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Elliptic Systems in Generalized Morrey Spaces

D.K. Palagachev, L.G. Softova*

Abstract. We obtain local regularity in generalized Morrey spaces for the strong solutions to $2b$-order linear elliptic systems with discontinuous coefficients.

Key Words and Phrases: generalized Morrey spaces, elliptic systems, VMO, a priori estimates.

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1. Introduction

We obtain local regularity result for the following uniformly elliptic systems with bounded and discontinuous coefficients

$$\mathfrak{L}(x,D)\mathbf{u} := \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x) D^\alpha \mathbf{u}(x) = \mathbf{f}(x).$$

In our previous papers [11, 13, 14] a *Calderón–Zygmund* type theory has been developed for linear and quasi-linear elliptic and parabolic systems in the framework of the classical Morrey spaces $L^{p,\lambda}$, assuming the principal coefficients of the operator to be essentially bounded functions of *vanishing mean oscillation (VMO).* On the other hand, in the recent years an exhaustive Calderón–Zygmund theory has been elaborated both for elliptic and parabolic equations/systems in *divergence form* with VMO-coefficients in the framework of the *generalized Morrey spaces* $L^{p,\omega}$ (cf. [5, 6] and the survey [4]). This last generalization of the spaces allows finer control on the local oscillation properties of a function near its singular points and that is why regularity results in $L^{p,\omega}$ of solutions to PDEs with discontinuous coefficients are of great importance in the applications to differential geometry, stochastic control, nonlinear optimization, adaptive discontinuous

*Corresponding author.

Galerkin FEMs, etc. As it concerns regularity results in other function spaces, we can mention also the recent results [1, 2] that consider linear higher order elliptic equations in Grand Lebesgue spaces.

In the present work we are going to extend the results obtained in [13, 15, 18] to uniformly elliptic systems with discontinuous coefficients in the framework of generalized Morrey spaces.

In what follows we use the standard notation:

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $r > 0$ and $\mathcal{B}_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$.
- $\Omega \subset \mathbb{R}^n, n \geq 3$, is a bounded domain, $|\Omega|$ is the Lebesgue measure of Ω, $\Omega_r(x) = \Omega \cap \mathcal{B}_r(x)$.
- $\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : \ |\xi| = 1\}$ is the unit sphere in $\mathbb{R}^n$;
- $\mathcal{M}^{m\times m}$ is the set of $m \times m$-matrices.
- For $\mathbf{u} = (u^1, \dots, u^m) : \Omega \to \mathbb{R}^m$ we write $|\mathbf{u}|^2 = \sum_{j\leq m} |u^j|^2$.
- For any function f and any domain D with $f : D \to \mathbb{R}$ we write

$$f_D = \fint_D f(y)dy = \frac{1}{|D|} \int_D f(y)\, dy,$$
$$\|f\|^p_{p,D} = \|f\|^p_{L^p(D)} = \int_D |f(y)|^p\, dy.$$

- For $\mathbf{u} \in L^p(\Omega; \mathbb{R}^m)$ we write $\|\mathbf{u}\|_{p,\Omega}$ instead of $\|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^m)}$.

Throughout this paper, the standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various constants and may change from one occurrence to another.

2. Definitions and preliminary results

We are interested in operators with *discontinuous coefficients* a^{jk}_α belonging to the Sarason function class *VMO*.

Definition 1. *For $a \in L^1_{\rm loc}(\mathbb{R}^n)$ and any $R > 0$ set*

$$\gamma_a(R) := \sup_{\mathcal{B}_r, r\leq R} \frac{1}{|\mathcal{B}_r|} \int_{\mathcal{B}_r} |a(y) - a_{\mathcal{B}_r}|\, dy,$$

where $\mathcal{B}_r$ is any ball in $\mathbb{R}^n$. We say that

- *$a \in BMO$ if*

$$\|a\|_* = \sup_{R>0} \gamma_a(R) < \infty;$$

- *$a \in VMO$ with VMO-modulus γ_a if $a \in BMO$ and*

$$\lim_{R\to 0} \gamma_a(R) = 0.$$

For a matrix-valued function $\mathcal{A} \in \mathcal{M}^{m\times m}$ with entries $a^{jk} \in VMO$ we define the VMO-modulus of $\mathcal{A}$ as $\gamma_{\mathcal{A}} = \sum_{j,k=1}^m \gamma_{a^{jk}}$.

We call *weight* a measurable function $\omega\colon\ \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ and for any ball $\mathcal{B}_r(x)$ we write $\omega(x,r)$ instead of $\omega(\mathcal{B}_r(x))$. In addition we assume that there exist positive constants κ_1, κ_2 and κ_3 such that

$$\begin{aligned}
&\kappa_1 < \frac{\omega(x_0,s)}{\omega(x_0,r)} < \kappa_2 \quad \forall\ 0 < r \le s \le 2r, \quad x_0 \in \mathbb{R}^n;\\
&\int_r^\infty \frac{\omega(x_0,s)}{s^{n+1}}\,ds \le \kappa_3 \frac{\omega(x_0,r)}{r^n}.
\end{aligned} \tag{1}$$

Definition 2 ([12])**.** *A function $f \in L^p(\Omega)$ with $1 \le p < \infty$ belongs to the generalized Morrey space $L^{p,\omega}(\Omega)$ if the following norm is finite:*

$$\|f\|_{p,\omega;\Omega} = \left(\sup_{\mathcal{B}_r(x)} \frac{1}{\omega(x,r)} \int_{\Omega_r(x)} |f(y)|^p\,dy \right)^{1/p},$$

where the supremo is taken over all balls centered at $x \in \Omega$ and of radius $r \in$ $(0, \operatorname{diam}\Omega]$.

The generalized Sobolev–Morrey space $W^{2b}_{p,\omega}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ with generalized derivatives $D^\alpha u$, $|\alpha| \le 2b$, belonging to $L^{p,\omega}(\Omega)$ and endowed with the norm

$$\|u\|_{W^{2b}_{p,\omega}(\Omega)} = \sum_{s=0}^{2b} \sum_{|\alpha|=s} \|D^\alpha u\|_{p,\omega;\Omega}.$$

Analogously, $\mathbf{u} = \big(u_1, \dots, u_m\big) \in W^{2b}_{p,\omega}(\Omega;\mathbb{R}^m)$ means $u_k \in W^{2b}_{p,\omega}(\Omega)$ and the norm $\|\mathbf{u}\|_{W^{2b}_{p,\omega}(\Omega;\mathbb{R}^m)}$ is given by $\sum_{k=1}^m \|u_k\|_{W^{2b}_{p,\omega}(\Omega)}$.

Remark 1. It is clear that if $\omega(x,r) = r^\lambda$ with $\lambda \in (0,n)$, then $L^{p,\omega}$ gives rise to the classical Morrey space $L^{p,\lambda}$, while $L^{p,1} \equiv L^p$ and $W^{2b}_{p,1}$ reduces to the classical parabolic Sobolev space W^{2b}_p (cf. [14]) when $\omega \equiv 1$.

In what follows, we will use also a localized version $W^{2b}_{p,\omega,\mathrm{loc}}(\Omega;\mathbb{R}^m)$ of $W^{2b}_{p,\omega}(\Omega;\mathbb{R}^m)$, consisting of all functions $\mathbf{u}$ that belong to $\mathbf{u} \in W^{2b}_{p,\omega}(\Omega';\mathbb{R}^m)$ for each $\Omega' \Subset \Omega$.

Definition 3. *Let* $\mathcal{K}(x;\xi)\colon\ \mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})\to\mathbb{R}$ *be a variable Calderón–Zygmund kernel, i.e.*

1. *for each fixed* $x\in\mathbb{R}^n$, $\mathcal{K}(x;\cdot)$ *is a Calderón–Zygmund kernel:*

   (a) $\mathcal{K}(x;\cdot)\in C^\infty(\mathbb{R}^n\setminus\{0\})$

   (b) $\mathcal{K}(x;\mu\xi)=\mu^{-n}\mathcal{K}(x,\xi)\qquad\forall\,\mu>0$

   (c) $\displaystyle\int_{\mathbb{S}^{n-1}}\mathcal{K}(x;\xi)\,d\sigma_\xi=0\qquad\int_{\mathbb{S}^{n-1}}|\mathcal{K}(x;\xi)|\,d\sigma_\xi<\infty;$

2. *for every multi-index* β : $\sup_{\xi\in\mathbb{S}^{n-1}}|D_\xi^\beta\mathcal{K}(x;\xi)|\le C(\beta)$ *independently of* x, *where* $\mathbb{S}^{n-1}$ *is the unit sphere in* $\mathbb{R}^n$.

Given a function $f\in L^1(\Omega)$, define the singular integral operator

$$\mathfrak{K}f(x):=P.V.\int_{\mathbb{R}^n}\mathcal{K}(x;x-y)f(y)\,dy$$

and its commutator with multiplication by a function $a\in L^\infty(\mathbb{R}^n)$ as

$$\begin{aligned}\mathfrak{C}[a,f](x):=&P.V.\int_{\mathbb{R}^n}\mathcal{K}(x;x-y)[a(y)-a(x)]f(y)\,dy\\ =&\,\mathfrak{K}(af)(x)-a(x)\mathfrak{K}f(x).\end{aligned}$$

The L^p and $L^{p,\omega}$-boundedness of the operators $\mathfrak{K}$ and $\mathfrak{C}$ have been obtained in [3, 10] and [16, 17], respectively. For the sake of completeness, we summarize these results here.

Proposition 1. *Let* ω *be a weight satisfying* (1) *and* $f\in L^{p,\omega}(\Omega)$ *with* $p\in(1,\infty)$. *Then there exists a positive constant* $C=C(p,\omega,\mathcal{K})$ *such that*

$$\|\mathfrak{K}f\|_{p,\omega;\Omega}\le C\|f\|_{p,\omega;\Omega},\qquad\|\mathfrak{C}[a,f]\|_{p,\omega;\Omega}\le C\|a\|_*\|f\|_{p,\omega;\Omega}.$$

In addition, if $a\in VMO$, *then for each* $\varepsilon>0$ *there exists* $r_0=r_0(\varepsilon,\gamma_a)>0$ *such that for any* $r\in(0,r_0)$ *and any ball* $\mathcal{B}_r$ *the following inequality holds:*

$$\|\mathfrak{C}[a,f]\|_{p,\omega;\mathcal{B}_r}\le C\varepsilon\|f\|_{p,\omega;\Omega}.$$

3. Statement of the problem

Hereafter $\mathbf{u}\colon\ \Omega\to\mathbb{R}^m$, $m\ge1$, stands for the unknown function, $\mathbf{f}=(f_1,\dots,f_m)\colon\ \Omega\to\mathbb{R}^m$ is a given vector-valued function and the coefficient matrix $\mathbf{A}_\alpha(x)\in\mathcal{M}^{m\times m}$ has entries $\{a_\alpha^{jk}\}_{j,k=1}^m$, $a_\alpha^{jk}\colon\ \Omega\to\mathbb{R}$, which are measurable functions. Fixed an integer $b\ge1$, we deal with the $2b$-order linear system

$$\mathfrak{L}(x,D)\mathbf{u}:=\sum_{|\alpha|=2b}\mathbf{A}_\alpha(x)D^\alpha\mathbf{u}(x)=\mathbf{f}(x)\qquad\text{a.e. in }\Omega,\tag{2}$$

that is equivalent to the system of differential equations

$$\sum_{k=1}^{m}\sum_{|\alpha|=2b} a_\alpha^{jk} D^\alpha u^k = \sum_{k=1}^{m} l^{jk}(x,D)u^k = f^j(x), \quad j=1,\dots,m. \tag{3}$$

The entries $l^{jk}(x,D)$ of the matrix differential operator $\mathfrak{L}(x,D)$ are homogeneous polynomials of degree $2b$, that is,

$$l^{jk}(x,\xi) := \sum_{|\alpha|=2b} a_\alpha^{jk}(x)\xi^\alpha, \quad \xi\in\mathbb{R}^n, \quad \xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_n^{\alpha_n}. \tag{4}$$

The operator $\mathfrak{L}(x,D)$ is supposed to be *uniformly elliptic* that means the characteristic determinant of $\mathfrak{L}(x,\xi)$ is non-vanishing for a.a. $x\in\Omega$ and all $\xi\neq 0$. Due to the homogeneity of l^{jk} this condition can be written as

$$\exists\,\delta>0\colon \quad \det\left\{\sum_{|\alpha|=2b}\mathbf{A}_\alpha(x)\xi^\alpha\right\} \ge \delta|\xi|^{2bm} \tag{5}$$

for almost all $x\in\Omega$ and all $\xi\in\mathbb{R}^n$.

Fix the coefficients of (2) at $x_0\in\Omega$ and consider the constant coefficients operator

$$\mathfrak{L}(x_0,D) := \sum_{|\alpha|=2b}\mathbf{A}_\alpha(x_0)D^\alpha.$$

Then the $2bm$-order differential operator

$$L(x_0,D) := \det\mathfrak{L}(x_0,D) = \det\left\{\sum_{|\alpha|=2b} a_\alpha^{jk}(x_0)D^\alpha\right\}_{j,k=1}^{m} \tag{6}$$

is elliptic as it follows from (5), and let $\widetilde{\Gamma}(x_0;x-y)$ be its fundamental solution. If the space dimension n is *odd*, then

$$\widetilde{\Gamma}(x_0;x-y) = |x-y|^{2bm-n}P\Big(x_0;\frac{x-y}{|x-y|}\Big) \tag{7}$$

with $P(x_0;\xi)$ being a real analytic function of $\xi\in\mathbb{S}^{n-1}$. If n is *even*, it is enough to introduce a fictitious new variable x_{n+1} and extend all functions as constants with respect to it (see [9]). Let $\{L_{jk}(x_0,\xi)\}_{j,k=1}^m$ be the *cofactor matrix* of $\{l^{jk}(x_0,\xi)\}_{j,k=1}^m$. Then $L_{jk}(x_0,D)$ are differential operators of order up to $2b(m-1)$ or identically zero. Since

$$\sum_{k=1}^{m} l^{ik}(x_0,\xi)L_{jk}(x_0,\xi) = \delta_{ij}L(x_0,\xi) \tag{8}$$

with the Kronecker symbol δ_{ij}, the *fundamental matrix* of $\mathfrak{L}(x_0, D)$ is given by

$$\mathbf{\Gamma}(x_0; x) = \{\Gamma^{jk}(x_0; x)\}_{j,k=1}^m = \{L_{kj}(x_0, D)\widetilde{\Gamma}(x_0; x)\}_{j,k=1}^m.$$

Let $\mathcal{B}_r \Subset \Omega$ be such that $x_0 \in \mathcal{B}_r$, $\mathbf{v} \in C_0^\infty(\mathcal{B}_r)$ and let us write

$$\mathfrak{L}(x_0, D)\mathbf{v}(x) = (\mathfrak{L}(x_0, D) - \mathfrak{L}(x, D))\mathbf{v}(x) + \mathfrak{L}(x, D)\mathbf{v}(x)\,.$$

Using the standard approach [7, 8, 9] we obtain an explicit representation formula for $\mathbf{v}$ via *Newtonian potentials*

$$\begin{aligned} \mathbf{v}(x) = & \int_{\mathcal{B}_r} \mathbf{\Gamma}(x_0; x-y)\mathfrak{L}\mathbf{v}(y)\, dy \\ & + \int_{\mathcal{B}_r} \mathbf{\Gamma}(x_0; x-y)\big(\mathfrak{L}(x_0, D) - \mathfrak{L}(y, D)\big)\mathbf{v}(y)\, dy. \end{aligned} \tag{9}$$

Taking the α-derivatives with $|\alpha| = 2b$ and then unfreezing the coefficients putting $x_0 = x$ we get

$$\begin{aligned} D^\alpha \mathbf{v}(x) = {} & p.v. \int_{\mathcal{B}_r} D^\alpha \mathbf{\Gamma}(x; x-y)\mathfrak{L}\mathbf{v}(y)\, dy \\ & + \sum_{|\alpha'|=2b} p.v. \int_{\mathcal{B}_r} D^\alpha \mathbf{\Gamma}(x; x-y)(\mathbf{A}_{\alpha'}(x) - \mathbf{A}_{\alpha'}(y))D^{\alpha'}\mathbf{v}(y)\, dy \\ & + \int_{\mathbb{S}^{n-1}} D^{\beta^s}\mathbf{\Gamma}(x; y)\nu_s\, d\sigma_y\ \mathfrak{L}\mathbf{v}(x) \\ & =: \mathfrak{K}_\alpha(\mathfrak{L}\mathbf{v})(x) + \sum_{|\alpha'|=2b} \mathfrak{C}_\alpha[\mathbf{A}_{\alpha'}, D^{\alpha'}\mathbf{v}](x) + \mathfrak{L}\mathbf{v}(x)\mathfrak{Q}_\beta(x), \end{aligned} \tag{10}$$

where the derivatives $D^\alpha\mathbf{\Gamma}(\cdot;\cdot)$ and $D^{\beta^s}\mathbf{\Gamma}(\cdot;\cdot)$ are taken with respect to the second variable, the multi-indices β^s are such that

$$\beta^s = (\alpha_1, \ldots, \alpha_{s-1}, \alpha_s - 1, \alpha_{s+1}, \ldots, \alpha_n), \quad |\beta^s| = 2b - 1,$$

and $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal to $\mathbb{S}^{n-1}$. Let us note that $\mathfrak{K}_\alpha$ are Calderón–Zygmund type singular integral operators, $\mathfrak{C}_\alpha$ are commutators of $\mathfrak{K}_\alpha$ with VMO functions, and $\mathfrak{Q}_\beta$ are bounded integrals (cf. [7, 8, 13]).

4. Main result

Our main result is given in the following theorem.

Theorem 1. *Suppose* (5), $\mathbf{A}_\alpha = \{a_\alpha^{jk}\} \in VMO(\Omega) \cap L^\infty(\Omega)$ *and let* $\mathbf{u} \in W^{2b}_{p,\mathrm{loc}}(\Omega;\mathbb{R}^m)$ *be a strong solution to* (2) *with* $p \in (1,\infty)$*. Let* $\mathbf{f} \in L^{p,\omega}(\Omega;\mathbb{R}^m)$ *with* ω *satisfying* (1)*. Then* $\mathbf{u} \in W^{2b}_{p,\omega,\mathrm{loc}}(\Omega;\mathbb{R}^m)$ *and*

$$\|\mathbf{u}\|_{W^{2b}_{p,\omega}(\Omega';\mathbb{R}^m)} \le C\big(\|\mathbf{f}\|_{p,\omega;\Omega} + \|\mathbf{u}\|_{p,\omega;\Omega''}\big) \tag{11}$$

for all $\Omega' \Subset \Omega'' \Subset \Omega$*, where the constant* C *depends on* $n, p, m, b, \omega, \|\mathbf{A}_\alpha\|_{\infty;\Omega}$*, the* VMO*-moduli* $\gamma_{\mathbf{A}_\alpha}$ *of the coefficients and on* $\operatorname{dist}(\Omega', \partial\Omega'')$.

The proof of Theorem 1 relies on some real analysis results regarding boundedness of Calderón–Zygmund type singular integral operators and their commutators, obtained in [13, 16, 17].

Proof. Fix an arbitrary $x_0 \in \operatorname{supp}\mathbf{u}$ and let $\mathcal{B}_r \equiv \mathcal{B}_r(x_0) \Subset \Omega$. Consider $\mathbf{v} \in W_0^{2b,p}(\mathcal{B}_r(x_0))$ (the closure of $C_0^\infty(\mathcal{B}_r(x_0))$ with respect to the norm in $W^{2b,p}(\mathcal{B}_r(x_0))$) with $\operatorname{supp}\mathbf{v} \subset \mathcal{B}_r(x_0)$. Then (10), Proposition 1 and $\mathbf{A}_\alpha \in VMO(\Omega)$ imply that for each $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon, \gamma_{\mathbf{A}_\alpha})$ such that

$$\|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{B}_r} \le C\big(\|\mathfrak{L}\mathbf{v}\|_{p,\omega;\mathcal{B}_r} + \varepsilon\|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{B}_r}\big)$$

whenever $r < r_0$. Choosing ε small enough we obtain

$$\|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{B}_r} \le C\|\mathfrak{L}\mathbf{v}\|_{p,\omega;\mathcal{B}_r}. \tag{12}$$

Let $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > 0$ and define the cut-off function $\varphi(x) \in C_0^\infty(\mathcal{B}_r)$ such that

$$\varphi(x) = \begin{cases} 1 & x \in \mathcal{B}_{\theta r}(x_0) \\ 0 & x \notin \mathcal{B}_{\theta' r}(x_0). \end{cases}$$

Since $\theta' - \theta = \theta(1-\theta)/2$, direct calculations give

$$|D^s\varphi| \le C(s)[\theta(1-\theta)r]^{-s}, \qquad \forall\, s = 1,2,\ldots,2b.$$

Setting $\mathbf{v} = \varphi\mathbf{u}$ in (12) we obtain

$$\begin{aligned} \|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta r}} &\le \|D^{2b}\mathbf{v}\|_{p,\omega;\mathcal{B}_{\theta' r}} \le C\|\mathfrak{L}\mathbf{v}\|_{p,\omega;\mathcal{B}_{\theta' r}} \\ &\le C\left(\|\mathbf{f}\|_{p,\omega;\mathcal{B}_{\theta' r}} + \sum_{s=1}^{2b-1}\frac{\|D^{2b-s}\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta' r}}}{[\theta(1-\theta)r]^s} + \frac{\|\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta' r}}}{[\theta(1-\theta)r]^{2b}}\right). \end{aligned}$$

Because of the choice of θ' we have $\theta(1-\theta) \le 2\theta'(1-\theta')$ that implies

$$\begin{aligned} [\theta(1-\theta)r]^{2b}\|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta r}} &\le C\Big([\theta'(1-\theta')r]^{2b}\|\mathbf{f}\|_{p,\omega;\mathcal{B}_{\theta' r}} \\ &+ \sum_{s=1}^{2b-1}[\theta'(1-\theta')r]^s\|D^s\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta' r}} + \|\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta' r}}\Big). \end{aligned} \tag{13}$$

Setting $\Theta_s = \sup_{0<\theta<1}[\theta(1-\theta)r]^s\|D^s\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta r}}$ we can rewrite (13) as

$$\Theta_{2b} \le C\left(r^{2b}\|\mathbf{f}\|_{p,\omega;\mathcal{B}_r} + \sum_{s=1}^{2b-1}\Theta_s + \Theta_0\right). \tag{14}$$

In order to estimate the seminorms Θ_s we need the following *interpolation inequality* which follows from [13, 19].

Lemma 1. *There is a constant C, independent of r, such that*

$$\Theta_s \le \varepsilon\Theta_{2b} + \frac{C}{\varepsilon^{s/(2b-s)}}\Theta_0 \qquad \textit{for each } \varepsilon \in (0,2). \tag{15}$$

Proof. Let $\theta_0 \in (0,1)$ be such that

$$\Theta_s \le 2[\theta_0(1-\theta_0)r]^s\|D\mathbf{u}\|^s_{p,\omega;\mathcal{B}_{\theta_0 r}}.$$

By interpolation and scaling arguments we obtain

$$\|D^s\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta_0 r}} \le \delta^{2b-s}\|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta_0 r}} + \frac{C'}{\delta^s}\|\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta_0 r}}$$

and hence

$$\Theta_s \le 2[\theta_0(1-\theta_0)r]^s\delta^{2b-s}\|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta_0 r}} + \frac{2C'[\theta_0(1-\theta_0)r]^s}{\delta^s}\|\mathbf{u}\|_{p,\omega;\mathcal{B}_{\theta_0 r}}.$$

◀

Turning back to (14), choosing suitable $\varepsilon \in (0,2)$, and applying (15) we get

$$\Theta_{2b} \le C\big(r^{2b}\|\mathbf{f}\|_{p,\omega;\mathcal{B}_r} + \Theta_0\big).$$

Fixing $\theta = 1/2$ at the seminorm Θ_s we obtain the following Caccioppoli-type estimate:

$$\|D^{2b}\mathbf{u}\|_{p,\omega;\mathcal{B}_{r/2}} \le C\big(\|\mathbf{f}\|_{p,\omega;\Omega} + Cr^{-2b}\|\mathbf{u}\|_{p,\omega;\mathcal{B}_r}\big). \tag{16}$$

The desired estimate (11) follows now by means of standard covering arguments with balls $\mathcal{B}_{r/2}$ for $r < \operatorname{dist}(\Omega', \partial\Omega'')$ and partition of unity over Ω' subordinated to this covering.

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Dian K. Palagachev
Department of Mechanics, Mathematics and Management, Polytechnic University of Bari, 70125 Bari, Italy
E-mail: dian.palagachev@poliba.it

Lubomira G. Softova
Department of Mathematics, University of Salerno, 84084 Salerno, Italy
E-mail: lsoftova@unisa.it

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