) if and only if

$$Lu\in -Su, \qquad (23)$$

where $\quad S := A + N : W_0^{1,p}(\Omega, w) \to 2^{W^{-1,p'}(\Omega,w^*)}$.
Thanks to Lemma 4, the operator S is bounded, u.s.c and quasimonotone. Furthermore, in light of the properties of the operator K given in Lemma 3 and by using the Minty-Browder Theorem (see [28, Theorem 26 A]), the inverse operator $T := K^{-1} : W^{-1,p'}(\Omega, w^*) \to W_0^{1,p}(\Omega, w)$ is bounded, continuous and satisfies condition (S_+).
Consequently, equation (23) is equivalent to

$$u = Tv \quad \text{and} \quad v \in -STv. \tag{24}$$

To solve equations (24), we will apply the degree theory introduced in Section 2. To do this, we first prove that the set

$$B := \Big\{ v \in W^{-1,p'}(\Omega, w^*) \mid v \in -tSoTv \quad \text{for some} \quad t \in [0,1] \Big\}$$

is bounded. Indeed, let $v \in B$, that is, $v + ta = 0$ for some $t \in [0,1]$, where $a \in STv$. Setting $u := Tv$, we write $a = Au + z \in Su$, where $z \in Nu$, that is,

$$\langle z, u\rangle = \int_\Omega h(x)u(x)dx,$$

for some $h \in L^{p'}(\Omega, \sigma^*)$ with $\underline{\psi}(x, u(x)) \leq h(x) \leq \overline{\psi}(x, u(x))$ for almost all $x \in \Omega$. According to (2), (7), (8) and (H_2), the Hölder inequality, the Young inequality, we have

$$\begin{aligned}
|||Tv|||^p = |||u|||^p &= \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i\, dx \\
&\leq \frac{1}{\varrho}\sum_{i=1}^N \int_\Omega |\partial_i u|^p dx = \frac{1}{\varrho}\langle Ku, u\rangle = \frac{1}{\varrho}\langle v, Tv\rangle \\
&\leq \frac{t}{\varrho}|\langle a, Tv\rangle| = \frac{t}{\varrho}\Big|\int_\Omega (u+h)u\, dx\Big| \\
&\leq \frac{t}{\varrho}\int_\Omega |u|^2 dx + \frac{t}{\varrho}\int_\Omega |h\, u|\sigma^{\frac{1-p'}{p'}}\sigma^{\frac{p'-1}{p'}} dx \\
&\leq C_1 \int_\Omega |u|^2 \sigma^{\frac{2}{p}}\sigma^{-\frac{2}{p}} dx + C_2\Big(\int_\Omega |h|^{p'}\sigma^{1-p'} dx\Big)^{\frac{1}{p'}} + C_3\Big(\int_\Omega |u|^p \sigma dx\Big)^{\frac{1}{p}} \\
&\leq C_1\Big(\int_\Omega |u|^p\sigma dx\Big)^{\frac{2}{p}}\Big(\int_\Omega \sigma^{\frac{-2}{p-2}} dx\Big)^{\frac{p-2}{p}} + C_2\Big(\int_\Omega |b(x)|^{p'}\sigma^*(x) + c^{p'}|u|^p\sigma dx\Big)^{\frac{1}{p'}} \\
&\quad + C_3\Big(\int_\Omega |u|^p\sigma dx\Big)^{\frac{1}{p}}
\end{aligned}$$

$$\leq C'\Big(\int_\Omega |u|^p\sigma dx\Big)^{\frac{2}{p}} + C_2\Big(\int_\Omega |b(x)|^{p'}\sigma^*(x)dx\Big)^{\frac{1}{p'}} + C''\Big(\int_\Omega |u|^p\sigma dx\Big)^{\frac{1}{p'}}$$
$$+ C_3\Big(\int_\Omega |u|^p\sigma dx\Big)^{\frac{1}{p}}$$
$$\leq \mathrm{Const}\Big(\Big(\sum_{i=1}^N\int_\Omega |\partial_i u|^p w_i dx\Big)^{\frac{2}{p}} + \Big(\sum_{i=1}^N\int_\Omega |\partial_i u|^p w_i dx\Big)^{\frac{1}{p'}} + \Big(\sum_{i=1}^N\int_\Omega |\partial_i u|^p w_i dx\Big)^{\frac{1}{p}}\Big)$$
$$\leq \mathrm{Const}\Big(|||Tv|||^2 + |||Tv|||^{p/p'} + |||Tv|||\Big).$$

As a result, $\{Tv \mid v \in B\}$ is bounded.
Since the operator S is bounded, it is obvious from (24) that the set B is bounded in $W^{-1,p'}(\Omega, w^*)$. Consequently, we can now choose a positive constant R such that

$$\|v\|_{W^{-1,p'}(\Omega,w^*)} < R \quad \text{for all} \quad v \in B.$$

It follows that

$$v \in -tSTv \quad \text{for all} \quad v \in \partial B_R(0) \quad \text{and all} \quad t \in [0,1].$$

From Lemma 1 we have

$$I + ST \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = KT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Consider an affine homotopy $H : [0,1] \times \overline{B_R(0)} \to 2^{W^{-1,p'}(\Omega,w^*)}$ given by

$$H(t,v) := (1-t)Iv + t(I+ST)v \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

From the homotopy invariance and normalization property of the degree d stated in Theorem 1, we obtain

$$d(I + SoT,\ B_R(0),\ 0) = d(I,\ B_R(0),\ 0) = 1.$$

Hence there exists a point $v \in B_R(0)$ such that

$$v \in -STv.$$

Finally, we conclude that $u = Tv$ is a weak solution of (1). This completes the proof. ◄

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Received 11 April 2020
Accepted 01 January 2021