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On Zeros of the Modified Bessel Function of the First Kind

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Abstract. Zeros of the modified Bessel function $I_\nu(z)$ of the first kind, considered as a function of index ν are studied. It is proved that for each ε, $\varepsilon > 0$ outside the band $|Im\nu| < \varepsilon$ the function $I_\nu(z)$ can only have a finite number of zeros. Real zeros of the function $I_\nu(z)$ are located in the intervals $(-2k, -(2k-1))$, $k = 1, 2, \ldots.$

Key Words and Phrases: Bessel functions, zeros of Bessel functions, Schrödinger equation, eigenvalues.

2010 Mathematics Subject Classifications: 34A55, 34B24, 58C40, 47E05

1. Introduction and main result

Consider the modified Bessel equation

$$z^2u'' + zu' - \left(z^2 + \nu^2\right)u = 0, \tag{1}$$

where ν is a complex parameter. It is well known [1] that this equation has a solution $I_\nu(z)$, representable in the form

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(\nu + k + 1)}, \tag{2}$$

where $\Gamma(\cdot)$ is a gamma function. Function $I_\nu(z)$ is called the modified Bessel function of the first kind and has numerous applications in many natural and technical sciences (especially in physics and mechanics). Zeros of conventional and modified Bessel functions of the first kind $J_\nu(z)$ and $I_\nu(z)$, as well as the functions themselves, have numerous applications in the problems of physics,

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mechanics, etc. (see [2, 3, 4]). It should be noted that the zeros of Bessel functions have been studied in more detail in case they are considered as functions of their arguments, i.e. with a fixed index (see [5, 6, 7, 8, 9, 10] and the references therein). The situation is different when Bessel functions are considered as functions of the index for a fixed argument. In this direction, we note the work [11], in which it is shown that for the positive z zeros ν_k of the Bessel function of the first kind $J_\nu(z)$ is real, simple and asymptotically close to the negative integers. A similar problem for the modified Bessel function of the second kind was studied in the works [12, 13, 14, 15]. In the works [1, 2] it was shown that the modified function $I_\nu(z)$ of the first kind, as a function of order ν, does not have zeros in the right half-plane. The analysis of ν-zeros of the function $I_\nu(z)$ in the left half-plane is also of interest. The last question, as far as we know, has not yet been studied.

In this paper, the distribution of ν-zeros function $I_\nu(z)$ in the left half-plane is studied. The main result of this work is the following theorem.

Theorem 1. *For each fixed $z > 0$, the function $I_\nu(z)$ outside the band $|Im\nu| < \varepsilon$, $\varepsilon > 0$ can only have a finite number of $\nu-$zeros. The real zeros of the function $I_\nu(z)$ can only be located in the intervals $(-2k, -(2k-1))$, $k = 1, 2, ...,$ and for large values of k, the function $I_\nu(z)$ has exactly two zeros in the interval $(-2k, -(2k-1))$, $k = 1, 2, ...$*

2. Proof of Theorem 1

Consider the equation (1). If we put $z = e^{c-x}$, $y(x) = u(e^{c-x})$, $\nu = i\lambda$, where c is any finite number, then equation (1) will take the form

$$-y'' + q(x)y = \lambda^2 y,\ q(x) = e^{2(c-x)}. \tag{3}$$

Consider the boundary problem, generated on the half-axis by $0 \le x < \infty$, differential equation (3) and a boundary condition

$$y(0) = 0. \tag{4}$$

From the above substitutions it follows that the function

$$f(x,\lambda) = I_{-i\lambda}\left(e^{c-x}\right) \tag{5}$$

is a solution of the equation (3). It is known that $I_{-i\lambda}(z)$ for a fixed $z > 0$ is an entire function of λ. Then, by (5), with each fixed x, $0 \le x < +\infty$, the solution $f(x,\lambda)$ of the equation (3) is as an entire function with respect to λ. From the known relation [16]

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \Gamma^{-1}(\nu+1)(1+o(1)),\ z \to 0,$$

it follows that

$$f(x,\lambda) = 2^{i\lambda} e^{-ic\lambda} e^{i\lambda x} \Gamma^{-1}(1-i\lambda)(1+o(1)),\ x \to +\infty. \tag{6}$$

On the other hand, as is known [17], the equation (3) has a unique solution $e(x,\lambda)$ with asymptotics $e(x,\lambda) = e^{i\lambda x}(1+o(1)),\ x \to +\infty$. Moreover, the following triangular representation is true:

$$e(x,\lambda) = e^{i\lambda x} + \int_x^{+\infty} K(x,t)\, e^{i\lambda t} dt, \tag{7}$$

where the kernel $K(x,t)$ is a continuously differentiable function and satisfies the relation

$$K(x,t) = O\left(e^{2c-x-t}\right),\ x+t \to +\infty. \tag{8}$$

From (5) - (8) it follows that

$$e(x,\lambda) = 2^{-i\lambda} e^{ic\lambda} \Gamma(1-i\lambda)\, I_{-i\lambda}\left(e^{c-x}\right). \tag{9}$$

According to the general theory (see [17]), the boundary problem (3) - (4) has a continuous spectrum that fills the half-axis $[0,+\infty)$. The eigenvalues of this problem coincide with the squares of the zeros of the function $e(0,\lambda)$, located in the upper half-plane. As is known [17], for $\lambda > 0$ the function$e(0,\lambda)$ has no zeros. In addition, by virtue of (2), (9), we have $e(0,0) \neq 0$. Further, it is known [17] that in the general case, i.e. for the real potential $q(x)$ from the class $\int_0^{+\infty} x\,|q(x)|\,dx < \infty$, the function $e(0,\lambda)$ in the half-plane $Im\lambda > 0$ can only have a finite number of zeros λ_n, lying on the imaginary axis: $\lambda_n = i\theta_n$, $\theta_n > 0$. In this case, the numbers $-\theta_n^2$ are the eigenvalues of the boundary problem (3) - (4). However, due to the positivity of the potential $q(x) = e^{2(c-x)}$, the boundary problem (3) - (4) cannot have [17] negative eigenvalues. Consequently, $e(0,\lambda) \neq 0$ for $Im\lambda \geq 0$. Then, (9) implies that for all ν, $Re\nu \geq 0$, the relation $I_\nu(e^c) \neq 0$ holds.

Next, we replace in equation (3) x by $-ix$ and put

$$z(x) = y(-ix),\ \lambda = ik. \tag{10}$$

As a result, we obtain the equation

$$-z'' - e^{2c} e^{2ix} z = k^2 z. \tag{11}$$

As shown in the works [18, 19], the equation (11) has the following solution:

$$f(x,k) = e^{ikx}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n+2k} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right).$$

In this case, the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n+1}^{\infty}\alpha\left(\alpha-n\right)\left|V_{n\alpha}\right|,\ \sum_{n=1}^{\infty}n\left|V_{n\alpha}\right|,\ \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\left|V_{n\alpha}\right|$$

converge. Moreover, in the case of equation (3) for odd n the equality $V_{n\alpha}=0,\ \alpha\geq n$ is valid. Then from (10), (11) it follows that the solution $e\left(x,\lambda\right)$ of equation (3) admits the representation

$$e\left(x,\lambda\right)=e^{i\lambda x}\left(1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n-i\lambda}\sum_{\alpha=n}^{\infty}U_{n\alpha}e^{i\alpha x}\right), \tag{12}$$

where $U_{n\alpha}=V_{2n,\alpha}$. Consequently, the function $e\left(x,\lambda\right)$ for each $x\geq 0$ is meromorphic with respect to λ, the poles of which are located at points $\lambda=-in, n\geq 1$. In addition, formulas (9), (12) show that with each real value of c zeros of the function $I_{\nu}\left(e^{c}\right)$ coincide with zeros of the meromorphic function

$$g\left(\nu\right)=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{A_n}{n+\nu},\ \ A_n=\sum_{\alpha=n}^{\infty}U_{n\alpha}. \tag{13}$$

Let $\nu=-\mu+i\eta,\mu>0$ be a zero of the function $g\left(\nu\right)$. Then from (13) we obtain

$$1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{n-\mu}{\left(n-\mu\right)^2+\eta^2}A_n=0, \tag{14}$$

$$\eta\sum_{n=1}^{\infty}\frac{1}{\left(n-\mu\right)^2+\eta^2}A_n=0. \tag{15}$$

Note that

$$\left|\sum_{n=1}^{\infty}\frac{n-\mu}{\left(n-\mu\right)^2+\eta^2}A_n\right|\leq\sum_{n=1}^{\infty}\frac{\left|n-\mu\right|}{2\left|n-\mu\right|\cdot\left|\eta\right|}\left|A_n\right|\leq\frac{1}{\left|\eta\right|}\sum_{n=1}^{\infty}\left|A_n\right|\to 0,\eta\to\infty. \tag{16}$$

It follows that the zeros of the functions $g\left(\nu\right)$ can only be located in some band $\left|Im\nu\right|=\left|\eta\right|<L$. Suppose that ε is an arbitrary positive number and $\left|\eta\right|\geq\varepsilon$. Then we have

$$\begin{aligned}&\left|\sum_{n=1}^{\infty}\tfrac{n-\mu}{(n-\mu)^2+\eta^2}A_n\right|\leq\sum_{n=1}^{\infty}\tfrac{|n-\mu|}{(n-\mu)^2+\eta^2}\left|A_n\right|=\sum_{n=1}^{\left[\frac{\mu}{2}\right]}\tfrac{|n-\mu|}{(n-\mu)^2+\eta^2}\left|A_n\right|+\\&+\sum_{n=\left[\frac{\mu}{2}\right]+1}^{\infty}\tfrac{|n-\mu|}{(n-\mu)^2+\eta^2}\left|A_n\right|\leq\sum_{n=1}^{\left[\frac{\mu}{2}\right]}\tfrac{\mu}{\left(\frac{\mu}{2}\right)^2+\eta^2}\left|A_n\right|+\sum_{n=\left[\frac{\mu}{2}\right]+1}^{\infty}\tfrac{|n-\mu|}{2|\eta|\cdot|n-\mu|}\left|A_n\right|\leq\\&\leq\tfrac{4}{\mu}\sum_{n=1}^{\infty}\left|A_n\right|+\tfrac{1}{2\varepsilon}\sum_{n=\left[\frac{\mu}{2}\right]+1}^{\infty}\left|A_n\right|\to 0,\mu\to\infty.\end{aligned} \tag{17}$$

By virtue of (17), outside any band $|Im\nu| < \varepsilon$, the function $g(\nu)$ can only have a finite number of zeros.

Finally assume that $v > 0$ and $I_{-\nu}(e^c) = 0$. If ν takes integer values, then $I_{-\nu}(e^c) = I_\nu(e^c) \neq 0$. With the remaining positive values of ν from the formula $K_\nu(e^c) = \frac{\pi}{2}\frac{I_{-\nu}(e^c)-I_\nu(e^c)}{\sin\nu\pi}$ (see [16]) it follows that

$$\sin\nu\pi = -\frac{\pi}{2}\frac{I_\nu(e^c)}{K_\nu(e^c)}.$$

Considering that [16] $\frac{I_\nu(e^c)}{K_\nu(e^c)} > 0$ for $\nu > 0$, we find from the last equation that the zeros of the functions $I_\nu(e^c)$ can be located in the interval $(-2k, -(2k-1))$, $k = 1, 2, ...$

Further, we use asymptotic formulas [5]

$$I_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \exp\left(\frac{z^2}{4\nu}\right), \tag{18}$$

$$K_\nu(z) \sim \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu} \Gamma(\nu) \exp\left(-\frac{z^2}{4\nu}\right), \tag{19}$$

true for $|z| \leq C\nu^{\frac{1}{2}}$, $|\arg z| \leq \frac{\pi}{2}, \nu \to \infty$, where $C > 0$ is some constant. On the other hand, for $|\arg\nu| \leq \pi - \varepsilon, \varepsilon > 0$, $\nu \to \infty$ the asymptotic formula

$$\Gamma(\nu+b) \sim \sqrt{2\pi}\exp(-\nu)\, v^{\nu+b-\frac{1}{2}} \tag{20}$$

holds. The last three relations lead us to the approximate equality

$$\sin\nu\pi = -\frac{1}{2}\left(\frac{ez}{2\nu}\right)^{2\nu}.$$

Since $\left(\frac{ez}{2\nu}\right)^{2\nu}$ strictly decreases for large values of ν, from the last equation it follows that for large values k, the function $I_\nu(z)$ has exactly two zeros in the interval $(-2k, -(2k-1))$.

Thus, the proof of the theorem is completed.

Note. As Hurwitz [5] and McDonald [6] showed, if $\nu < -1$ and $2k+1 < -\nu < 2k+2$, where $k \geq 0$ is an integer, then $I_\nu(z)$ as a function of z, has $4k+2$ zeros, two of which lie on the real axis. Thus, the results of this work are consistent with the results of [5, 6].

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