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On Stability of Bases From Perturbed Exponential Systems in Orlicz Spaces

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Abstract. In this article, perturbed exponential system $\{e^{i\lambda_n t}\}_{n\in Z}$, (where $\{\lambda_n\}$ is some sequence of real numbers), is considered in the Orlicz space $L_M(-\pi,\pi)$. We find a condition on the sequence $\{\lambda_n\}$, which is sufficient for the above system to form a basis for $L_M(-\pi,\pi)$. We establish an analogue of classical Levinson theorem on the replacement of a finite number of elements of this system by other elements. Our results are the analogues of the corresponding results obtained for Lebesgue spaces L_p, $1\le p\le +\infty$. We also establish an analogue of classical Levinson theorem on the completeness of above system in the spaces L_p, $1\le p\le +\infty$.

Key Words and Phrases: Orlicz space, Levinson theorem, basicity.

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1. Introduction

Consider perturbed systems of sines

$$\{\sin \lambda_n x\}_{n\in N}\,, \tag{1}$$

and cosines

$$\{\cos \lambda_n x\}_{n\in Z_+}\,, \tag{2}$$

where N is a set of all positive integers, $Z_+=\{0\}\bigcup N$, and $\{\lambda_n\}\subset R$ is some sequence of real numbers. These systems are the natural perturbations of classical systems of sines and cosines, and they are also the eigenfunctions of second order ordinary differential operator with integral boundary condition. Moreover, it should be noted that the frame theory originates from the research by Duffin R.J. and Schaeffer A.C. [25] (see also [26,27]) dedicated to the frame properties

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of such systems in the spaces L_2. That's why there is great interest in studying basis properties of these systems in different kinds of function spaces. First results in this field belong probably to Paley-Wiener [1] and N. Levinson [2]. The well-known " Kadets 1/4 " theorem also belongs to this field (see [3]). When λ_n has a constant shift $\lambda_n = n + \alpha sign\, n$ ($\alpha \in R$), the systems (1) and (2) arise in the solution of mixed or elliptic type differential equations by the Fourier method (see, e.g., [4-6]). In view of this, many authors have studied the basis properties of the systems (1) and (2) (see, e.g., [4,6-15,23,24]). All above-mentioned works treat basis properties in the Lebesgue spaces.

Orlicz spaces were introduced by W. Orlicz and Z. Birnbaum in the beginning of 1930 s in connection with orthogonal decomposition. Orlicz spaces have wide applications in different fields of mathematics such as approximation, stochastic analysis, nonlinear differential equations, Fourier analysis, etc. Numerous facts of classical analysis are transferred to these spaces.

In this work we consider a perturbed exponential system $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ in the Orlicz space $L_M(-\pi,\pi)$. We establish an analogue of classical Levinson theorem on the replacement of a finite number of elements of this system by other elements. We find a condition on the sequence λ_n, which is sufficient for this system to form a basis for $L_M(-\pi,\pi)$. Our results are the analogues of the corresponding results obtained for Lebesgue spaces L_p, $1 \le p \le +\infty$ (see, e.g., [16]). We also establish an analogue of classical Levinson theorem on the completeness of above system in the spaces L_p, $1 \le p \le +\infty$.

2. Needful information

We will use following notations. N will denote the set of positive integers, $Z_+ = \{0\}\bigcup N$; $Z = \{-N\}\bigcup Z_+$, $X_M(\cdot)$ will be the characteristic function of the set M; R will stand for the set of real numbers, C will denote the set of complex numbers, $\widehat{M}$ will stand for the closure of the set M in the corresponding norm and $\overline{(\cdot)}$ will denote the complex conjugation.

Definition 1. *Continuous convex function $M(u)$ in R is called an N-function, if it is even and satisfies the conditions*

$$\lim_{u\to 0}\frac{M(u)}{u} = 0;\ \lim_{u\to\infty}\frac{M(u)}{u} = \infty\,.$$

Definition 2. *Let M be an N-function. The function*

$$M^*(v) = \max_{u\ge 0}\left[u\,|v\ - M(u)\right],$$

is called an N-function complementary to $M(\cdot)$.

The function $M^*(\cdot)$ can be characterized as follows. Let the function $p(\cdot): R_+ \to R_+ = [0; +\infty)$ be right continuous for $t \geq 0$, positive for $t > 0$, non-decreasing and satisfy the conditions $p(0) = 0$, $p(\infty) = \lim_{t\to\infty} p(t) = \infty$. Define

$$q(s) = \sup_{p(t)\leq s} t, s \geq 0.$$

The function $q(\cdot)$ has the same properties as the function $p(\cdot)$: it is positive for $s > 0$, right continuous for $s \geq 0$, non-decreasing and satisfies conditions $q(0) = 0$, $q(\infty) = \lim_{s\to\infty} q(s) = \infty$. The functions

$$M(u) = \int_0^{|u|} p(t)\,dt,\ M^*(v) = \int_0^{|v|} q(s)\,ds,$$

are called N-functions complementary to each other.

Definition 3. *N-function $M(\cdot)$ satisfies Δ_2-condition for large values of u, if $\exists k > 0 \wedge \exists u_0 \geq 0$:*

$$M(2u) \leq kM(u),\ \forall u \geq u_0.$$

Δ_2-condition is equivalent to requiring that, for $\forall l > 1$, $\exists k(l) > 0 \wedge \exists u_0 \geq 0$:

$$M(lu) \leq k(l)\,M(u),\ \forall u \geq u_0.$$

Now let's define the Orlicz space. Let $M(\cdot)$ be some N-function, $G \subset R$ be a (Lebesgue) measurable finite-dimensional set. Denote by $L_0(G)$ the set of all functions measurable in G. Let

$$\rho_M(u) = \int_G M[u(x)]\,dx,$$

and

$$L_M(G) = \{u \in L_0(M) : \rho_M(u) < +\infty\}.$$

$L_M(G)$ is called an Orlicz class.

Let $M(\cdot)$ and $M^*(\cdot)$ be N-functions complementary to each other. Let

$$L_M^*(G) = \{u \in L_0(M) : |u, v| < +\infty,\ \forall u(\cdot) \in L_{M^*}(G)\},$$

where

$$(u, v) = \int_G u(x)\,\overline{v(x)}dx.$$

$L_M^*(G)$ is called an Orlicz space. With the norm:

$$\|u\|_M = \sup_{\rho_{M^*(v)}\leq 1} |(u, v)|,$$

$L_M^*(G)$ becomes a Banach space. Note that in $L_M^*(G)$, the following norm is equivalent to the norm $\|u\|_M$:

$$\|u\|_M = \inf\left\{k > 0 : \rho_M\left(\frac{u}{k}\right) \le 1\right\}.$$

$\|\cdot\|_M$ is called a Luxemburg norm. Let's recall the following well known fact.

Statement 1. *If N-function $M(\cdot)$ satisfies Δ_2-condition, then $L_M^*(G) = L_M(G)$ and the closure of the set of bounded (including continuous) functions coincides with $L_M^*(G)$.*

Statement 2. *If N-function $M(\cdot)$ satisfies Δ_2-condition, then $L_M^*(G)$ is separable.*

More details on the above facts can be found in [17,18].

In the sequel, as G we will consider the interval $G = [-\pi, \pi]$, and for simplicity, we will always omit the letter G in the notations (for example $L_M^*(G) = L_M(G)$, etc.). Later we will need some facts about Fourier analysis in Orlicz spaces. Let's first define the following characteristic of the space L_M.

Let $M(u)$ and $N(v)$ be N-functions complementary to each other. Let $v(x)$ be a function in $L_N(-\pi,\pi)$ such that $\rho(v;N) \le 1$. Then according to the Jensen integral inequality

$$M\left\{\frac{\int_G u(x)\,dx}{mes\,G}\right\} \le \frac{\int_G M[u(x)]\,dx}{mes\,G},\ u(x) \in L_M,$$

we have

$$N\left(\frac{1}{m(E_n)}\int_{E_n} v(x)\,dx\right) \le \frac{1}{m(E_n)}\int_{E_n} N(v(x))\,dx \le \frac{1}{m(E_n)},$$

from which it follows that

$$\int_{E_n} v(x)\,dx \le m(E_n)\,N^{-1}\left(\frac{1}{m(E_n)}\right), \tag{3}$$

where $N^{-1}(\cdot)$ is the inverse function of $N(\cdot)$.

For the function $v_0(x)\,N^{-1}\left(\frac{1}{m(E_n)}\right)\chi_{E_n}$, satisfying the condition $\rho(v_0;N) = 1$, we have

$$\int_{E_n} v_0(x)\,dx \le m(E_n)\,N^{-1}\left(\frac{1}{m(E_n)}\right). \tag{4}$$

By the definition of the norm, we have

$$\|\chi_{E_n}\|_M = \sup_{\rho(v;N)\leq 1} \left| \int_G \chi_{E_n} v\,(x)\,dx \right| = \sup_{\rho(v;N)\leq 1} \left| \int_{E_n} v\,(x)\,dx \right|.$$

Hence, by virtue of (3) and (4), we obtain the formula for the norm of the characteristic function:

$$\|\chi_{E_n}\|_M = m\,(E_n)\,N^{-1}\left(\frac{1}{m\,(E_n)}\right).$$

Definition 4. *We shall say that the function $u \in L_M^*$ has an absolute continuous norm if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that*

$$\|u\chi_{E_n}\|_{L_M(-\pi,\pi)} = \sup_{\rho(v;N)\leq 1} \left| \int_{E_n} u\,(x)\,v\,(x)\,dx \right| < \varepsilon,$$

provided $m\,(E_n) < \delta\,(E_n \subset (-\pi,\pi))$.

More details on the above facts can be found in [17,18].

So, let $M\,(\cdot)$ be some N-function and $M^{-1}\,(\cdot)$ be its inverse on $[0,+\infty)$. Let

$$h\,(t) = \limsup_{x\to\infty} \frac{M^{-1}\,(x)}{M^{-1}\,(tx)}, t > 0,$$

and define the numbers

$$\alpha_M = -\lim_{t\to\infty} \frac{\ln h\,(t)}{\ln t};\ \beta_M = -\lim_{t\to 0+} \frac{\ln h\,(t)}{\ln t}.$$

The numbers α_M and β_M are called upper and lower Boyd indices for the Orlicz space L_M.

The following relationship holds:

$$0 \leq \alpha_M \leq \beta_M \leq 1;$$

$$\alpha_M + \beta_{M^*} \equiv 1;\ \alpha_{M^*} + \beta_M \equiv 1.$$

The space L_M is reflexive if and only if $0 < \alpha_M \leq \beta_M < 1$. If $1 \leq q < \frac{1}{\beta_M} \leq \frac{1}{\alpha_M} < p \leq +\infty$, then the continuous embeddings $L_p\,(-\pi,\pi) \subset L_M \subset L_q\,(-\pi,\pi)$ hold.

We will also need the following

Theorem 3. *For every p and q such that*

$$1 \leq p < \frac{1}{\beta_M} \leq \frac{1}{\alpha_M} < q \leq +\infty,$$

we have

$$L_q \subset L_M \subset L_p,$$

with the inclusion maps being continuous.

More details regarding these concepts can be found in [19-22].

We will need some concepts and facts from the theory of Banach function spaces (see e.g. [24,25]). Let $(R;\mu)$ be a measure space, and M^+ be the cone of μ-measurable functions on R the values of which lie in $[0, +\infty]$. Denote the characteristic function of a μ-measurable subset of R by χ_E.

Definition 5. *A mapping $\rho : M^+ \to [0, +\infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, n \in N$ in M^+, for all constants $a \geq 0$ and for all μ-measurable subsets $E \subset R$, the following properties hold:*

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho(af) = a\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f$ μ-a.e.$\Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \uparrow f\mu$-a.e.$\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P4) $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;

(P5) $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$, for some constant $C_E : 0 < C_E < +\infty$ depending on E and ρ, but independent of f.

Let M denote the set of all extended scalar-valued (real or complex) μ-measurable functions and $M_0 \subset M$ denote the subclass of functions that are finite μ-a.e. .

Definition 6. *Let ρ be a function norm. The set $X = X(\rho)$ of all functions f in M for which $\rho(|f|) < +\infty$ is called a Banach function space. For each $f \in X$, define $\|f\|_X = \rho(|f|)$.*

The following theorem is true.

Theorem 4. *Let ρ be a function norm and let $X = X(\rho)$ and $\|\cdot\|_X$ be as above. Then under the natural vector space operations, $(X; \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$M_s \subset X \subset M$$

hold, where M_s is the set of μ-simple functions. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges pointwise μ-a.e. to f.

Let

$$\rho(g) = \sup\left\{\int_\gamma f(\tau)\, g(\tau)\, |dt| : f \in M^+; \rho(f) \le 1\right\}, \forall g \in M^+.$$

A space

$$X' = \{g \in M : \rho'(|g|) < +\infty\}$$

is called an associate space (Kothe dual) of X.

The functions $f; g \in M_0$ are called equimeasurable if

$$|\{\tau \in \gamma : |f(\tau)| > \lambda\} = \{\tau \in \gamma : |g(\tau)| > \lambda\}|, \forall\lambda \ge 0.$$

Banach function norm $\rho : M^+ \to [0, \infty]$ is called rearrangement invariant if for arbitrary equimeasurable functions $f; g \in M_0$ the relation $\rho(f) = \rho(g)$ holds. In this case, Banach function space X with the norm $\|\cdot\|_X = \rho(|\cdot|)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Orlicz, Lorentz, Lorentz-Orlicz spaces are r.i.s.

Theorem 5. *The Banach space dual X^* of a b.f.s. X is canonically isometrically isomorphic to the associate space X' if and only if X has absolutely continuous norm.*

We will also use the following statement from [22, p.14].

Statement 6. *Let X be a b.f.s. over $(M; \mu)$with norm $\|\cdot\|_X$. A function $f \in X$ has absolutely continuous norm if and only if $\|f_{\chi E_n}\| \downarrow 0$ for every sequence $\{E_n\}_{n\in N}$ satisfying $E_n \downarrow \emptyset$ μ-a.e. .*

For more details about these facts see, e.g., [22].

We will also need the following fact concerning the basicity properties of systems from the [27] (see also [11]).

Statement 7. *Suppose a finite number of elements in the basis of some Banach space are replaced by the other elements of this space. Then the following properties are equivalent for the newly obtained system:*

i) it forms a basis;

ii) it is complete;

iii) it is minimal.

3. Sufficient condition for separation of $\{\lambda_n\}$

For obtaining our main results, we need some concepts and facts.

Definition 7. *A system* $\{f_n\}_{n\in N}\subset L_M(-\pi,\pi)$ *is called-Hilbert* ($q>0$) *if there exists absolute constant* $m>0$ *such that for every finite set of complex numbers* $\{C_n\}$ *the inequality*

$$\left(\sum_n |C_n|^q\right)^{\frac{1}{q}} \le m\left\|\sum_n C_n f_n\right\|_M$$

holds.

Definition 8. *A sequence* $\{\lambda_n\}$ *is called separated if* $\inf\limits_{i\neq j}|\lambda_n-\lambda_k|>0$.

The following simple lemma is true.

Lemma 1. *Let* $\{\lambda_n\}_{n\in Z}\subset R$ *be some sequence of real numbers. If the system* $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ *is* q *-Hilbert in* $L_M(-\pi,\pi)$, *then* $\{\lambda_n\}_{n\in Z}$ *is separated.*

Proof. From the definition of q-Hilbertness, we obtain

$$(2)^{\frac{1}{q}} \le m\left\|e^{i\lambda_n x}-e^{i\lambda_k x}\right\|_{L_M(-\pi,\pi)},\ k\neq n. \tag{5}$$

Taking into account the inequality

$$\left|e^{i\lambda_n x}-e^{i\lambda_k x}\right| \le \left|2\sin\left(\frac{\lambda_n-\lambda_k}{2}x\right)\right| \le \pi\left|\lambda_n-\lambda_k\right|,$$

we have

$$\left\|e^{i\lambda_n x}-e^{i\lambda_k x}\right\|_{L_M(-\pi,\pi)} = \sup_{\rho(v;N)\le 1}\left|\int_{-\pi}^{\pi}\left(e^{i\lambda_n x}-e^{i\lambda_k x}\right)v(x)\,dx\right| \le$$

$$\le \sup_{\rho(v;N)\le 1}\int_{-\pi}^{\pi}\left|\left(e^{i\lambda_n x}-e^{i\lambda_k x}\right)\right||v(x)|\,dx \le$$

$$\le \sup_{\rho(v;N)\le 1}\int_{-\pi}^{\pi}\pi\left|\lambda_n-\lambda_k\right||v(x)|\,dx = \pi\left|\lambda_n-\lambda_k\right| \sup_{\rho(v;N)\le 1}\int_{-\pi}^{\pi}|v(x)|\,dx =$$

$$= 2\pi^2 N^{-1}\left(\frac{1}{2\pi}\right)\left|\lambda_n-\lambda_k\right|.$$

Consequently

$$(2)^{\frac{1}{q}} \le 2\pi^2 m N^{-1}\left(\frac{1}{2\pi}\right)\left|\lambda_n-\lambda_k\right|,$$

$$\delta = \frac{(2)^{\frac{1}{q}}}{2\pi^2 m N^{-1}\left(\frac{1}{2\pi}\right)} \Rightarrow \left|\lambda_n-\lambda_k\right| > \delta.$$

The rest follows directly from (5).

The lemma is proved. ◄

The lemma below can be proved in exactly the same way.

Lemma 2. *Let $\{\lambda_n\}_{n\in Z} \subset R$ be some sequence. If the system* (1) *(or* (2)*) is q-Hilbert in $L_M(-\pi,\pi)$, then $\{\lambda_n\}_{n\in Z}$ is separated.*

4. The L_M -analogue of Levinson theorem

In this section, we establish an analogue of Levinson theorem in L_M. Denote by L'_M the associate space of L_M, i.e.

$$(L_M(a,b))' = \left\{g \in F(a,b) : \rho'_M(|g|) < +\infty\right\},$$

where

$$\rho'_M(g) = \sup\left\{\int_a^b fgdt : f \in F^+(a,b);\ \|f\|_M \leq 1\right\},$$

$F(a,b)$ are Lebesgue-measurable functions on (a,b) and $F^+(a,b) = \{f \in F(a,b) : f \geq 0\}$.

The following analogue of Levinson theorem is true.

Theorem 8. *Suppose that the N-function $M(u)$ satisfies the Δ_2 − condition. Let $\{\lambda_k\}_{k\in N} \subset C$ be some sequence. In order for the exponential system $\left\{e^{i\lambda_k x}\right\}_{k\in N}$ to be not complete in $L_M(-\pi,\pi)$, it is necessary and sufficient that there exist an entire function $F(\lambda)$ vanishing at all points λ_k, $k \in N$ and admitting representation*

$$F(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda x}\overline{v(x)}dx,$$

where $v(x) \in L_N(-\pi;\pi)$ is some function.

Proof. Let the system $\left\{e^{i\lambda_k x}\right\}_{k\in N}$ be not complete in $L_M(-\pi,\pi)$. Then it is clear that there exists a non-zero functional $V \in L_N(-\pi,\pi)$ such that

$$V\left(e^{i\lambda_k x}\right) = \int_{-\pi}^{\pi} e^{i\lambda_k x}\overline{v(x)}dx = 0, \forall k \in N.$$

Let's show that the spaces $L_N(-\pi,\pi)$ and $L'_M(-\pi,\pi)$ are isometrically isomorphic, i.e. they can be equated with each other. By Theorem 5, to show this, it suffices to prove that $L_M(-\pi,\pi)$ has absolutely continuous norm. Let

$u \in L_M(-\pi,\pi)$ be an arbitrary function. As $C[-\pi,\pi]$ (a space of continuous functions on $[-\pi,\pi]$) is dense in $L_M(-\pi,\pi)$, for $\forall\varepsilon>0$, $\exists u_0 \in C[-\pi,\pi]$ we have

$$\|u-u_0\|_{L_M(-\pi,\pi)}<\varepsilon.$$

Let $\{E_n\}_{n\in N}\subset(-\pi,\pi)$ be an arbitrary sequence of (Lebesgue) measurable sets such that $E_n\downarrow\emptyset$ m-a.e. (m is a Lebesgue measure.) Recall that $E_n\downarrow\emptyset$ m-a.e. means $\chi_{E_n}\downarrow 0$ m-$a.e.$

Let's show that $\|u\chi_{E_n}\|_{L_M(-\pi;\pi)}\downarrow 0$. So, let $\varepsilon>0$ be an arbitrary number. We have

$$\|u\chi_{E_n}\|_{L_M(-\pi,\pi)}=\|(u-u_0)\chi_{E_n}+u_0\chi_{E_n}\|_{L_M(-\pi,\pi)}\le$$

$$\le\|(u-u_0)\chi_{E_n}\|_{L_M(-\pi,\pi)}+\|u_0\chi_{E_n}\|_{L_M(-\pi,\pi)}\le$$

$$\le\varepsilon+\|u_0\chi_{E_n}\|_{L_M(-\pi,\pi)}.$$

Therefore

$$\|u\chi_{E_n}\|_{L_M(-\pi,\pi)}\le\varepsilon+\|u_0\chi_{E_n}\|_{L_M(-\pi;\pi)}. \qquad (6)$$

Let $c=\|u_0\|_{L_\infty(-\pi;\pi)}$. We have

$$\|u_0\chi_{E_n}\|_{L_M(-\pi;\pi)}=\sup_{\rho(v;N)\le 1}\left|\int_{-\pi}^{\pi}u_0\chi_{E_n}v(x)\,dx\right|\le$$

$$\le\|u_0\|_{L_\infty(-\pi,\pi)}\|\chi_{E_n}\|_M=c\|\chi_{E_n}\|_M=$$

$$=c\sup_{\rho(v;N)\le 1}\left|\int_{-\pi}^{\pi}\chi_{E_n}v(x)\,dx\right|=c\sup_{\rho(v;N)\le 1}\left|\int_{E_n}v(x)\,dx\right|=$$

$$=cm(E_n)N^{-1}\left(\frac{1}{m(E_n)}\right),$$

where m is a Lebesgue measure. Obviously, $\lim\limits_{n\to\infty}E_n=\bigcap_{n=1}^{\infty}E_n=\emptyset$, m-a.e. Consequently,

$$\lim_{n\to\infty}|E_n|=\left|\lim_{n\to\infty}E_n\right|=0.$$

Then from (6) it follows that $\|u\chi_{E_n}\|_{L_M(-\pi,\pi)}\to 0$, $n\to\infty$.

Thus, by Statement 6, the space $L_M(-\pi,\pi)$ has absolutely continuous norm. Then from Theorem 5 it follows that $L_N(-\pi,\pi)=L'_M(-\pi,\pi)$. Hence, it is clear that

$$\exists v(x)\in L_N(-\pi,\pi):V(f)=\int_{-\pi}^{\pi}f(x)\overline{v(x)}dx,\ \forall f\in L_M(-\pi,\pi).$$

Let

$$F(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda x} \overline{v(x)} dx, \; \lambda \in C. \tag{7}$$

Obviously, $F(\cdot)$ is an entire function and $F(\lambda_k) = 0$, $\forall k \in N$.

The theorem is proved. ◄

Theorem 9. *Suppose that the N-function $M(u)$ satisfies the Δ_2-condition. If the entire function $F(\cdot)$ is represented in the form (7), $v(x) \in L_N(-\pi, \pi)$, $F(\lambda_0) = 0$ and $\mu \in C$ is an arbitrary number, then the function*

$$F_1(\lambda) = \frac{\lambda - \mu}{\lambda - \lambda_0} F(\lambda)$$

is also represented in the form (7).

Proof. Absolutely similar to the proof of Levinson theorem, let

$$\varphi(x) = v(x) + i(\mu - \lambda_0) e^{-i\lambda_0 x} \int_{-\pi}^{x} e^{i\lambda_0 y} v(y) \, dy. \tag{8}$$

By multiplying both sides by $e^{i\lambda x}$ and integrating from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} e^{i\lambda x} \varphi(x) \, dx = F(\lambda) + i(\mu - \lambda_0) \int_{-\pi}^{\pi} e^{i(\lambda - \lambda_0)x} \left(\int_{y}^{\pi} e^{i\lambda_0 y} v(y) \, dy \right) dx.$$

Changing the order of integration, we have $\int_{-\pi}^{\pi} e^{i\lambda x} \varphi(x) \, dx = \frac{\lambda - \mu}{\lambda - \lambda_0} F(\lambda) = F_1(\lambda)$ (for more details about these facts see, e.g., [2]). So we have

$$F_1(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda x} \varphi(x) \, dx.$$

It remains to show that $\varphi(x) \in L_N(-\pi, \pi)$. It is absolutely clear that $\left| e^{i\lambda x} \right| \leq const \leq \infty$, $\forall x \in [-\pi, \pi]$. Therefore, from the expression (8) for $\varphi(\cdot)$ it follows that it now suffices to prove $\int_{-\pi}^{x} |v(y)| \, dy \in L_N(-\pi, \pi)$. But this is obvious, because $\int_{-\pi}^{x} |v(y)| \, dy \in C(-\pi, \pi)$.

The theorem is proved. ◄

This theorem has the following direct corollary.

Corollary 1. *Let the system $\left\{ e^{i\lambda_k x} \right\}_{k \in N}$ be complete in $L_M(-\pi, \pi)$. If n arbitrary functions are removed from this system and n other functions $\left\{ e^{i\mu_j x} \right\}$, $j = 1, 2, ..., n$, where $\mu_1, ..., \mu_n$ are arbitrary complex numbers different from any of λ_k, are added instead of them, then the newly obtained system will be complete in $L_M(-\pi, \pi)$.*

5. On stability of exponential bases in L_M.

The following main theorem is true.

Theorem 10. *Suppose that the N-function $M(u)$ satisfies the Δ_2-condition. Let $M(\cdot)$, $M^*(\cdot)$ be N-functions complementary to each other and the numbers α_M and β_M are upper and lower Boyd indices for the Orlicz space L_M. Let $\{\lambda_n\}_{n\in Z}$; $\{\mu_n\}_{n\in Z} \subset R$ be some sequences, $\lambda_i \neq \lambda_j$, $\mu_i \neq \mu_j$ for $i \neq j$. Let*

$$\sum_{n=-\infty}^{n=+\infty} |\lambda_n - \mu_n|^{\gamma} < +\infty,$$

where $\gamma = \min\left(\frac{1}{\beta_M}; \frac{1}{\beta_{M^*}}\right)$, $\alpha_M + \beta_{M^*} \equiv 1$, $\alpha_{M^*} + \beta_M \equiv 1$.

If the system $\left\{e^{i\lambda_n x}\right\}_{n\in N}$ forms a basis for $L_M(-\pi,\pi)$, equivalent to the basis $\left\{e^{inx}\right\}_{n\in Z}$, then the system $\left\{e^{i\mu_n x}\right\}_{n\in Z}$ also forms a basis for $L_M(-\pi,\pi)$, equivalent to $\left\{e^{inx}\right\}_{n\in Z}$.

Proof. We first consider the case $\frac{1}{\beta_{M^*}} \geq 2$.

$$0 < \alpha_M \leq \beta_M < 1 \Rightarrow 1 < \frac{1}{\beta_M} \leq \frac{1}{\alpha_M} < +\infty \Rightarrow$$

$$0 < 1 - \beta_{M^*} \leq 1 - \alpha_{M^*} < 1 \Rightarrow 0 < \alpha_{M^*} \leq \beta_{M^*} < 1 \Rightarrow$$

$$\frac{1}{\beta_{M^*}} \geq 2 \Rightarrow \frac{1}{2} \geq \beta_{M^*} \geq \alpha_{M^*} \Rightarrow 0 < \alpha_{M^*} \leq \frac{1}{2} \Rightarrow$$

$$\frac{1}{2} \leq 1 - \alpha_{M^*} < 1 \Rightarrow \frac{1}{2} \leq \beta_M < 1 \Rightarrow 1 < \frac{1}{\beta_M} \leq 2.$$

Then it is clear that $\gamma = \frac{1}{\beta_M}$. Let $\varphi_n(x) = e^{i\lambda_n x}$, $\psi_n(x) = e^{i\mu_n x}$. We have

$$|\varphi_n(x) - \psi_n(x)| \leq \pi |\lambda_n - \lambda_k|$$

$$\|\varphi_n(x) - \psi_n(x)\|^p_{L_M(-\pi,\pi)} \leq c |\lambda_n - \lambda_k|^p,$$

where $c > 0$ is a constant independent of n. We choose $p = \frac{1}{\beta_M}$, then $1 < p \leq 2$. Consequently

$$\sum_{n=-\infty}^{n=+\infty} \|\varphi_n(x) - \psi_n(x)\|^p_{L_M(-\pi,\pi)} < +\infty.$$

Let $\{C_n\}$ be an arbitrary finite set of numbers $C_n \in C$. Then from the Hausdorff-Young theorem we obtain

$$\|\{C_n\}\|_{l_q} \le C \left\| \sum_n C_n e^{inx} \right\|_{L_p(-\pi,\pi)},$$

where $C > 0$ is a constant independent of C_n. We choose numbers $p, q \in [1, +\infty]$ such that

$$1 \le p < \frac{1}{\beta_M} \le \frac{1}{\alpha_M} < q \le +\infty.$$

Then according to Theorem 3 we have

$$L_q \subset L_M \subset L_p,$$

with the inclusion maps being continuous. Consequently $\forall f \in L_M$, $\|f\|_p \le C \|f\|_M$.

Therefore

$$\|\{C_n\}\|_{l_q} \le C \left\| \sum_n C_n e^{inx} \right\|_{L_M(-\pi,\pi)}. \tag{9}$$

As the bases $\{\varphi_n(x)\}$ and $\left\{e^{inx}\right\}_{n\in Z}$ are equivalent, from (9) it follows

$$\left\| \sum_n C_n e^{inx} \right\|_{L_M(-\pi,\pi)} \le C \left\| \sum_n C_n \varphi_n(x) \right\|_{L_M(-\pi,\pi)} \Rightarrow$$

$$\|\{C_n\}\|_{l_q} \le C \left\| \sum_n C_n \varphi_n(x) \right\|_{L_M(-\pi,\pi)}.$$

Let's take some number $m \in N$ and let

$$f_n = \begin{cases} \varphi_n, & |n| < m, \\ \psi_n, & |n| \ge m. \end{cases}$$

We have

$$\left\| \sum_n C_n (f_n - \varphi_n) \right\|_{L_M(-\pi,\pi)} \le \sum_n |C_n| \, \|f_n - \varphi_n\|_{L_M(-\pi,\pi)} \le$$

$$\le \|\{C_n\}\|_{l_q} \left(\sum_n \|f_n - \varphi_n\|^p_{L_M(-\pi,\pi)} \right)^{\frac{1}{p}} \le$$

$$\leq C\left(\sum_{|n|\geq m}\|\psi_n-\varphi_n\|^p_{L_M(-\pi,\pi)}\right)^{\frac{1}{p}}\left\|\sum_n C_n\varphi_n(x)\right\|_{L_M(-\pi,\pi)}$$

$$=C(m)\left\|\sum_n C_n\varphi_n(x)\right\|_{L_M(-\pi,\pi)}, \tag{10}$$

where

$$C(m)=C\left(\sum_{|n|\geq m}\|\psi_n-\varphi_n\|^p_{L_M(-\pi,\pi)}\right)^{\frac{1}{p}}.$$

It is absolutely clear that $\lim\limits_{m\to\infty}C(m)=0$, and therefore, for large m we have $0<c(m)<1$. Then from Paley-Wiener theorem (for Banach case; see, e.g., [27, p. 187] and the relation (10) it follows that the system $\{f_n\}_{n\in Z}$ forms a basis for $L_M(-\pi,\pi)$, equivalent to $\{\varphi_n\}_{n\in Z}$. From the completeness of the system $\{f_n\}_{n\in Z}$ in $L_M(-\pi,\pi)$ and Corollary 1 it follows that the system $\{\psi_n\}_{n\in Z}$ is also complete in $L_M(-\pi,\pi)$. Then, by Statement 7, the system $\{\psi_n\}_{n\in Z}$ also forms a basis for $L_M(-\pi,\pi)$, equivalent to $\{\varphi_n\}_{n\in Z}$.

Now let's consider the case $\frac{1}{\beta_M}>2$.

$$\alpha_M+\beta_{M^*}\equiv 1,\alpha_{M^*}+\beta_M\equiv 1.$$

$$0<\alpha_M\leq\beta_M<1\Rightarrow 1<\frac{1}{\beta_M}\leq\frac{1}{\alpha_M}<+\infty\Rightarrow$$

$$\beta^*_M=1-\alpha_M\Rightarrow\frac{1}{\beta^*_M}=\frac{1}{1-\alpha_M}\Rightarrow$$

$$\frac{1}{\beta_M}>2\Rightarrow\frac{1}{\alpha_M}>2\Rightarrow\alpha_M<\frac{1}{2}$$

$$\alpha_M<\frac{1}{2}\Rightarrow-\alpha_M>-\frac{1}{2}\Rightarrow 1-\alpha_M>\frac{1}{2}\Rightarrow\frac{1}{1-\alpha_M}<2\Rightarrow\frac{1}{\beta^*_M}<2.$$

Then we have $\frac{1}{\beta^*_M}<2$ and $\gamma=\frac{1}{\beta^*_M}$. We choose $q=\frac{1}{\beta_{M^*}}$. Then we have $q<\frac{1}{\beta_M}$. Let $\{C_n\}$ be an arbitrary finite set of numbers $C_n\in C$. Then from the Hausdorff-Young theorem we obtain

$$\|\{C_n\}\|_{l_p}\leq C\left\|\sum_n C_n e^{inx}\right\|_{L_q(-\pi,\pi)},$$

where $C>0$ is a constant independent of C_n. We choose numbers $p,q\in[1,+\infty]$ such that

$$1\le q<\frac{1}{\beta_M}\le\frac{1}{\alpha_M}<p\le+\infty.$$

Then according to Theorem 3 we have

$$L_p\subset L_M\subset L_q,$$

with the inclusion maps being continuous. Consequently, $\forall f\in L_M$, $\|f\|_q\le C\|f\|_M$.

Then

$$\left\|\sum_n C_n e^{inx}\right\|_{L_q(-\pi,\pi)}\le C\left\|\sum_n C_n e^{inx}\right\|_{L_M(-\pi,\pi)},\ C>0.$$

As the bases $\{\varphi_n(x)\}$ and $\left\{e^{inx}\right\}_{n\in Z}$ are equivalent, we have

$$\left\|\sum_n C_n e^{inx}\right\|_{L_M(-\pi,\pi)}\le C\left\|\sum_n C_n\varphi_n(x)\right\|_{L_M(-\pi,\pi)}.$$

Let's take some number $m\in N$ and let

$$f_n=\begin{cases}\varphi_n, & |n|<m\\ \psi_n, & |n|\ge m\end{cases}$$

We have

$$\left\|\sum_n C_n(f_n-\varphi_n)\right\|_{L_M(-\pi,\pi)}\le\sum_n|C_n|\,\|f_n-\varphi_n\|_{L_M(-\pi,\pi)}\le$$

$$\le\|\{C_n\}\|_{l_p}\left(\sum_n\|f_n-\varphi_n\|^q_{L_M(-\pi,\pi)}\right)^{\frac{1}{q}}\le$$

$$\le C\left(\sum_{|n|\ge m}\|\psi_n-\varphi_n\|^q_{L_M(-\pi,\pi)}\right)^{\frac{1}{q}}\left\|\sum_n C_n e^{inx}\right\|_{L_q(-\pi,\pi)}\le$$

$$\le C\left(\sum_{|n|\ge m}\|\psi_n-\varphi_n\|^q_{L_M(-\pi,\pi)}\right)^{\frac{1}{q}}\left\|\sum_n C_n e^{inx}\right\|_{L_M(-\pi,\pi)}\le$$

$$\leq C\left(\sum_{|n|\geq m}\|\psi_n-\varphi_n\|^q_{L_M(-\pi,\pi)}\right)^{\frac{1}{q}}\left\|\sum_n C_n\varphi_n(x)\right\|_{L_M(-\pi,\pi)}$$

$$\leq C(m)\left\|\sum_n C_n\varphi_n(x)\right\|_{L_M(-\pi,\pi)}. \tag{11}$$

It is absolutely clear that $\lim_{m\to\infty} C(m)=0$ and therefore, for large m we have $0<c(m)<1$. Then from Paley-Wiener theorem (for Banach case; see, e.g., [27, p. 187] and the relation (11) it follows that the system $\{f_n\}_{n\in Z}$ forms a basis for $L_M(-\pi,\pi)$, equivalent to $\{\varphi_n\}_{n\in Z}$. From the completeness of the system $\{f_n\}_{n\in Z}$ in $L_M(-\pi,\pi)$ and Corollary 1 it follows that the system $\{\psi_n\}_{n\in Z}$ is also complete in $L_M(-\pi,\pi)$. Then, by Statement 7, the system $\{\psi_n\}_{n\in Z}$ also forms a basis for $L_M(-\pi,\pi)$, equivalent to $\{\varphi_n\}_{n\in Z}$.

Proceeding absolutely similar to case $\frac{1}{\beta_M}\leq 2$, we now establish the basicity of the system $\{\psi_n\}_{n\in Z}$ *for* $L_M(-\pi,\pi)$.

The theorem is proved. ◄

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