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Existence of Three Positive Solutions for Higher Order Separated and Lidstone Type Boundary Value Problems with p-Laplacian

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Abstract. In this paper, we consider a higher order p-Laplacian differential equations,

$$(-1)^n[\phi_p(u^{(2n-2)}(t) + k^2u^{(2n-4)}(t))]'' = f(t, u(t)), \quad 0 \le t \le 1,$$

associated with the boundary conditions

$$u^{(2i)}(0) = 0 = u^{(2i)}(1), \quad 1 \le i \le n-1,$$
$$a_1u(0) - a_2u'(0) = 0 \qquad b_1u(1) + b_2u'(1) = 0,$$

where $n \ge 2$ and $k \in (0, \frac{2}{3}\pi)$ is a constant. By applying Five functional fixed point theorem, we establish sufficient conditions for the existence of triple positive solutions.

Key Words and Phrases: Green's function, p-Laplacian, boundary value problem, positive solutions, cone, five functional fixed point theorem.

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1. Introduction

The p-Laplace equation has been much studied during the last fifty years and its theory is by now rather developed. The p-Laplace equation is a degenerate or singular elliptic equation in divergence form. And defined as

$$-div(|\nabla u|^{p-2}\nabla u) = 0 \quad 1 < p < \infty. \tag{1}$$

When $p = 2$, the equation (1) reduces to the Laplace equation. When $p \neq 0$, the equation (1) is nonlinear and degenerates at the zeros of the gradient of u.

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Consequently, in this case the solution, commonly referred to as p-harmonic functions, need not be smooth, nor even C^2 [13]. The particular case of (1) is the classical one dimensional p-Laplacian operator and is given by $\phi_p(s) = |s|^{p-2}s$, where $p > 1$, $\phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. These type of problems appears in mathematical modeling of viscoelastic flows, image processing, turbulent filtration in porous media, biophysics, plasma physics, rheology, glaciology, radiation of heat, plastic molding etc. Recent advanced research indicated that even the Brownian motion has its counter part and a mathematical game 'tug of war' leads to the case $p = \infty$. For more details on applications and origin of p-Laplacian, we refer to [6, 9]. Due to wide mathematical and physical background, the existence of positive solutions for nonlinear boundary value problems with p-Laplacian operators have received great attention in recent years. To mention a few paper along these lines Wang [25], Lian and Wong [19], Agarwal et al. [2], Li and Ge [16], Liu and Ge [20], Avery and Henderson [3], Li and Shen [18] and for further development in the topic, see [11, 12, 22, 29, 31, 26, 27, 30].

Motivated by the papers mentioned above, in this paper, we establish existence of triple positive solutions for higher order p-Laplacian differential equations of the form

$$(-1)^n[\phi_p(u^{(2n-2)}(t) + k^2u^{(2n-4)}(t))]'' = f(t, u(t)), \quad 0 \le t \le 1, \tag{2}$$

with the separated boundary conditions

$$\begin{aligned} a_1u(0) - a_2u'(0) = 0 \\ b_1u(1) + b_2u'(1) = 0 \end{aligned} \tag{3}$$

and Lidstone type boundary condition

$$u^{(2i)}(0) = 0 = u^{(2i)}(1), \quad i = 1, 2, 3, \dots, n-1, \tag{4}$$

where $n \ge 2, a_1, b_1 \ge 0, a_2, b_2 > 0$, $k \in (0, \frac{2}{3}\pi)$ is a constant, $k^2 \le \frac{a_1b_1}{a_2b_2}$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, by applying five functional fixed point theorem. In the past few decades for $k = 0$ and $p = 2$, a lot of work has been done on the existence of one and multiple positive solutions of the boundary value problems associated with differential equations by using various methods, see [7, 8, 5, 10, 28, 24] and for $k \neq 0$ and $p = 2$, most of the authors focused on the existence of positive solutions of second order differential equations satisfying Neumann and Sturm-Liouville boundary conditions, see [14, 23, 17, 19].

The rest of the paper is organized as follows. In Section 2, we express the solution of the boundary value problem (2)-(4) in to an equivalent integral equation involving Green's functions as a kernel and estimate bounds for these Green's functions. In Section 3, we establish criteria for the existence of triple positive solution for the boundary value problem (2)-(4). Finally as an application, we give an example to illustrate our results.

2. Green's function and bounds

In this section, we express the solution of the boundary value problem (2)-(4) in to an equivalent integral equation involving Green functions and we estimate bounds for these Green functions.

Let $v = \Big((-1)^{n-1}\big(\phi_p(u^{(2n-2)}(t) + k^2u^{(2n-4)}(t))\big)\Big)$. Then we construct the Green's function $k_1(t,s)$ for the homogeneous problem

$$-v''(t) = 0, \quad 0 \leq t \leq 1, \tag{5}$$

$$v(0) = 0 = v(1). \tag{6}$$

Lemma 1. *Let $y(t) \in L^1[0,1]$. Then the problem (5)-(6) has a unique positive solution*

$$v(t) = \int_0^1 k_1(t,s)y(s)ds, \tag{7}$$

where

$$k_1(t,s) = \begin{cases} t(1-s), & t \leq s \\ s(1-t), & s \leq t. \end{cases} \tag{8}$$

Clearly, ϕ_p is an odd function, from Lemma 1

$$\phi_p\Big((-1)^{n-1}\big(u^{(2n-2)}(t) + k^2u^{(2n-4)}(t)\big)\Big) = \int_0^1 k_1(t,s)f(s,u(s))ds.$$

Since $\phi_p^{-1} = \phi_q$, it follows that

$$(-1)^{n-2}\big[-(u''(t) + k^2u(t))^{(2(n-2))}\big] = \phi_q\Big(\int_0^1 k_1(t,s)f(s,u(s))ds\Big).$$

Lemma 2. *Let $\phi_q\Big(\int_0^1 k_1(t,s)f(s,u(s))ds\Big) \in L^1[0,1]$ and $x(t) = -(u''(t) + k^2u(t))$. Then the BVP*

$$(-1)^{n-2}x^{(2n-4)}(t) = 0, \qquad 0 \leq t \leq 1, \tag{9}$$

$$x^{(2i)}(0) = 0 = x^{(2i)}(1), \quad 2 \leq i \leq n-3 \tag{10}$$

has a unique solution

$$x(t) = \int_0^1 k_{n-2}(t,s)\phi_q(\int_0^1 k_1(s,r)f(r,u(r))dr)ds. \tag{11}$$

The Green's function for the homogeneous boundary value problem (9), (10) *as described in [1, 28] is* $k_{n-2}(t,s)$, *where* $k_{n-2}(t,s)$ *is defined recursively as*

$$k_j(t,s) = \int_0^1 k_{j-1}(t,\xi)k_1(\xi,s)d\xi, \text{ for } 2 \le j \le n-2, \tag{12}$$

and $k_1(t,s)$ *was defined in* (8).

Combining the assumptions of Lemma 2 and (11) we get

$$-(u''(t) + k^2u(t)) = \int_0^1 k_{n-2}(t,s)\phi_q(\int_0^1 k_1(s,r)f(r,u(r))dr)ds. \tag{13}$$

Lemma 3. *The boundary value problem* (2)–(4) *has a unique solution*

$$u(t) = \int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau)ds)dr, \tag{14}$$

where $h(t,s)$ *is the Green's function for the corresponding homogeneous BVP* (13) *with the boundary conditions* (3) *and given by*

$$h(t,s) = \begin{cases} \frac{1}{d}(a_1\sin kt + a_2k\cos kt)(b_1\sin k(1-s) + b_2k\cos k(1-s)), & t \le s, \\ \frac{1}{d}(a_1\sin ks + a_2k\cos ks)(b_1\sin k(1-t) + b_2k\cos k(1-t)), & s \le t. \end{cases} \tag{15}$$

where $d = k(a_1b_1 - a_2b_2k^2)\sin k + k^2(a_1b_2 + a_2b_1)\cos k \neq 0$.

Equation (14) is the equivalent integral equation for the boundary value problem (2)–(4).

Lemma 4. *The Green's function* $h(t,s)$ *satisfies the following inequalities:*

(i) $h(t,s) > 0$, *for all* $t,s \in (0,1)$,

(ii) $h(t,s) \le Mh(s,s)$, *for all* $(t,s) \in [0,1]\times[0,1]$,

(iii) $\frac{1}{M}h(s,s) \le h(t,s)$, *for all* $(t,s) \in [0,1]\times[0,1]$,

where $M = max\left\{\frac{a_1+a_2k}{a_2k\cos k}, \frac{b_1+b_2k}{b_2k\cos k}\right\}$.

Proof. For the proof we refer to [21]. ◄

Lemma 5. *[28] The Green's function* $k_1(t,s)$ *in* (8) *satisfies the following inequalities:*

(i) $k_1(t,s) \geq 0$, *for all* $t, s \in [0,1]$,

(ii) $k_1(t,s) \leq k_1(s,s)$, *for all* $t, s \in [0,1]$,

(iii) $k_1(t,s) \geq \frac{1}{4} k_1(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Lemma 6. *[28] The Green's function* $k_{n-2}(t,s)$ *in* (12) *satisfies the following inequalities:*

(i) $k_{n-2}(t,s) \geq 0$, *for all* $t, s \in [0,1]$,

(ii) $k_{n-2}(t,s) \leq \frac{1}{6^{n-3}} k_1(s,s)$, *for all* $t, s \in [0,1]$,

(iii) $k_{n-2}(t,s) \geq \frac{1}{4^{n-2}} \left(\frac{11}{96}\right)^{n-3} k_1(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

For the reader's convenience, we present some necessary definitions and theorems we may use throughout the entire paper.

Definition 1. *Let* X *be a Banach space over* $\mathbb{R}$. *A nonempty, closed set* $P \subset X$ *is a* cone, *provided*

(a) $a_1\mathbf{u} + a_2\mathbf{v} \in P$ *for all* $\mathbf{u}, \mathbf{v} \in P$ *and all* $a_1, a_2 \geq 0$, *and*

(b) $\mathbf{u}, -\mathbf{u} \in P$ *implies* $\mathbf{u} = \mathbf{0}$.

Definition 2. *The map* α *is said to be nonnegative continuous concave functional on* P *provided that* $\alpha : P \to [0, \infty)$, *is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$, *and* $0 \leq t \leq 1$. *Similarly, we say the map* β *is a nonnegative continuous convex functional on* P *provided that* $\beta : P \to [0, \infty)$, *is continuous and*

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$, *and* $0 \leq t \leq 1$.

Definition 3. *An operator T is called completely continuous, if it is continuous and maps bounded sets into precompact sets.*

Let $\gamma, \beta,$ and θ be nonnegative, continuous, convex functionals on P and α, ψ be nonnegative, continuous, concave functionals on P. Then, for nonnegative real numbers h, a, b, d and c, we define the convex sets

$$\begin{aligned}
&P(\gamma, c) = \{x \in P : \gamma(x) < c\},\\
&P(\gamma, \alpha, a, c) = \{x \in P : a \le \alpha(x), \gamma(x) \le c\},\\
&Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \le d, \gamma(x) \le c\},\\
&P(\gamma, \theta, \alpha, a, b, c) = \{x \in P : a \le \alpha(x), \theta(x) \le b, \gamma(x) \le c\}, \text{ and}\\
&Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : h \le \psi(x), \beta(x) \le d, \gamma(x) \le c\}.
\end{aligned}$$

The following Avery Five Functional Fixed Point Theorem [4] which was the generalization of Leggett-Williams Fixed Point Theorem [15], will be used to prove our main results.

Theorem 1. *Let P be a cone in real Banach space E. Suppose there exist positive numbers c and e, nonnegative, continuous, concave functionals α and ψ on P, and nonnegative, continuous, convex functionals $\gamma, \beta,$ and θ on P, with*

$$\alpha(x) \le \beta(x) \textit{and } \|x\| \le e\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$$

is completely continuous and there exist nonnegative numbers h, a, k, b with $0 < a < b$ such that:

(i) $\{x \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Ax) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, k, c)$,

(ii) $\{x \in Q(\gamma, \beta, \psi, h, a, c) : \beta(x) < a\} \neq \emptyset$ *and* $\beta(Ax) < a$ *for* $x \in Q(\gamma, \beta, \psi, h, a, c)$,

(iii) $\alpha(Ax) > b$ *provided* $x \in P(\gamma, \alpha, b, c)$ *with* $\theta(Ax) > k$,

(iv) $\beta(Ax) < a$ *provided* $x \in Q(\gamma, \beta, a, c)$ *with* $\psi(Ax) < h$. *Then A has at least*

three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < a, \; b < \alpha(x_2), \; \textit{and } a < \beta(x_3) \textit{ with } \alpha(x_3) < b.$$

3. Existence of three positive solutions

In this section, we will impose conditions on $f(t,u(t))$ to establish the existence of at least three positive solutions for nonlinear p-Laplacian boundary value problem (2)-(4) by applying Five Functional Fixed Point Theorem.

Let $B=\{u|u\in C[0,1]\}$ be a Banach space with the norm $\|u\|=\max_{t\in[0,1]}|u|$, and let

$$P=\{u\in B|u(t)>0,t\in[0,1] \text{ and } \min_{t\in I}|u(t)|\geq\eta\|u\|\},$$

where $\eta=\left(\frac{1}{M^2}\right)\left(\frac{11^{n-3}}{2^{6n-16}}\right)$. We note that P is a cone in B. Define an integral operator $T:P\to B$ by

$$Tu(t)=\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau)dsdr,\ t\in[0,1]. \quad (16)$$

Now it's time to define the nonnegative continuous concave functionals α,ψ and the nonnegative continuous convex functionals β,θ,γ on $\mathcal{P}$ by

$$\alpha(u)=\min_{t\in I}|u(t)|,$$
$$\psi(u)=\min_{t\in J}|u(t)|,$$
$$\beta(u)=\max_{t\in I}u(t),$$
$$\gamma(u)=\theta(u)=\max_{t\in[0,1]}u(t),$$

$$\begin{aligned}\alpha(u)&=\min_{t\in I}|u(t)|\leq\max_{t\in I}u(t)=\beta(u),\\ \|u\|&\leq\frac{1}{\eta}\min_{t\in I}u(t)\leq\frac{1}{\eta}\max_{t\in[0,1]}u(t)=\frac{1}{\eta}\gamma(u),\end{aligned} \quad (17)$$

where $J=[t_1,t_2],\frac{1}{4}<t_1,t_2<\frac{3}{4}$. We denote

$$\begin{aligned}\mathcal{L}&=\left(\frac{M}{6^{n-3}}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q\big(k_1(\tau,\tau)d\tau\big)dsdr\right)^{-1}\\ \mathcal{M}&=\left(\eta\frac{M}{6^{n-3}}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q\big(k_1(\tau,\tau)d\tau\big)dsdr\right)^{-1}.\end{aligned} \quad (18)$$

The p-Laplacian BVP (2)-(4) has a solution $u(t)$ if and only if $u(t)$ is a fixed point of the operator T defined in the cone P.

We assume the following conditions hold throughout this paper:

(A1) $0<\int_0^1 k_1(t,s)ds<\infty$,

(A2) $f : [0,1] \times [0,\infty] \to [0,\infty]$ is a continuous function.

Lemma 7. *The operator $T : P \to B$ defined by* (16) *is a self map on P.*

Proof. From (A1) and the positivity of the Green's function $h(t,s)$ and $k_1(t,s)$ in Lemma 4 and 5, respectively, $u \in P$, $Tu(t) \geq 0$ on $t \in [0,1]$. Now, for $u \in P$ and by Lemmas 4–6, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau)dsdr,\\ &\leq \frac{M}{6^{n-3}}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q(\int_0^1 k_1(\tau,\tau)f(\tau,u(\tau))d\tau)dsdr.\end{aligned}$$

So that

$$\|Tu\| \leq \frac{M}{6^{n-3}}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau)dsdr. \tag{19}$$

Then by Lemmas 4– 6, for $u \in P$ we have

$$\min_{t\in I}(Tu)(t) = \min_{t\in I}\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau)dsdr$$

$$\geq \frac{1}{M}\int_0^1 h(r,r)(\frac{1}{4^{n-2}}(\frac{11}{96})^{n-3})\Big(\int_0^1 k_1(s,s)\phi_q(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau)ds\Big)dr$$

$$\geq \eta\|Tu\|.$$

So $(Tu)(t) \in P$. Hence $T : P \to P$. ◀

Further, the operator T is completely continuous by an application of the Arzela–Ascoli theorem.

Theorem 2. *Suppose that there exist $0 < b < k < \frac{k}{\eta} < c$, such that f satisfies the following conditions:*

W1. $f(t,u(t)) < \phi_p(b\mathcal{L})$, $t \in [0,1]$ and $u \in [\eta b, b]$,

W2. $f(t,u(t)) > \phi_p(k\mathcal{M})$, $t \in I$ and $u \in [k, \frac{k}{\eta}]$,

W3. $f(t,u(t)) < \phi_p(c\mathcal{L})$, $t \in [0,1]$ and $u \in [0,c]$.

Then the p-Laplacian BVP (2)-(4) *has at least three positive solutions, u_1, u_2, and u_3 such that $\beta(u_1) < b, k < \alpha(u_2)$ and $b < \beta(u_3)$ with $\alpha(u_3) < k$.*

Proof. Recall T defined in (16):

$$Tu(t) = \int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau)dsdr,\ t\in[0,1].$$

Hence we need to prove the existence of three fixed points for T which satisfy the conditions of Theorem 1. As shown in Lemma 7, the operator T is a self map on P, and also T is completely continuous. From (17), for each $u \in P, \alpha(u) \leq \beta(u)$, and $\|u\| \leq \frac{1}{\eta}\gamma(u)$. It is shown that $T : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$. Let $u \in \overline{P(\gamma,c)}$. Then $0 < \gamma(u) = \max_{t\in[0,1]} u(t) = \|u\| \leq c$. By using condition $W3$,

$$\gamma(Tu(t)) = \max_{t\in[0,1]} Tu(t)$$

$$\begin{aligned}
&= \max_{t\in[0,1]}\Big[\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr\Big]\\
&\leq \int_0^1 Mh(r,r)\int_0^1 \frac{1}{6^{n-3}}k_1(s,s)\phi_q\big(\int_0^1 k_1(\tau,\tau)\phi_p(c\mathcal{L})d\tau\big)dsdr\\
&< c\mathcal{L}\frac{M}{6^{n-3}}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q\big(\int_0^1 k_1(\tau,\tau)d\tau\big)dsdr < c. \qquad (20)
\end{aligned}$$

Thus, $Tu(t) \in \overline{P(\gamma,c)}$, we have $T : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$. From this end we claim condition (i) and (ii) of Theorem 1. It is clear that

$$\frac{k+\frac{k}{\eta}}{2} \in \{u \in P(\gamma,\theta,\alpha,k,\frac{k}{\eta},c), \alpha(u) > k\} \neq \emptyset, \qquad (21)$$

$$\frac{\eta b + b}{2} \in \{u \in Q(\gamma,\beta,\psi,\eta b,b,c), \beta(u) < b\} \neq \emptyset. \qquad (22)$$

Now, let $u \in P(\gamma,\theta,\alpha,k,\frac{k}{\eta},c)$ or $u \in Q(\gamma,\beta,\psi,\eta b,b,c)$. Then $k \leq |u(t)| \leq \frac{k}{\eta}$ and $\eta b \leq |u(t)| \leq b$. By using the hypothesis $(W2)$ we get

$$\alpha(Tu(t)) = \min_{t\in I} Tu(t)$$

$$\begin{aligned}
&= \min_{t\in I}\Big[\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr\Big]\\
&\geq \int_0^1 \frac{1}{M}h(r,r)\int_0^1 \frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3} k_1(s,s)\phi_q\big(\int_0^1 k_1(s,\tau)\phi_p(k\mathcal{M})d\tau\big)dsdr\\
&> \frac{k\mathcal{M}}{M}\frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q\big(\frac{1}{4}\int_0^1 k_1(\tau,\tau)d\tau\big)dsdr > k. \qquad (23)
\end{aligned}$$

By considering condition $W1$,

$$\beta(Tu(t)) = \max_{t\in J} Tu(t)$$

$$= \max_{t\in J}\Big[\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr\Big]$$
$$\le \int_0^1 Mh(r,r)\int_0^1 \frac{1}{6^{n-3}}k_1(s,s)\phi_q\big(\int_0^1 k_1(s,\tau)\phi_p(b\mathcal{L})d\tau\big)dsdr$$
$$< b\mathcal{L}\frac{M}{6^{n-3}}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q\big(\int_0^1 k_1(\tau,\tau)d\tau\big)dsdr < b. \tag{24}$$

Thus, conditions (i) and (ii) of Theorem 1 are verified. Let's prove (iii). Let $u\in P(\gamma,\alpha,k,c)$ with $\theta(Tu(t)) > \frac{k}{\eta}$. Then

$$\alpha(Tu(t)) = \min_{t\in I} Tu(t)$$

$$= \min_{t\in I}\Big[\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr\Big]$$
$$\ge \int_0^1 \frac{1}{M}h(r,r)\int_0^1 \frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3}k_1(s,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr$$
$$\ge \frac{1}{M}\frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr$$
$$\ge \frac{11^{n-3}}{M^2 2^{6n-16}6^{n-3}}\int_0^1 h(t,r)\int_0^1 k_1(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr$$
$$= \eta\frac{1}{6^{n-3}}\int_0^1 h(t,r)\int_0^1 k_1(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr$$
$$\ge \eta\max_{t\in[0,1]}\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr = \eta\theta(Tu(t)) > k. \tag{25}$$

Finally, to assert (iv) of Theorem 1, let $u\in Q(\gamma,\beta,b,c)$ with $\psi(Tu(t)) < \eta b$.

$$\beta(Tu(t)) = \max_{t\in J} Tu(t)$$

$$= \max_{t\in J}\Big[\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr\Big]$$
$$\le \max_{t\in[0,1]}\Big[\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr\Big]$$

$$\leq \int_0^1 Mh(r,r)\int_0^1 \frac{1}{6^{n-3}}k_1(s,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr$$

$$= \frac{1}{\eta}\frac{\eta M}{6^{n-3}}\int_0^1 h(r,r)\int_0^1 k_1(s,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr$$

$$\leq \frac{1}{\eta}\min_{t\in I}\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr$$

$$\leq \frac{1}{\eta}\min_{t\in J}\int_0^1 h(t,r)\int_0^1 k_{n-2}(r,s)\phi_q\big(\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau\big)dsdr = \frac{1}{\eta}\psi(Tu(t)) < b. \quad (26)$$

From (23)-(26), all the conditions of Theorem 1 satisfied, then by Theorem 1 the assertion of Theorem 2 follows. ◀

4. Example

Let us consider an example to illustrate our results.

Example 1. *Consider the boundary value problem*

$$(-1)^3[\phi_p\big(u^{(4)}(t)+k^2u''(t)\big)]'' = f(t,u(t)), \quad 0\leq t\leq 1, \quad (27)$$

$$\left.\begin{aligned} u(0)-2u'(0)=0,\\ u(1)+3u'(1)=0,\end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned} u''(0)=0=u''(1),\\ u^{(4)}(0)=0=u^{(4)}(1).\end{aligned}\right\} \quad (29)$$

Set $p=2$ *and* $k=\frac{1}{3}$. *The solution of the BVP* (28)-(29) *is given by*

$$u(t) = \int_0^1 h(t,r)\int_0^1 k_1(r,s)\int_0^1 k_1(s,\tau)f(\tau,u(\tau))d\tau dsdr,$$

where

$$h(t,s) = \frac{1}{\frac{1}{9}\sin(1/3)+\frac{5}{9}\cos(1/3)}\begin{cases}(\sin(\frac{t}{3})+\frac{2}{3}\cos(\frac{t}{3}))(\sin(\frac{1}{3}(1-s))+\frac{2}{3}\cos(\frac{1}{3}(1-s)))\\ (\sin(\frac{s}{3})+\frac{2}{3}\cos(\frac{s}{3}))(\sin(\frac{1}{3}(1-t))+\frac{2}{3}\cos(\frac{1}{3}(1-t)))\end{cases}$$

and

$$f(t,u(t)) = \begin{cases} e^t+\frac{545u}{100}, t\in[0,1], u\in[0,1],\\ e^t+\frac{545}{100}+271(u^2-1), t\in[1/4,3/4], u\in[1,43],\\ e^t+\frac{50108445}{100}, t\in[0,1], u\in[43,\infty).\end{cases}$$

By computing the value of the parameters we get $\eta = 0.0356, \mathcal{L} = 8.1778, \mathcal{M} = 229.713$. $f(t, u(t))$ *satisfies all the conditions of Theorem 2. Then the p-Laplacian BVP* (27)-(29) *has at least three positive solutions* u_1, u_2, *and* u_3 *such that* $\max_{t\in[\frac{1}{4},\frac{3}{4}]} u_1 < 1, 3/2 < \min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2$ *and* $1 < \max_{t\in[\frac{1}{4},\frac{3}{4}]} u_3$ *with* $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_3 < 3/2$ *by Theorem 1.*

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