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Some Generalizations of the Riemann Operator Method

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Abstract. The generalized Bergman [3] and Vekua [9] complex operators are a single apparatus for constructing solutions of various linear partial differential equations with three independent variables, that associate holomorphic functions with two complex variables and solutions of these equations.
Integral operators are constructed that convert arbitrary analytic functions into regular solutions of partial differential equations of different types (elliptical, parabolic, hyperbolic) in three-dimensional case. A method for finding an integral representation of solutions of iterative partial differential equations of different types is developed. As an example, the Cauchy problem for the Helmholtz equation of 4th order is solved.

Key Words and Phrases: differential equation, Riemann operator method, Cauchy problem, Helmholtz equation.

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1. Introduction

The theory of analytic functions is a developed branch of analysis, and the Riemann operator method makes it possible to use it to study differential equations. Riemann's method is reduced to the derivation of an integral formula that explicitly expresses the desired solution of the Cauchy problem through the initial data and at the same time directly proves the uniqueness of the solution.

To construct the theory of solvability of boundary value problems for a second order equation of elliptic type

$$\Delta U + a_1(x,y)\frac{\partial U}{\partial x} + a_2(x,y)\frac{\partial U}{\partial y} + a_3(x,y)U = 0, \qquad (*)$$

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where a_1, a_2, a_3 are analytic functions, I.N.Vekua [9] essentially applied the following integral representation for all regular solutions of equation (*):

$$U(x,y) = Re\left[\alpha\left(z,\bar{z}\right)\varphi(z) + \int\limits_0^z \beta\left(z,\bar{z},t\right)\varphi(t)dt\right], \qquad (**)$$

where $\varphi(z)$ is an arbitrary analytic function, α and β are functions (each of them is a so-called Riemann function) with the coefficients a_1, a_2, a_3. The method of constructing the formula (**), called the Riemann-Vekua method, is the simplest, clearest and most constructive one.

A generalization of the Riemann operator method is proposed, by means of which linear equations of stationary and nonstationary processes can be considered from a single position. The study of the properties of regular solutions and the use of the apparatus of special functions made it possible to obtain a representation for the solutions of the equations in a form convenient for research. The generalization will be based on the Helmholtz equation.

2. Main results

Partial differential equations containing differential operators of the form

$$L_S = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + S$$

play an important role in the mathematical modeling of various processes. In particular, they are used in modeling of diffusion processes, as well as biological and environmental phenomena.

Methods for solving such equations involve the creation of integral and differential operators that determine the solution of equations of elliptic and hyperbolic types [1, 2].

As is known, the solution of the Cauchy problem for an equation of elliptic type with analytic coefficients exists and it is unique [4, 5].

Let D be a simply connected domain in the plane $z = x+iy$, that is symmetric with respect to the real axis, and let $f(z)$ and $g(z)$ be arbitrary analytic functions in D. Then the function $U(x,y)$ defined by equality [10]

$$\begin{aligned} U(x,y) = \frac{1}{2}\left[f(z) + f(\bar{z})\right] + \frac{\alpha\left(z-\bar{z}\right)}{8}\int\limits_{\bar{z}}^{z} f(\sigma)\frac{J_1\left(\frac{\alpha}{2}\sqrt{\left(z-\sigma\right)\left(\bar{z}-\sigma\right)}\right)}{\sqrt{\left(z-\sigma\right)\left(\bar{z}-\sigma\right)}}d\sigma + \\ + \frac{1}{2i}\int\limits_{\bar{z}}^{z} g(\sigma)J_0\left(\frac{\alpha}{2}\sqrt{\left(z-\sigma\right)\left(\bar{z}-\sigma\right)}\right)d\sigma, \end{aligned} \qquad (1)$$

where the integration from point $\bar{z}$ to z is carried out along any directional contour L in D, and $J_\nu(z)$ is the Bessel function, is a regular solution in D of the following Cauchy problem:

$$\begin{gathered}\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\alpha^2}{4}U=0, \quad \alpha - const > 0,\\ U\big|_{y=0}=f(x), \quad \frac{\partial U}{\partial y}\big|_{y=0}=g(x).\end{gathered} \tag{2}$$

Due to the uniqueness of the solution of the Cauchy problem, formula (1) is a general integral representation for all regular solutions of the Helmholtz equation (2) in D through two arbitrary analytic functions in this domain. That is, formula (1) establishes a one-to-one correspondence between the regular solutions of equation (2) and the analytic functions in D.

Let G be an arbitrary star domain with respect to $z=0$, $z^*\in G^*=\{x-iy|x+iy\in G\}$, τ be a real or complex variable, $\tau\in T$, $f(z,\tau)$ be holomorphic in G and continuous in $\overline{G}$ function. Consider the differential equation of the form

$$L_S U=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+S\right)U=0, \tag{3}$$

where $U=U(x,y,\tau)$ and S is a linear operator depending only on τ.

In accordance with the above statement (1) the integral representation for the solutions of equation (3) will be sought in the form

$$\begin{gathered}U\left(z,z^*,\tau\right)=\frac{1}{2}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}\right)\int\limits_{z^*}^{z} f(\sigma)J_0\left(\sqrt{S(z-\sigma)\left(z^*-\sigma\right)}\right)d\sigma+\\ +\frac{1}{2i}\int\limits_{z^*}^{z} g(\sigma)J_0\left(\sqrt{S(z-\sigma)\left(z^*-\sigma\right)}\right)d\sigma.\end{gathered} \tag{4$'$}$$

Let in (4′) $z^*=\bar{z}$, $\sigma=r\cos\theta+ir\sin\theta\cos t=x+iy\cos t$, $(z-\sigma)\left(z^*-\sigma\right)=y^2\sin^2 t$. Then

$$\begin{gathered}U(x,y,\tau)=\frac{1}{2}\frac{\partial}{\partial y}\int\limits_{0}^{\pi} f(x+iy\cos t,\tau)J_0\left(\sqrt{S}y\sin t\right)y\sin t dt+\\ +\frac{1}{2i}\int\limits_{0}^{\pi} g(x+iy\cos t,\tau)J_0\left(\sqrt{S}y\sin t\right)y\sin t dt.\end{gathered} \tag{5}$$

Rewrite (5) in the form

$$\begin{gathered}U(x,y,\tau)=\frac{1}{2}\frac{\partial}{\partial y}\int\limits_{0}^{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}\sin^{2n+1}t}{(n!)^2 2^{2n}}S^n f(x+iy\cos t,\tau)dt+\\ +\frac{1}{2i}\int\limits_{0}^{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{(n!)^2 2^{2n}}\sin^{2n+1}t\ S^n g(x+iy\cos t,\tau)dt\end{gathered} \tag{6}$$

or

$$U_1(z,\bar{z},\tau)=\frac{1}{2}\frac{\partial}{\partial y}\int\limits_0^{\pi} Ef(x+iy\cos t,\tau)dt, \tag{6'}$$

$$U_2(z,\bar{z},\tau)=\frac{1}{2i}\int\limits_0^{\pi} Eg(x+iy\cos t,\tau)dt, \tag{6''}$$

where $E=\sum\limits_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}\sin^{2n+1}t}{(n!)^2 2^{2n}}S^n$.

Let the operator S and the functions f and g be such that the series under the integral (6′), (6″) is uniformly convergent $\forall z=x+iy\in G$, $\tau\in T_0\subset T$. Prove that $U_1(z,\bar{z},\tau)$ satisfies equation (3). Substitute formula (6’) into (3).

Consider the operator $\Delta U_1=\frac{\partial^2 U_1}{\partial x^2}+\frac{\partial^2 U_1}{\partial y^2}$

$$\Delta U_1=\frac{1}{2}\frac{\partial}{\partial y}\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\int\limits_0^{\pi} y^{2n+1}\sin^{2n+1}tf(x+iy\cos t,\tau)dt=$$
$$=\frac{1}{2}\frac{\partial}{\partial y}\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\int\limits_0^{\pi}\big(y^{2n+1}f''\sin^2 t+(2n+2)2ny^{2n-1}f+$$
$$+2(2n+1)y^{2n}f'i\cos t\big)\sin^{2n+1}tdt.$$

After transformations under the sign of the sum, we obtain

$$\Delta U_1=\frac{1}{2}\frac{\partial}{\partial y}\int\limits_0^{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}y^{2n+2}\sin^{2n+1}t}{(n!)^2 2^{2n}}S^{n+1}fdt=-SU_1.$$

In the same way $\Delta U_2=-SU_2$.

Therefore, formula (4′) is true. Thus the following theorem is proved:

Theorem 1. *For all functions $f(z,\tau)$, $g(z,\tau)$ holomorphic in G and continuous in $\overline{G}$ such that the series in (6) is uniformly convergent $\forall z\in G$, $\tau\subset T_0\subset T$,*

$$\begin{aligned}U(z,\bar{z},\tau)=&\frac{1}{2}\frac{\partial}{\partial y}\int\limits_0^{\pi} f(x+iy\cos t,\tau)J_0\left(\sqrt{S}y\sin t\right)y\sin tdt+\\&+\frac{1}{2i}\int\limits_0^{\pi} g(x+iy\cos t,\tau)J_0\left(\sqrt{S}y\sin t\right)y\sin tdt\end{aligned} \tag{4}$$

is a solution of equation (3) $\forall\tau\in T_0$, z, $\bar{z}$ from the neighborhood of $z=0$, $\bar{z}=0$.

Remark 1. *We will consider further $g(z)=0$.*

The integral equation (4) with the Bessel function in the kernel can be solved. For this purpose we consider domains of two types:

a) domain G, which is symmetric with respect to the real axis and contains a complete segment connecting any two of its points with the same abscissa;

b) domain G which contains a complete segment of a line drawn from an infinitely distant point to any point parallel to the axis OY.

In the class of domains where f takes on real values, formula (4) will look like

$$U(x,y,\tau) = \frac{\partial}{\partial y}\int\limits_0^y Re\ f(x+i\xi,\tau)J_0\left(\sqrt{S\left(y^2-\xi^2\right)}\right)d\xi, \qquad (7)$$

$$U(x,y,\tau) = -\frac{\partial}{\partial y}\int\limits_y^\infty Re\ f(x+i\xi,\tau)I_0\left(\sqrt{S\left(\xi^2-y^2\right)}\right)d\xi, \qquad (8)$$

where

$$f(z,\tau)\cos\left(\sqrt{S}z\right)z^{-\frac{1}{2}} = O\left(\frac{1}{|z|^\varepsilon}\right)\ as\ |z| \to \infty. \qquad (9)$$

The integral operators (7), (8) map the analytic function $f(z,\tau)$ in G into the solutions of equation (3).

Considering equations (7), (8) as integral convolution-type equations with a Bessel function in the kernel, we obtain their solutions according to [7].

$$Re\, f(x+iy,\tau) = \frac{\partial}{\partial y}\int\limits_0^y U(x,\xi,\tau)I_0\left(\sqrt{S\left(y^2-\xi^2\right)}\right)d\xi, \qquad (7')$$

$$Re\, f(x+iy,\tau) = -\frac{\partial}{\partial y}\int\limits_y^\infty U(x,\xi,\tau)J_0\left(\sqrt{S\left(\xi^2-y^2\right)}\right)d\xi. \qquad (8')$$

1^0. If $S = \dfrac{\alpha^2}{4}$, then equation (3) becomes the Helmholtz equation (2), and the integral image (4) is the integral reflection of the solutions of the Helmholtz equation. That is, we come to formula (1).

The obtained formulas of inversion (7′) and (8′) allow to reduce the boundary value problems for the Helmholtz equation to the corresponding boundary-value problems for analytic functions.

Problem. In the right half-plane $z = x + iy$, find the regular solution of equation (2) with the assumption

$$U(x,y)\big|_{x=0} = \Phi(y), \quad -\infty < y < \infty.$$

In this case $\Phi(y)$ is an even continuous function. We do not formulate any assumptions about the behavior of the function $U(x,y)$ as $z \to \infty$ in advance.

The solution of the problem, as shown by formula (7), is sought in the form

$$U(x,y) = \varphi(x,y) - \frac{\alpha}{2} y \int\limits_0^y \varphi(x,\xi) \frac{J_1\left(\frac{\alpha}{2}\sqrt{y^2-\xi^2}\right)}{\sqrt{y^2-\xi^2}} d\xi, \quad \varphi = Re\, f(z). \tag{10}$$

By the formula of inversion (7′), for $-\infty < y < \infty$ we obtain

$$\varphi(0,y) = \Phi(y) + \frac{\alpha}{2} y \int\limits_0^y \Phi(\xi) \frac{I_1\left(\frac{\alpha}{2}\sqrt{y^2-\xi^2}\right)}{\sqrt{y^2-\xi^2}} d\xi. \tag{11}$$

We assume that the function $\varphi(0,y)$ on the entire y-axis is correctly continuous and in the neighborhood of an infinitely distant point satisfies the condition $H(v)$, that is

$$|\varphi(0,y_2) - \varphi(0,y_1)| \le A \left| \frac{1}{y_2} - \frac{1}{y_1} \right|^v, \quad (A = const > 0,\ v = const > 0)$$

with large enough $|y_1|$, $|y_2|$. Then the solution of the Dirichlet problem in the half-plane $x > 0$ will look like [6]

$$\varphi(x,y) = -Re \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{\varphi(0,\xi) - a}{\xi - (y - ix)} d\xi + a,$$

where $a = \varphi(0,\infty)$. Given that $\varphi(0,y) = \varphi(0,-y)$, the last equality can be rewritten in the form

$$\varphi(x,y) = \frac{x}{\pi} \int\limits_0^{\infty} \varphi^*(\xi) \left[\frac{1}{(\xi-y)^2 + x^2} + \frac{1}{(\xi+y)^2 + x^2} \right] d\xi + a, \tag{12}$$

where $\varphi^*(\xi) = \varphi(0,\xi) - a$. Substituting (12) into (10), we obtain the solution of the Dirichlet problem

$$U(x,y) = a \cos \frac{\alpha}{2} y + \frac{x}{\pi} \frac{\partial}{\partial y} \int\limits_0^{\infty} \varphi^*(t) dt \int\limits_0^y J_0\left(\frac{\alpha}{2}\sqrt{y^2-\xi^2}\right) \times$$
$$\times \left[\frac{1}{(t-\xi)^2 + x^2} + \frac{1}{(t+\xi)^2 + x^2} \right] d\xi.$$

3. Integral representation for solutions of parabolic equations

Let $S = -\left(a + b\dfrac{\partial}{\partial\tau}\right)$, where $a, b - const.$

Then equation (3) becomes an equation of parabolic type

$$U_{xx} + U_{yy} - bU_\tau - aU = 0. \tag{13}$$

The operator $Ef(z,\tau)$ takes the form

$$Ef(x+iy\cos t,\tau) = y\sin t\sum_{n,k=0}^{\infty}\left(\frac{y^2\sin^2 t}{2^2}\right)^{n+k}\frac{1}{((n+k)!)^2}C_{n+k}^k a^n b^k\frac{\partial^k f}{\partial\tau^k} =$$
$$= y\sin t\sum_{n,k=0}^{\infty}\left(\frac{y^2\sin^2 t}{2^2}\right)^{n+k}\frac{a^n b^k}{n!k!(n+k)!}\frac{k!}{2\pi i}\oint_K\frac{f(x+iy\cos t,\xi)}{(\xi-\tau)^{k+1}}d\xi =$$
$$= \frac{1}{2\pi i}\oint_K\frac{f(x+iy\cos t,\xi)}{\xi-\tau}y\sin t\sum_{n,k=0}^{\infty}\left(\frac{ay^2\sin^2 t}{2^2}\right)^n\frac{1}{n!}\frac{1}{(n+k)!}\times$$
$$\times\left(\frac{by^2\sin^2 t}{2^2}\right)^k\frac{d\xi}{(\xi-\tau)^k} = \frac{1}{2\pi i}\oint_K f(x+iy\cos t,\xi)H(x,y,\tau,\xi,t)\frac{d\xi}{\xi-\tau}.$$

Here K is a circle in T_0 centred at the point $\xi = \tau$,

$$H(x,y,\tau,\xi,t) = y\sin t\sum_{n,k=0}^{\infty}\frac{1}{n!(n+k)!}\left(a\left(\frac{y}{2}\right)^2\sin^2 t\right)^n\left(\frac{b\left(\frac{y}{2}\right)^2\sin^2 t}{\xi-\tau}\right)^k =$$
$$= \Phi_3\left(1, 1, \frac{b\left(\frac{y}{2}\right)^2\sin^2 t}{\xi-\tau}, a\left(\frac{y}{2}\right)^2\sin^2 t\right)y\sin^2 t,$$

where $\Phi_3(\beta,\gamma,\omega,z) = \sum\limits_{n,k=0}^{\infty}\dfrac{(\beta)_k}{(\gamma)_{n+k}}\dfrac{\omega^k z^n}{k!n!}$ is a degenerated hypergeometric function of two variables, and $(\beta)_k$, $(\gamma)_{n+k}$ are Pochhammer symbols.

The following theorem is proved.

Theorem 2. *For functions $f(z,\tau)$, that are holomorphic in G and continuous in $\overline{G}$, $\forall\tau\in T_0\subset T$ $U(x,y,\tau) = \dfrac{1}{2\pi i}\oint\limits_K\left(\dfrac{1}{2}\dfrac{\partial}{\partial y}\int\limits_0^\pi f(x+iy\cos t,\xi)H(x,y,\tau,\xi,t)dt\right)\dfrac{d\xi}{\xi-\tau}$ is a solution of equation (13).*

If $b = 0$, $a = -\dfrac{\alpha^2}{4}$, then

$$H(x,y,\tau,\xi,t) = \Phi_3\left(1,1,0,-\frac{\alpha^2}{4}\left(\frac{y}{2}\right)^2\sin^2 t\right) y\sin t =$$
$$= \sum_{n=0}^{\infty}\frac{(1)_0}{(1)_n}\left(-\frac{\alpha^2}{4}\left(\frac{y}{2}\right)^2\sin^2 t\right)^n\frac{1}{n!} = y\sin t\, J_0\left(\frac{\alpha}{2}y\sin t\right).$$

We come to equation (2) and to the known representation of its solutions – formula (10).

If $a = 0$ ($b \neq 0$), then we have the equation

$$U_{xx} + U_{yy} - bU_\tau = 0.$$

The integral representation of the solutions of this equation will be

$$U(x,y,\tau) = \frac{1}{2\pi i}\oint_K\left(\frac{1}{2}\frac{\partial}{\partial y}\int_0^\pi f(x+iy\cos t,\xi)y\sin t\,\ell^{\frac{by^2\sin^2 t}{4(\xi-\tau)}}\,dt\right)\frac{d\xi}{\xi-\tau}.$$

4. Integral representation of solutions of an equation of hyperbolic type

Let $S = -\left(b\frac{\partial}{\partial\tau}\right)^2$, where b is a constant.

The differential equation (3) takes the form

$$U_{xx} + U_{yy} - b^2U_{\tau\tau} = 0. \tag{14}$$

Equation (14) is an equation of hyperbolic type for $b \in R$ ($b \neq 0$) and an equation of elliptical type for b purely imaginary.

Let b be a real number. Then Ef will be as follows:

$$Ef(z,\tau) = \sum_{n=0}^{\infty}\frac{y^{2n+1}\sin^{2n+1}t\,b^n}{(n!)^2 2^{2n}}\frac{(2n)!}{2\pi i}\oint_K\frac{f(z,\xi)d\xi}{(\xi-\tau)^{2n+1}},$$

where K is a circle in T_0 centred at the point $\xi = \tau$.

Given that [8]

$$\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}\left(\frac{b\left(\frac{y}{2}\right)^2\sin^2 t}{(\xi-\tau)}\right)^n = \frac{1}{\sqrt{1-\frac{by^2\sin^2 y}{(\xi-\tau)^2}}}$$

with condition $-\dfrac{1}{4} \le \dfrac{b\left(\dfrac{y}{2}\right)^2 \sin^2 t}{(\xi-\tau)^2} \le \dfrac{1}{4}$ we have the following theorem.

Theorem 3. *For all functions $f(z,\tau)$ that are holomorphic in G and continuous in $\overline{G}$*

$$U(x,y,\tau) = \frac{1}{2\pi i} \oint\limits_K \left(\frac{1}{2} \frac{\partial}{\partial y} \int\limits_0^\pi f(x+iy\cos t, \xi) \frac{y \sin t dt}{\sqrt{(\xi-\tau)^2 - by^2 \sin^2 t}} \right) d\xi$$

is a solution of equation (14).

The statement of this theorem is obtained by direct verification.

Consider the differential equation of the form

$$L_S^n U = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + S \right)^n U = 0, \qquad (L_S^n)$$

where $U = U(x,y,\tau)$, S is a linear operator dependening only on $\tau \in T$.

Lemma 1. *If $U_r(z,\bar{z},\tau)$ is a $\left(r=\overline{0,n-1}\right) \in 2(r+1)$ times continuously differentiable solution of the equation (L_S), then the function defined by*

$$U(x,y,\tau) = \sum_{r=0}^{n-1} U_r(x,y,\tau) y^r$$

satisfies the equation (L_S^n).

Proof. We prove this lemma by the method of mathematical induction. Let's show that $L_S^2 U = 0$:

$$\begin{aligned} L_S^2 U &= L_S(L_S U) = L_S(L_S(U_0 + yU_1)) = L_S(L_S y U_1) = \\ &= L_S\left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + S \right) y U_1 \right) = 2\frac{\partial}{\partial y}(L_S U_1) = 0. \end{aligned}$$

Let the lemma be valid for $n-1$, that is $L_S^{n-1}\left(\sum\limits_{r=0}^{n-2} U_r y^r \right) = 0$.

Prove, that $L_S^n U = 0$.

$L_S^n U = L_S(L_S^{n-1} U) = L_S\left(L_S^{n-1}\left(\sum\limits_{r=0}^{n-2} U_r y^r + U_{n-1} y^{n-1} \right)\right) = L_S^n\left(y^{n-1} U_{n-1}\right).$

Next we prove that

$$L_S^n\left(y^{n-1} U_{n-1}\right) = 0. \qquad (15)$$

Let $U_{n-1} = \varphi$. Then $L_S^n\left(y^{n-1}\varphi\right) = L_S^{n-1}\left(L_S y^{n-1}\varphi\right)$. Consider

$$L_S\left(y^{n-1}\varphi\right) = y^{n-1}\left(\frac{\partial^2\varphi}{\partial x^2} + S\varphi\right) + \frac{\partial^2}{\partial y^2}\left(y^{n-1}\varphi\right) = \\ = y^{n-1}L_S\varphi + (n-1)(n-2)y^{n-3}\varphi + 2(n-1)y^{n-2}\frac{\partial\varphi}{\partial y}. \tag{16}$$

Equation (15) is also proved by the method of mathematical induction: When $n = 1$, under the condition of the lemma $L_S U_0 = 0$.

Assume that (15) is true for $r < n$, that is

$$L_S^r\left(y^{r-1}\varphi\right) = 0 \Rightarrow L_S^n\left(y^{r-1}\varphi\right) = 0 \quad r < n. \tag{17}$$

Actually, $L_S^n\left(y^{r-1}\varphi\right) = L_S^{n-r}\left(L_S^r\left(y^{r-1}\varphi\right)\right) = 0$.

Using (16), (17) we prove the validity of (15) for n, that is

$$L_S^n\left(y^{n-1}\varphi\right) = 2(n-1)L_S^{n-1}\left(y^{n-1}\frac{\partial\varphi}{\partial y}\right) + (n-1)(n-2)L_S^{n-1}\left(y^{n-3}\varphi\right) = 0. \blacktriangleleft$$

The above considerations prove the following theorem.

Theorem 4. *For all functions $f_r(z,\tau)$ $\left(r = \overline{0, n-1}\right)$ that are holomorphic in G and continuous in $\overline{G}$*

$$U(x,y,\tau) = \sum_{r=0}^{n-1} y^r \frac{1}{2}\frac{\partial}{\partial y}\int_0^\pi f_r(x + iy\cos t, \tau) J_0\left(\sqrt{S}y\sin t\right) y\sin t dt \tag{18}$$

is a solution of the equation (L_S^n) for arbitrary $\tau = T_0$, z, $\bar{z}$ from the neighborhood of $z = 0$, $\bar{z} = 0$.

4^0. $S = \dfrac{\alpha^2}{4}$, $n = 2$

5. Cauchy problem

In the domain $0 < x, y < \infty$, find four times continuously differentiable solution of the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\alpha^2}{4}\right)^2 U = 0, \tag{19}$$

that satisfies the conditions

$$\left.\frac{\partial^m U}{\partial y^m}\right|_{y=0} = f_m(x), \quad m = \overline{0,3}, \tag{20}$$

where $f_m(x)$ is a given function continuously differentiable a sufficient number of times.

We look for the solution of the problem in the form (18) for $n = 2$:

$$U(x, y) = U_0(x, y) + yU_1(x, y).$$

Since U_0 and U_1 satisfy equation (2), with the boundary conditions (20) fulfilled, we come to the corresponding boundary value problems for $U_0(x, y)$ and $U_1(x, y)$.

Finally, the solution of the Cauchy problem (19), (20) for the Helmholtz equation of fourth order will look like

$$\begin{aligned}
U(x, y) = {}& \frac{1}{2}\left[f_0(z) + f_0(\bar{z})\right] - \\
& - \frac{z - \bar{z}}{8} \int\limits_{\bar{z}}^{z} \left(\frac{\alpha^2}{4} f_0(\sigma) + f_0''(\sigma) + f_2(\sigma)\right) J_0\left(\frac{\alpha}{2}\sqrt{(z-\sigma)(\bar{z}-\sigma)}\right) d\sigma + \\
& + \frac{\alpha(z - \bar{z})}{8} \int\limits_{\bar{z}}^{z} f_0(\sigma) \frac{J_1\left(\frac{\alpha}{2}\sqrt{(z-\sigma)(\bar{z}-\sigma)}\right)}{\sqrt{(z-\sigma)(\bar{z}-\sigma)}} d\sigma - \frac{i}{2}\int\limits_{\bar{z}}^{z} f_1(\sigma) J_0\left(\frac{\alpha}{2}\sqrt{(z-\sigma)(\bar{z}-\sigma)}\right) d\sigma - \\
& - i\int\limits_{\bar{z}}^{z} \left(\frac{\alpha f_1(\sigma)}{8} + \frac{2}{\alpha} f_1''(\sigma) + \frac{2}{\alpha} f_3(\sigma)\right) \sqrt{(z-\sigma)(\bar{z}-\sigma)} J_1\left(\frac{\alpha}{2}\sqrt{(z-\sigma)(\bar{z}-\sigma)}\right) d\sigma.
\end{aligned}$$

5^0. $S = -\left(a + b\frac{\partial}{\partial \tau}\right)$. Based on the lemma and using Theorem 2, we come to the following theorem.

Theorem 5. *For all functions $f_r(z, \tau)$ $\left(r = \overline{0, n-1}\right)$ that are holomorphic in G and continuous in $\overline{G}$*

$$U(x, y, \tau) = \sum_{r=0}^{n-1} y^r \frac{1}{2\pi i} \oint\limits_{K} \left(\frac{1}{2}\frac{\partial}{\partial y} \int\limits_{0}^{\pi} f_r(x + iy\cos t, \xi) H(x, y, \tau, \xi, t) dt\right) \frac{d\xi}{\xi - \tau}$$

is a solution of the n-th order equation of parabolic type

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} - b\frac{\partial}{\partial \tau} - a\right)^n U = 0$$

for arbitrary $\tau \in T_0$, z, $\bar{z}$ from the neighborhood of $z = 0$, $\bar{z} = 0$.

6^0. Based on the lemma and using Theorem 3, we have the following statement:

Theorem 6. *For all functions* $f_r(z,\tau)$ $\left(r=\overline{0,n-1}\right)$ *that are holomorphic in* G *and continuous in* $\overline{G}$

$$U(x,y,\tau)=\sum_{r=0}^{n-1} y^r \frac{1}{2\pi i}\oint_K \left(\frac{1}{2}\frac{\partial}{\partial y}\int_0^{\pi} f_r(x+iy\cos t,\xi)\frac{y\sin t dt}{\sqrt{(\xi-\tau)^2-by^2\sin^2 t}}\right)d\xi$$

is a solution of the n*-th order equation of hyperbolic type*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial t^2}-b^2\frac{\partial^2}{\partial \tau^2}\right)^n U=0$$

for arbitrary $\tau\in T_0$, z, $\bar{z}$ *from the neighborhood of* $z=0$, $\bar{z}=0$.

6. Conclusion

New representations for the solutions of some iterative equations of elliptic, parabolic and hyperbolic types are obtained. The generalization of the Riemann operator method allows studying iterative equations of stationary and nonstationary processes from a single position.

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