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The Logarithm of the Modulus of an Entire Function as a Minorant for a Subharmonic Function outside a Small Exceptional Set

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Abstract. Let $u \not\equiv -\infty$ be a subharmonic function on the complex plane $\mathbb{C}$. In 2016, we obtained a result on the existence of an entire function $f \neq 0$ satisfying the estimate $\log|f| \leq \mathsf{B}_u$ on $\mathbb{C}$, where the functions B_u are integral averages of u for rapidly shrinking disks as it approaches infinity. We give another equivalent version of this result with $\log|f| \leq u$ outside a very small exceptional set if u is of finite order.

Key Words and Phrases: subharmonic function, entire function, exceptional set, Riesz measure, integral average, covering of sets, type and order of function.

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1. Introduction

1.1. Definitions and notations. Preliminary result

We consider the set $\mathbb{R}$ of *real numbers* mainly as the *real axis* in the *complex plane* $\mathbb{C}$, and $\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geq 0\}$ is the *positive semiaxis* in $\mathbb{C}$. Besides, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is *the extended real line* with the natural order $-\infty \leq x \leq +\infty$ for every $x \in \overline{\mathbb{R}}$, $\overline{\mathbb{R}}^+ := \mathbb{R}^+ \cup \{+\infty\}$, $x^+ := \sup\{x, 0\}$ for each $x \in \overline{\mathbb{R}}$. For an *extended real function* $f \colon S \to \overline{\mathbb{R}}$, its *positive part* is the function $f^+ \colon s \underset{s \in S}{\longmapsto} \bigl(f(s)\bigr)^+$.

For $z \in \mathbb{C}$ and $r \in \mathbb{R}^+$, we denote by $D(z,r) := \{z' \in \mathbb{C} \colon |z'-z| < r\}$ *the open disk centered at* z *and of radius* r, where $D(z,0)$ is the empty set $\varnothing$, $D(r) := D(0,r)$, $\overline{D}(z,r) := \{z' \in \mathbb{C} \colon |z'-z| \leq r\}$ *is a closed disk centered at* z *and of radius* r, $\overline{D}(r) := \overline{D}(0,r)$, and the circle $\partial\overline{D}(z,r) := \overline{D}(z,r) \setminus D(z,r)$

centered at z and of radius r, $\partial\overline{D}(r) := \partial\overline{D}(0,r)$. For a function $v\colon \overline{D}(z,r) \to \overline{\mathbb{R}}$, we define the *integral averages* on circles and disks as

$$\mathsf{C}_v(z,r) := \frac{1}{2\pi}\int_0^{2\pi} v(z+re^{i\theta})\,\mathrm{d}\theta, \qquad \mathsf{C}_v^{\mathrm{rad}}(r) := \mathsf{C}_v(0,r), \tag{1C}$$

$$\mathsf{B}_v(z,r) := \frac{2}{r^2}\int_0^{r} \mathsf{C}_v(z,t)t\,\mathrm{d}t, \qquad \mathsf{B}_v^{\mathrm{rad}}(r) := \mathsf{B}_v(0,r); \tag{1B}$$

$$\mathsf{M}_v(z,r) := \sup_{z'\in\partial\overline{D}(z,r)} v(z'), \qquad \mathsf{M}_v^{\mathrm{rad}}(r) := \mathsf{M}_v(0,r), \tag{1M}$$

where $\mathsf{M}_v(z,r) = \sup\limits_{z'\in\overline{D}(z,r)} v(z')$ if v is subharmonic on $\mathbb{C}$ [1, Definition 2.6.7], [2].

The following result [3, Corollary 2] of 2016 found several useful applications [4, Lemma 5.1], [5], [6, Proposition 2], [7], [8, Lemma 6.3], [9], [10, 7.1] for entire functions on the complex plane:

Theorem 1 ([3, Corollary 2], see also [4, Lemma 5.1])**.** *Let $u \not\equiv -\infty$ be a subharmonic function on $\mathbb{C}$, and $q \in \mathbb{R}^+$ be a number with the corresponding function*

$$Q\colon z \underset{z\,\in\,\mathbb{C}}{\longmapsto} \frac{1}{(1+|z|)^q} \leq 1. \tag{2}$$

Then there is an entire function $f_q \not\equiv 0$ on $\mathbb{C}$ such that

$$\log\bigl|f_q(z)\bigr| \leq \mathsf{B}_u\bigl(z,Q(z)\bigr) \leq \mathsf{C}_u\bigl(z,Q(z)\bigr) \leq \mathsf{M}_u\bigl(z,Q(z)\bigr) \quad \textit{for each } z\in\mathbb{C}. \tag{3}$$

In this article, we obtain another equivalent version of Theorem 1 for subharmonic functions of finite order. This version may be useful in situations that we are not discussing here.

1.2. Main result for minorants outside an exceptional set

For an extended real function $m\colon \mathbb{R}^+ \to \overline{\mathbb{R}}$, we define [11], [12], [8, 2.1, (2.1t)]

$$\mathsf{ord}[m] := \limsup_{r\to+\infty} \frac{\log\bigl(1+m^+(r)\bigr)}{\log r} \in \overline{\mathbb{R}}^+, \tag{4}$$

the *order of growth of m*; for $p \in \mathbb{R}^+$,

$$\mathsf{type}_p[m] := \limsup_{r\to+\infty} \frac{m^+(r)}{r^p} \in \overline{\mathbb{R}}^+, \tag{5}$$

the *type of growth of m at the order p*. Thus, it is easy to see that

$$\mathsf{order}[m] = \inf\bigl\{p\in\mathbb{R}^+\colon \mathsf{type}_p[m] < +\infty\bigr\}, \quad \inf\varnothing := +\infty. \tag{6}$$

If u is a subharmonic function on $\mathbb{C}$, then

$$\mathsf{order}[u] \overset{(1\mathrm{M})}{:=} \mathsf{order}[\mathsf{M}_u^{\mathrm{rad}}], \quad \mathsf{type}_p[u] \overset{(1\mathrm{M})}{:=} \mathsf{type}_p[\mathsf{M}_u^{\mathrm{rad}}], \tag{7}$$

and, under the condition $\mathsf{type}_p[u] < +\infty$, the following 2π-periodic function

$$\mathsf{ind}_p[u](s) := \limsup_{r\to+\infty} \frac{u(re^{is})}{r^p} \in \mathbb{R}, \quad s \in \mathbb{R}, \tag{8}$$

is called the *indicator of the growth of u at the order p* [12, 3.2].

For a ray or a circle on $\mathbb{C}$, we denote by mes the *linear Lebesgue measure* on this ray or the *measure of length* on this circle.

Theorem 2. *Let $u \not\equiv -\infty$ be a subharmonic function on the complex plane, and $\mathsf{order}[\mathsf{B}_u^{\mathrm{rad}}] \overset{(1\mathrm{B})}{<} +\infty$. Then the conclusion* (3) *of Theorem* 1 *with arbitrary positive numbers $q \in \mathbb{R}^+$ is equivalent to the following statement:*

For any positive $q \in \mathbb{R}^+$, there are an entire function $f_q \not\equiv 0$ and a no-more-than countable set of disks $D(z_k, t_k)$, $k = 1, 2, \dots,$ such that

$$\log\bigl|f_q(z)\bigr| \le u(z) \quad \textit{for each } z \in \mathbb{C} \setminus E_q, \textit{ where} \tag{9I}$$

$$E_q := \bigcup_k D(z_k, r_k), \quad \sup_k t_k \le 1, \quad \sum_{|z_k|\ge R} t_k = O\Bigl(\frac{1}{R^q}\Bigr) \textit{ as } R \to +\infty. \tag{9E}$$

If $\mathsf{ord}[u] \overset{(7)}{<} +\infty$, then the statements of Theorem 1 *or the statements* (9) *of this Theorem* 2 *can be supplemented by the following restrictions:*

$$\mathsf{ord}\bigl[\log|f_q|\bigr] \overset{(4),(6),(7)}{\le} \mathsf{ord}[u], \tag{10o}$$

$$\mathsf{type}_p\bigl[\log|f_q|\bigr] \overset{(5),(7)}{\le} \mathsf{type}_p[u] \quad \textit{for each } p \in \mathbb{R}^+, \tag{10t}$$

$$\mathsf{ind}_p\bigl[\log|f_q|\bigr] \overset{(8)}{\le} \mathsf{ind}_p[u] \quad \textit{for each } q \in \mathbb{R}^+. \tag{10i}$$

Besides, for any ray $L \subset \mathbb{C}$, we have

$$\mathsf{mes}\Bigl(L \setminus \bigl(E_q \cup D(R)\bigr)\Bigr) = O\Bigl(\frac{1}{R^q}\Bigr) \quad \textit{as } R \to +\infty, \tag{11}$$

and also

$$\mathsf{mes}\Bigl(E_q \bigcap \partial\overline{D}(R)\Bigr) = O\Bigl(\frac{1}{R^q}\Bigr) \quad \textit{as } R \to +\infty. \tag{12}$$

Theorem 2 will be proved in Sec. 3 after some preparations.

2. Preparatory results

2.1. On exceptional sets

For a Borel measure μ on $\mathbb{C}$, we set

$$\mu(z,t) := \mu\bigl(\overline{D}(z,t)\bigr), \quad z \in \mathbb{C},\ t \in \mathbb{R}^+. \tag{13}$$

For a function $d\colon \mathbb{C} \to \mathbb{R}^+$, $S \subset \mathbb{C}$ and $r\colon \mathbb{C} \to \mathbb{R}$, we define

$$\begin{aligned} S^{\cup d} &:= \bigcup_{z\in S} D\bigl(z, d(z)\bigr) \subset \mathbb{C},\\ r^{\vee d}&\colon z \underset{z\,\in\,\mathbb{C}}{\longmapsto} \sup\Bigl\{r(z')\colon z' \in D\bigl(z, d(z)\bigr)\Bigr\} \in \overline{\mathbb{R}}, \end{aligned}$$

and denote *the indicator function* of the set S by

$$\mathbf{1}_S\colon z \underset{z\,\in\,\mathbb{C}}{\longmapsto} \begin{cases} 1 & \text{if } z \in S,\\ 0 & \text{if } z \notin S. \end{cases}$$

Lemma 1 (cf. [13, Normal Points Lemma], [14, § 4. Normal points, Lemma])**.** *Let $r\colon \mathbb{C} \to \mathbb{R}^+$ be a Borel function such that*

$$d := 2\sup\{r(z)\colon z \in \mathbb{C}\} < +\infty, \tag{14}$$

and μ be a Borel positive measure on $\mathbb{C}$ with

$$E_{\mu,r} := \left\{ z \in \mathbb{C}\colon \int_0^{r(z)} \frac{\mu(z,t)}{t}\,\mathrm{d}t > 1 \right\} \subset \mathbb{C}. \tag{15}$$

Then there exists a no-more-than countable set of disks $D(z_k, t_k)$, $k = 1, 2, \dots,$ such that

$$\begin{aligned} z_k \in E_{\mu,r}, \quad t_k \le r(z_k), \quad E_{\mu,r} \subset \bigcup_k D(z_k, t_k),\\ \sup_{z\in\mathbb{C}} \#\bigl\{k\colon z \in D(z_k, t_k)\bigr\} \le 2020, \end{aligned} \tag{16}$$

i. e., the multiplicity of this covering $\{D(z_k, t_k)\}_{k=1,2,\dots}$ of set $E_{\mu,r}$ is not greater than 2020, *and, for every μ-measurable subset $S \subset \bigcup_k D(z_k, t_k)$,*

$$\frac{1}{2020} \sum_{S\cap D(z_k,t_k)\neq\varnothing} t_k \le \int_{S^{\cup d}} r^{\vee r}\,\mathrm{d}\mu \le \int_{S^{\cup d}} r^{\vee d}\,\mathrm{d}\mu. \tag{17}$$

Proof. By definition (15), there is a number

$$t_z \in \bigl(0, r(z)\bigr) \quad \textit{such that} \quad 0 < t_z < r(z)\mu(z, t_z) \textit{ for each } z \in E_{\mu,r}. \tag{18}$$

Thus, the system $\mathcal{D} = \bigl\{D(z, t_z)\bigr\}_{z\in E}$ of these disks has a property

$$E_{\mu,r} \subset \bigcup_{z\in E} D(z, t_z), \quad 0 < t_z \le r(z) \overset{(14)}{\le} R.$$

By the Besicovitch Covering Theorem [15, 2.8.14], [16], [17], [18, I.1, Remarks], [19], [20] in the Landkof version [21, Lemma 3.2], we can select some no-more-than countable subsystem in $\mathcal{D}$ of disks $D(z_k, t_k) \in \mathcal{D}$, $k = 1, 2, \dots$, $t_k := t_{z_k}$, such that properties (16) are fulfilled. Consider a μ-measurable subset $S \subset \bigcup_k D(z_k, t_k)$. In view of (18) it is easy to see that

$$\bigcup \Bigl\{ D(z_k, t_k) \colon S \cap D(z_k, t_k) \neq \varnothing \Bigr\} \overset{(18),(14)}{\subset} \bigcup_{z\in S} D(z, d) = S^{\cup d}. \tag{19}$$

Hence, in view of (18) and (16), we obtain

$$\begin{aligned}
\sum_{S\cap D(z_k,t_k)\neq\varnothing} t_k &:= \sum_{S\cap D(z_k,t_k)\neq\varnothing} t_{z_k} \overset{(18)}{\le} \sum_{S\cap D(z_k,t_k)\neq\varnothing} r(z_k)\mu(z, t_k) \\
&= \sum_{S\cap D(z_k,t_k)\neq\varnothing} \int_{D(z_k,t_k)} r(z_k)\,\mathrm{d}\mu(z) \overset{(18)}{\le} \sum_{S\cap D(z_k,t_k)\neq\varnothing} \int_{D(z_k,t_k)} r^{\vee r}\,\mathrm{d}\mu \\
&\overset{(19)}{=} \sum_{S\cap D(z_k,t_k)\neq\varnothing} \int_{S^{\cup d}} \mathbf{1}_{D(z_k,t_k)} r^{\vee r}\,\mathrm{d}\mu \\
&= \int_{S^{\cup d}} \Biggl(\sum_{S\cap D(z_k,t_k)\neq\varnothing} \mathbf{1}_{D(z_k,t_k)} \Biggr) r^{\vee r}\,\mathrm{d}\mu \\
&\overset{(16)}{\le} 2020 \int_{S^{\cup d}} r^{\vee r}\,\mathrm{d}\mu \overset{(14)}{\le} 2020 \int_{S^{\cup d}} r^{\vee d}\,\mathrm{d}\mu.
\end{aligned}$$

Thus, we obtain (17). This completes the proof of Lemma 1. ◄

Lemma 2. *Let $\bigl\{D(z_j, t_j)\bigr\}_{j\in J}$ be a system of disks in $\mathbb{C}$, $d := 2\sup_{j\in J} t_j < +\infty$. Then, for each $z \in \mathbb{C}$, there is a positive number $r \le d$ such that*

$$\bigcup_{j\in J} D(z_j, t_j) \bigcap \partial\overline{D}(z, r) = \varnothing. \tag{20}$$

Proof. Consider a disk $\overline{D}(z,d)$, where, without loss of generality, we can assume that $z=0$. Then, by condition $d := 2\sup_{j\in J} t_j < +\infty$, the union

$$\bigcup_{j\in J}\Bigl(D(z_j,t_j)e^{-i\arg z_j}\Bigr)\bigcap[0,d] \tag{21}$$

of radial projections $\Bigl(D(z_j,t_j)e^{-i\arg z_j}\Bigr)\bigcap[0,d]$ of $D(z_j,t_j)$ onto the radius $[0,d]$ is not empty, i.e. there is a point $r\in[0,d]$ outside (21), which gives (20) for $z=0$.

Lemma 2 is proved. ◀

Lemma 2 has the following consequence:

Lemma 3. *Let* $\bigl\{D(z_k,t_k)\bigr\}_{k=1,2,\dots}$ *be a system of disks satisfying* (9E) *with a strictly positive number* $q\in\mathbb{R}^+\backslash\{0\}$*, and* $q'<q$ *be a positive number. Then there exists a number* $R_q\in\mathbb{R}^+$ *such that for any* $z\in\mathbb{C}$ *with* $|z|>R_q$ *there is a positive number* $r\le\bigl(1+|z|\bigr)^{-q'}$ *such that* (20) *holds for* $J=\{1,2,\dots\}$.

Proof. By condition (9E), there is a constant $C\in\mathbb{R}^+$ such that

$$\sum_{D(z_k,t_k)\backslash D(|z|-2)\neq\varnothing} t_k \le \frac{C}{(1+|z|)^q} \quad \text{for each } z\in\mathbb{C} \text{ with } |z|\ge 3, \tag{22}$$

and, for $|z|\ge 3$,

$$\textit{if } D(z_k,t_k)\setminus D(z,2)\neq\varnothing, \textit{ then } D(z_k,t_k)\backslash D(|z|-2)\neq\varnothing. \tag{23}$$

For $0\le q'<q$, we choose $R_q\ge 3$ so that

$$C(1+|z|)^{q'-q}\le\frac{1}{2} \quad \text{for all } |z|\ge R_q\ge 3. \tag{24}$$

It follows from (22)–(24) that

$$\sum_{D(z_k,t_k)\backslash D(z,2)\neq\varnothing} t_k \le \frac{C}{(1+|z|)^q} \overset{(24)}{\le} \frac{1}{2}\frac{1}{(1+|z|)^{q'}} \quad \text{for each } z\in\mathbb{C} \text{ with } |z|\ge R_q,$$

and

$$\sup_{D(z_k,t_k)\backslash D(z,2)\neq\varnothing} t_k \le \frac{1}{2}\frac{1}{(1+|z|)^{q'}} \quad \text{for each } z\in\mathbb{C} \text{ with } |z|\ge R_q. \tag{25}$$

For an *arbitrary fixed point* $z\in\mathbb{C}$ with $|z|\ge R_q$, we consider

$$J := \bigl\{k\colon D(z_k,t_k)\backslash D(z,2)\neq\varnothing\bigr\}, \quad \mathcal{D} := \bigl\{D(z_k,t_k)\bigr\}_{k\in J}.$$

By Lemma 2, with these J and $\mathcal{D}$ there is a circle $\partial\overline{D}(z,r)$ such that

$$0 \le r \overset{(25)}{\le} (1+|z|)^{-q'} \le 1, \quad \bigcup_{k\in J} D(z_k,t_k) \bigcap \partial\overline{D}(z,r) = \varnothing.$$

But, in view of (23), if $k \notin J$, then, as before, $D(z_k,t_k)\bigcap\partial\overline{D}(z,r) = \varnothing$.

Lemma 3 is proved. ◄

2.2. The order and the upper density for measures on $\mathbb{C}$

For a Borel positive measure μ on $\mathbb{C}$, a function

$$\mu^{\mathrm{rad}}\colon r \underset{r\,\in\,\mathbb{R}^+}{\overset{(13)}{\longmapsto}} \mu(0,r) \tag{26}$$

is called the *radial counting function of* μ, the quantity

$$\mathsf{ord}[\mu] \overset{(4),(6)}{:=} \mathsf{ord}\left[\mu^{\mathrm{rad}}\right]$$

is called the *order* of measure μ, and, for $p \in \mathbb{R}^+$, the quantity

$$\mathsf{type}_p[\mu] \overset{(5)}{:=} \mathsf{type}_p\left[\mu^{\mathrm{rad}}\right] \tag{27}$$

is called the *upper density* of measure μ *at the order* p.

If $u \not\equiv -\infty$ is a subharmonic function on $\mathbb{C}$ with the *Riesz measure*

$$\varDelta_u = \frac{1}{2\pi}\triangle u, \tag{28}$$

where the *Laplace operator* $\triangle$ acts in the sense of the theory of distributions or generalized functions [1], [2], then, by the Poisson–Jensen formula [1, 4.5], [2]

$$u(z) = \mathsf{C}_u(z,r) - \int_0^r \frac{\varDelta_u(z,t)}{t}\,\mathrm{d}t, \quad z \in \mathbb{C}, \tag{29}$$

in a disk $D(z,r)$ in the form [4, 3, (3.3)]

$$\mathsf{C}_u(r) - \mathsf{C}_u(1) = \int_1^r \frac{\varDelta_u^{\mathrm{rad}}(t)}{t}\,\mathrm{d}t,$$

and by (1B) together with

Lemma 4 ([22], [23, Theorem 3])**.** *If u is a subharmonic function on $\mathbb{C}$, then $\mathsf{B}(z,t) \le \mathsf{C}(z,t) \le \mathsf{B}(z,\sqrt{e}t)$ for each $z \in \mathbb{C}$ and for each $t \in \mathbb{R}^+$.*

we can easily obtain

Lemma 5. *Let $u \not\equiv -\infty$ be a subharmonic function on $\mathbb{C}$ with Riesz measure* (28). *Then, for each $r \geq 1$,*

$$\mathsf{B}_u(r) - \mathsf{C}_u(1) \leq \mathsf{C}_u(r) - \mathsf{C}_u(1) \leq \int_1^r \frac{\varDelta_u^{\mathrm{rad}}(t)}{t}\,\mathrm{d}t \leq \mathsf{C}_u(r) \leq \mathsf{B}_u(\sqrt{e}r). \tag{30}$$

In particular, we have the equalities

$$\mathsf{ord}[\varDelta_u] = \mathsf{ord}[\mathsf{C}_u] = \mathsf{ord}[\mathsf{B}_u],$$

and the equivalences

$$\bigl[\mathsf{type}_p[\varDelta_u] < +\infty\bigr] \iff \bigl[\mathsf{type}_p[\mathsf{C}_u] < +\infty\bigr] \iff \bigl[\mathsf{type}_p[\mathsf{B}_u] < +\infty\bigr]$$

for each strictly positive $p \in \mathbb{R}^+ \setminus \{0\}$.

3. The proof of Theorem 2

3.1. From Theorem 1 to (9)

Let $q' \in \mathbb{R}^+$. By Lemma 5, we have

$$a_u := \mathsf{ord}[\varDelta_u] \overset{(30)}{=} \mathsf{ord}[\mathsf{C}_u] < +\infty. \tag{31}$$

We choose

$$q := a_u + q' + 3 \geq 3. \tag{32}$$

and an entire function f_q from Theorem 1 with properties (2)–(3). Then, for entire function $e^{-1}f_q \not\equiv 0$, we obtain

$$\begin{aligned} \log\bigl|e^{-1}f_q(z)\bigr| &\leq \mathsf{C}_u\bigl(z, Q(z)\bigr) - 1 \\ &\overset{(29)}{=} u(z) + \int_0^{Q(z)} \frac{\varDelta_u(z,t)}{t}\,\mathrm{d}t - 1 \quad \text{for each } z \in \mathbb{C}\setminus(-\infty)_u, \end{aligned} \tag{33}$$

where $(-\infty)_u := \bigl\{z \in \mathbb{C} \colon u(z) = -\infty\bigr\}$ is a *minus-infinity G_δ polar set* [1, 3.5], and 1-dimensional Hausdorff measure of $(-\infty)_u$ is zero [2, 5.4]. Therefore, this set $(-\infty)_u$ can be covered by a system of disks as in (9E) with q' instead of q. By Lemma 1 with

$$r \overset{(2)}{:=} Q, \quad d \overset{(14)}{\leq} 2, \quad \mu \overset{(28)}{:=} \varDelta_u, \quad E_q \overset{(16)}{:=} \bigcup_k D(z_k, t_k) \overset{(15),(9\mathrm{E})}{\supset} E_{\mu,r}, \tag{34}$$

we have, in view of (33),

$$\log\bigl|e^{-1}f_q(z)\bigr| \overset{(33),(15)}{\leq} u(z), \quad \text{for each } z\in\mathbb{C}\setminus\bigl(E_q\cup(-\infty)_u\bigr). \tag{35}$$

If $S:=E_q\setminus D(R)$ and $R\geq 4$, then, by (17),

$$\begin{aligned}\frac{1}{2020}\sum_{|z_k|\geq R} t_k &\overset{(17)}{\leq} \int_{S^{\cup d}} r^{\vee d}\,\mathrm{d}\varDelta_u \overset{(34)}{\leq} \int_{|z|\geq R-2}\frac{1}{\bigl(1+(|z|-2)\bigr)^q}\,\mathrm{d}\varDelta_u(z)\\ &= \int_{R-2}^{+\infty}\frac{1}{(t-1)^q}\,\mathrm{d}\varDelta_u^{\mathrm{rad}}(t) \overset{(32)}{\leq} \int_{R-2}^{+\infty}\frac{\varDelta_u^{\mathrm{rad}}(t)}{(t-1)^{q-1}}\,\mathrm{d}t\\ &\overset{(6),(31)}{\leq} \text{const}\int_{R-2}^{+\infty}\frac{t^{a_u+1}}{(t-1)^{q-1}}\,\mathrm{d}t \overset{(32)}{=} O(R^{a_u+3-q}) \quad \text{as } R\to+\infty,\end{aligned}$$

where const $\in\mathbb{R}^+$ is independent of R, and $R^{a_u+3-q}\overset{(32)}{=}R^{-q'}$. The latter together with (35) gives the statements (9) of Theorem 2.

3.2. From (9) to Theorem 1

Let $q^*\in\mathbb{R}^+$. Suppose that the statements (9) of Theorem 2 are fulfilled with $q>q'>q^*\geq 0$. By Lemma 3 there exists a number $R_q\in\mathbb{R}^+$ such that for any $z\in\mathbb{C}$ with $|z|>R_q$ there is a positive number $r_z\leq\bigl(1+|z|\bigr)^{-q'}$ such that $E_q\cap\partial\overline{D}(z,r_z)=\varnothing$. Hence, by (9I), we obtain

$$\log\bigl|f_q(z+r_ze^{is})\bigr|\leq u(z+r_ze^{is}) \quad \text{for each } s\in\mathbb{R} \tag{36}$$

and for any $z\in\mathbb{C}$ with $|z|\geq R_q$. Therefore,

$$\log\bigl|f_q(z)\bigr|\leq \mathsf{C}_{\log|f_q|}(r_z)\overset{(36)}{\leq}\mathsf{C}_u(r_z)\leq\mathsf{C}\Bigl(z,\frac{1}{(1+|z|)^{q'}}\Bigr) \quad \text{if } |z|\geq R_q.$$

Hence there exist a sufficiently small number $a>0$ and a sufficiently large number $R_{q^*}\geq R_q$ such that

$$\log\bigl|af_q(z)\bigr|\leq\mathsf{C}\Bigl(z,\frac{1}{\sqrt{e}(1+|z|)^{q^*}}\Bigr) \quad \text{if } |z|\geq R_{q^*}.$$

The function $\log|af_q|$ is bounded from above on $D(R_{q^*})$, and the function

$$\mathsf{C}\Bigl(\cdot,\frac{1}{\sqrt{e}(1+R_{q^*})^{q^*}}\Bigr)\colon z\underset{z\,\in\,\mathbb{C}}{\longmapsto}\mathsf{C}\Bigl(z,\frac{1}{\sqrt{e}(1+R_{q^*})^{q^*}}\Bigr)$$

is continuous [24, Theorem 1.14]. Therefore, there exists a sufficiently small number $b > 0$ such that

$$\log\bigl|abf_q(z)\bigr| \leq \mathsf{C}\Bigl(z, \frac{1}{\sqrt{e}(1+|z|)^{q^*}}\Bigr) \quad \text{for all } z \in \mathbb{C}.$$

Hence, for $f_{q^*} := abf_q \neq 0$, by Lemma 4, we obtain (3) with $q^* \in \mathbb{R}^+$ instead of q in (2). Further, equalities (10o) and (10t) for orders and types are obvious consequences of (3) even for $q = 0$. Similarly, we obtain equality (10i), since indicators (8) of the growth of $\log|f_q|$ and u are continuous. Relations (11)–(12) are obvious particular cases of (9E).

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