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Characterization and Wavelet Packets Associated with VN-MRA on $L^2\left(K, \mathbb{C}^N\right)$

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Abstract. The concept of vector-valued nonuniform multiresolution analysis on local field of positive characteristic was considered by Shah and Bhat. We construct the associated wavelet packets for such an MRA and investigate their properties. Moreover, we show how to obtain several new bases of the space $L^2\left(K, \mathbb{C}^N\right)$ by constructing a series of subspaces of these wavelet packets.

Key Words and Phrases: vector-valued non-uniform multiresolution analysis, wavelet, wavelet packet, local field, Fourier transform.

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1. Introduction

In recent years there has been a considerable interest in the problem of constructing wavelet bases on various groups. Recently, R.L. Benedetto and J.J. Benedetto [5] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Since local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p-adic field $\not\lesssim_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. The concept of multiresolution analysis on a local field K of positive characteristic was introduced by

Jiang et al. [9]. They pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence.

It is well known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman et al. [7] introduced the notion of orthogonal univariate wavelet packets. Well known Daubechies orthogonal wavelets are a special case of wavelet packets. Chui and Li[6] generalized the concept of orthogonal wavelet packets to the case of compactly supported orthogonal vector-valued wavelet packets so that they can be employed to the spline wavelets and so on. Shen [14] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Mittal and Manchanda [10] constructed vector-valued nonuniform wavelet packets. The construction of wavelet packets and wavelet frame packets on local fields of positive characteristic were recently reported by Behera and Jahan in [2]. They proved lemma on the so-called splitting trick and several theorems concerning the Fourier transform of the wavelet packets and the construction of wavelet packets to show that their translates form an orthonormal basis of $L^2(K)$. Other notable generalizations are the vector-valued wavelets. More details on wavelet packets can be found in [1, 3, 4, 8, 11, 13] and the references therein.

Recently Shah and Bhat [12] have generalized the concept of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to vector-valued nonuniform multiresolution analysis on local fields of positive characteristic. They called it a *vector-valued nonuniform multiresolution analysis* (VNMRA) on local fields of positive characteristic.

Motivated and inspired by the concept of vector-valued nonuniform multiresolution analysis on local fields of positive characteristic, we construct the associated orthogonal wavelet packets for such an MRA on local fields of positive characteristic. More precisely, we show that the collection of all dilations and translations of the wavelet packets is an overcomplete system in $L^2\left(K, \mathbb{C}^N\right)$. Finally, we investigate certain properties of the vector-valued wavelet packets on local fields of positive characteristic by introducing a notion of decomposition of the space $L^2\left(K, \mathbb{C}^N\right)$.

This paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and also some results which are required in the subsequent sections. In Section 3, we introduce the notion of vector-valued wavelet packets on local field K and prove that they generate an orthonormal basis for $L^2\left(K,\mathbb{C}^N\right)$. In Section 4, we define vector-valued nonuniform wavelet packets and we study their properties on the space $L^2\left(K,\mathbb{C}^N\right)$.

2. Preliminaries on local fields

A local field K is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\nleq_p$ or its finite extension. If K is of positive characteristic, then K is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let K be a fixed local field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : K \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\,y| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in K : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in K. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exists a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \displaystyle\sum_{\ell=k}^{\infty} c_\ell\, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < \Theta^{-k}\right\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [15]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural numbers, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$, so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K^+, then, as it was proved in [15], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$, where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \eta_0, \eta_1, \eta_2, \dots, \eta_{c-1}\} \subset \mathfrak{D}^*$ such that $\text{span}\{\eta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \quad 0 \le a_k < p, \quad \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\eta_1 + \cdots + a_{c-1}\eta_{c-1})\,\mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2\Theta^2 + \cdots + b_s\Theta^s$, $n \in \mathbb{N}_0$, $0 \le H_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m + n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < \Theta^k$, then $u(r\Theta^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\eta_0, \eta_1, \eta_2, \dots, \eta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \ne 1. \end{cases}$$

The Fourier transform of $f \in L^1(K)$ is denoted by $\widehat{f}(\xi)$ and defined by

$$F\{f(x)\} = \widehat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx.$$

Note that

$$\widehat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)} dx = \int_K f(x)\chi(-\xi x)\, dx.$$

The properties of Fourier transforms on local field K are much similar to those on the classical field $\mathbb{R}$. In fact, the Fourier transform on local fields of positive characteristic have the following properties:

- The map $f \to \widehat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\widehat{f}\|_\infty \le \|f\|_1$.
- If $f \in L^1(K)$, then $\widehat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(K)$, then $\|\widehat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(K)$ is defined by

$$\widehat{f}(\xi) = \lim_{k\to\infty} \widehat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le\Theta^k} f(x)\overline{\chi_\xi(x)}\, dx,$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\widehat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx.$$

The series $\sum_{n\in\mathbb{N}_0} \widehat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \big|f(x)\big|^2 dx = \sum_{n\in\mathbb{N}_0} \Big|\widehat{f}\big(u(n)\big)\Big|^2.$$

Let $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$, where $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representations of $\mathfrak{D}$ in K^+. Then we define

$$\ell^2(\mathcal{Z}) = \left\{ z : \mathcal{Z} \to \mathbb{C} : \sum_{n\in\mathbb{N}_0} |z(u(n))|^2 < \infty \right\}$$

as a Hilbert space with inner product

$$\langle z, w\rangle = \sum_{n\in\mathbb{N}_0} z(u(n))\overline{w(u(n))}.$$

The *Fourier transform* on $\ell^2(\mathcal{Z})$ is the map $\widehat{\ }: \ell^2(\mathcal{Z}) \to L^2(\mathfrak{D})$ defined for $z \in \ell^2(\mathcal{Z})$ by

$$\widehat{z}(\xi) = \sum_{n\in\mathbb{N}_0} z(u(n))\chi_{u(n)}(\xi),$$

and the *Inverse Fourier transform* on $L^2(\mathfrak{D})$ is the map $\vee : L^2(\mathfrak{D}) \to \ell^2(\mathcal{Z})$ defined for $f \in L^2(\mathfrak{D})$ by

$$f^{\vee}(u(n)) = \left\langle f, \chi_{u(n)} \right\rangle = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx.$$

For $z \in \ell^2(\mathcal{Z})$, we have

$$\begin{aligned} (\widehat{z})^{\vee}(u(n)) &= \left\langle \widehat{z}, \chi_{u(n)} \right\rangle \\ &= \left\langle \sum_{m\in\mathbb{N}_0} z(u(m))\chi_{u(m)}, \chi_{u(n)} \right\rangle \\ &= \sum_{m\in\mathbb{N}_0} z(u(m)) \left\langle \chi_{u(m)}, \chi_{u(n)} \right\rangle \\ &= z(u(n)). \end{aligned}$$

Since $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ is an orthonormal basis for $L^2(\mathfrak{D})$. It is also clear that the function $\widehat{z}$ is an integral periodic function because for $m \in \mathbb{N}_0$, we have

$$\begin{aligned} \widehat{z}(\xi + u(m)) &= \sum_{n\in\mathbb{N}_0} z(u(n))\chi_{u(n)}(\xi), \chi_{u(n)}(u(m)). \\ &= \sum_{n\in\mathbb{N}_0} z(u(n))\chi_{u(n)}(\xi) \\ &= \widehat{z}(\xi). \end{aligned}$$

For $z, w \in \ell^2(\mathcal{Z})$, we have *Parseval's relation*:

$$\langle z, w\rangle = \sum_{n\in\mathbb{N}_0} z(u(n))\overline{w(u(n))} = \int_{\mathfrak{D}} \widehat{z}(\xi)\overline{\widehat{w}(\xi)}\, d\xi = \langle \widehat{z}, \widehat{w}\rangle\,,$$

and *Plancherel's formula*:

$$\|z\|^2 = \sum_{n\in\mathbb{N}_0} |z(u(n))|^2 = \int_{\mathfrak{D}} |\widehat{z}(\xi)|^2\, d\xi = \|\widehat{z}\|^2.$$

We now reconsider vector valued multiresolution on local fields as defined in [1]. Let M be a constant and $M \leq s \in \mathbb{Z}$. By $L^2(K, \mathbb{C}^M)$ we denote the set of all vector valued functions $f(x)$, i.e.

$$\begin{aligned} L^2(K, \mathbb{C}^M) = \{&\mathbf{f}(x) = (f_1(x), f_2(x), ..., f_M(x))^T : \\ &x \in K, \mathbf{f}_t(x) \in L^2(K), k = 1, 2, ..., s\}, \end{aligned}$$

where T means the transpose of the vector.

The space $L^2(K, \mathbb{C}^M)$ is called vector-valued function space. For $\mathbf{f} \in L^2(K, \mathbb{C}^M)$, $||f||$ denotes the norm of the vector-valued function $\mathbf{f}$ and is defined as

$$||\mathbf{f}|| = \left(\sum_{t=1}^{M} \int_K |f_t(x)|^2 dt \right)^{\frac{1}{2}}. \tag{1}$$

For a vector-valued function $\mathbf{f} \in L^2(K, \mathbb{C}^M)$ the integration of $\mathbf{f}(x)$ is defined as

$$\int_K f(x)dt = \left(\int_K f_1(x), \int_K f_2(x), ..., \int_K f_M(x) \right)^T.$$

The Fourier transform of $\mathbf{f}(x)$ is defined by

$$\hat{\mathbf{f}}(\zeta) = \int_K \mathbf{f}(x)\overline{\chi_\zeta(x)}dx.$$

For any two vector-valued functions $\mathbf{f}, \mathbf{g} \in L^2(K, \mathbb{C}^M)$ the inner product $\langle \mathbf{f}, \mathbf{g} \rangle$ is defined as

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_K \mathbf{f}(x)\overline{\mathbf{g}}(x)dx. \tag{2}$$

A sequence $\{\mathbf{f}_t(x)\} \in L^2(K, \mathbb{C}^M)$ is said to be orthonormal if it satisfies

$$\langle \mathbf{f}_s(\cdot), \mathbf{f}_t(\cdot) \rangle = \delta_{s,t} I_M, \quad s, t \in \mathbb{Z}, \tag{3}$$

where $\delta_{s,t}$ denotes the Kronecker symbol such that $\delta_{s,t} = 1$ when $s = t$ and $\delta_{s,t} = 0$ when $s \neq t$, I_M denotes the identity matrix of order $M \times M$.

Definition 1. *A sequence $\{\mathbf{f}_t(x)\} \in L^2(K, \mathbb{C}^M)$, $t \in \mathbb{Z}$ is called an orthonormal basis for $L^2(K, \mathbb{C}^M)$ if it satisfies* (3) *and, moreover, for any $\mathbf{f} \in L^2(K, \mathbb{C}^M)$ there exists a sequence of $M \times M$ constant matrices $\{F_k\}_{k \in \mathbb{Z}}$ such that*

$$\mathbf{f}(x) = \sum_{t \in \mathbb{Z}} F_t \mathbf{f}_t(x), \quad x \in K, \tag{4}$$

where the multiplication $F_t \mathbf{f}_t(x)$ for each fixed x is the $M \times 1$ matrix multiplication, and the convergence for infinite summation is as same as of the norm $||.||$ defined by (1) *for the vector-valued function space.*

Let $\{\mathbf{f}_t(x)\}_{k \in \mathbb{Z}}$ be an orthonormal basis for $L^2(K, \mathbb{C}^M)$. Then the expansion (4) for any $f \in L^2(K, \mathbb{C}^M)$ is unique and

$$F_k = \langle \mathbf{f}, \mathbf{f}_t \rangle, \quad t \in \mathbb{Z}. \tag{5}$$

We also call the expansion (4) the Fourier expansion of $\mathbf{f}$.

The corresponding Parseval equality is

$$\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{t\in\mathbb{Z}} F_t \overline{F_k}. \tag{6}$$

From Eq. (6) it is clear that $\langle \mathbf{f}, \mathbf{f} \rangle = \mathbf{0}$ if and only if $\mathbf{f} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector.

Let $\mathbf{\Phi}(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_M(x))^T \in L^2(K, \mathbb{C}^M)$ satisfy the following refinement equation:

$$\mathbf{\Phi}(x) = \sum_{k\in\mathbb{N}_0} H_k \mathbf{\Phi}(\mathfrak{p}^{-1}x - u(k)), \tag{7}$$

where $\{H_k\}_{k\in\mathbb{N}_0}$ is a $M \times M$ constant matrix sequence. Define a closed subspace $V_j \subset L^2(K, \mathbb{C}^M)$ by

$$V_j = clos_{L^2(K,\mathbb{C}^M)} \left(span\{\varphi(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{N}_0\} \right), j \in \mathbb{Z}. \tag{8}$$

Vector-valued multiresolution analysis defined by Abdullah on local fields [1] is as follows:

Definition 2. $\mathbf{\Phi}(x)$ *defined by* (7) *generates a vector-valued multiresolution analysis* $\{V_j\}_{j\in\mathbb{Z}}$ *of* $L^2(K, \mathbb{C}^M)$, *if the sequence* $\{V_j\}_{j\in\mathbb{Z}}$ *defined in* (8) *satisfies:*

1. $... \subset V_{-1} \subset V_0 \subset V_1 \subset ...,$

2. $\cap_{j\in\mathbb{Z}} V_j = \{bf0\}, \cup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(K, \mathbb{C}^M)$, *where* $\mathbf{0}$ *is the zero vector of* $L^2(K, \mathbb{C}^M)$,

3. $\mathbf{\Phi}(x) \in V_0$ *if and only if* $\mathbf{\Phi}(\mathfrak{p}^{-j}x) \in V_j \forall j \in \mathbb{Z}$,

4. there exists $\mathbf{\Phi}(x) \in V_0$ *such that the sequence* $\{\mathbf{\Phi}(x - u(k)), k \in \mathbb{N}_0\}$ *is an orthonormal basis of* V_0. *The vector-valued function* $\varphi(x)$ *is called a scaling function of the vector-valued multiresolution analysis.*

On taking the Fourier transform on both sides of (7), and assuming that $\hat{\mathbf{\Phi}}(\zeta)$ is continuous at zero, we have

$$\hat{\mathbf{\Phi}}(\zeta) = H(\mathfrak{p}\zeta)\hat{\mathbf{\Phi}}(\mathfrak{p}\zeta), \quad \zeta \in K, \tag{9}$$

where

$$H(\zeta) = q \sum_{k\in\mathbb{N}_0} P_k \overline{\chi_k(\zeta)}. \tag{10}$$

Let $W_j, j \in \mathbb{Z}$ denote the orthogonal complement of V_j in V_{j+1} and there exist a vector-valued function $\mathbf{\Psi}(\mathbf{x}) \in L^2(K, \mathbb{C}^M)$ such that the translations and dilations of $\mathbf{\Psi}(\mathbf{x})$ form a Riesz basis of W_j i.e.

$$W_j = clos_{L^2(K,\mathbb{C}^M)} \left(span\{\mathbf{\Psi}(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{N}_0\} \right), j \in \mathbb{Z}. \tag{11}$$

Since $\mathbf{\Phi}(x)\in W_0\subset V_1$, there exists a unique finitely supported sequence $\{G_k\}_{k\in\mathbb{N}_0}$ of $M\times M$ constant matrices such that

$$\mathbf{\Psi}(x)=\sum_{k\in\mathbb{N}_0} G_k\mathbf{\Phi}(\mathfrak{p}^{-1}x-u(k)). \tag{12}$$

Let

$$H(\zeta)=q\sum_{k\in\mathbb{N}_0} G_k\overline{\chi_k(\zeta)}. \tag{13}$$

Then the equation (12) becomes

$$\hat{\mathbf{\Psi}}(\zeta)=G_k(\mathfrak{p}\zeta)\hat{\mathbf{\Phi}}(\mathfrak{p}\zeta),\ \zeta\in K. \tag{14}$$

3. Vector-valued nonuniform multiresolution analysis

Vector-valued nonuniform multiresolution analysis on local fields defined in [12] is as follows:

Definition 3. *Given integers $N\geq 1$ and r odd with $1\leq r\leq qN-1$ such that r and N are relatively prime, we say that $\mathbf{\Phi}\in L^2(K,\mathbb{C}^M)$ generates a VNUMRA $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(K,\mathbb{C}^M)$, if the sequence $\{V_j\}_{j\in\mathbb{Z}}$ satisfies:*

(a) $...\subset V_{-1}\subset V_0\subset V_1\subset ...$,

(b) $\cup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(K,\mathbb{C}^M)$,

(c) $\cap_{j\in\mathbb{Z}}V_j=\{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector of $L^2(K,\mathbb{C}^M)$,

(d) $\mathbf{\Phi}(x)\in V_j$ if and only if $\{\mathbf{\Phi}(\mathfrak{p}^{-1}Nx)\in V_{j+1}\ \ \forall\ \ j\in\mathbb{Z}\}$,

(e) there exists $\mathbf{\Phi}(x)\in V_0$ such that the sequence $\{\mathbf{\Phi}(x-\lambda),\lambda\in\Lambda\}$ is an orthonormal basis of V_0, where $\Lambda=\{0,u(r)/N\}+\mathcal{Z}$. The vector valued function $\mathbf{\Phi}(x)$ is called a scaling function of the VNUMRA.

Note that when $N=1$, one recovers from the above definition, the definition of vector-valued multiresolution analysis on local fields of positive characteristic.

Let $\mathbf{\Phi}(x)=(\varphi_1(x),\varphi_2(x),...,\varphi_M(x))^T\in L^2(K,\mathbb{C}^M)$ satisfy the following refinement equation:

$$\mathbf{\Phi}(x)=\sum_{\lambda\in\Lambda} H_\lambda\varphi\left((\mathfrak{p}^{-1}N)x-\lambda\right), \tag{15}$$

where $\{H_\lambda\}_{\lambda\in\Lambda}$ is $M\times M$ constant matrix sequence that has only finite number of terms. Define a closed subspace $V_j\in L^2(K,\mathbb{C}^M)$ by

$$V_j=clos_{L^2(K,\mathbb{C}^M)}\left(span\{\mathbf{\Psi}\left((\mathfrak{p}^{-1}N)^jx-\lambda\right),\lambda\in\Lambda\}\right),\quad j\in\mathbb{Z}. \tag{16}$$

Given a VNUMRA, let W_m denote the orthogonal complement of V_m in V_{m+1}, for any integer m. It is clear from the conditions (a)-(c) of Definition 3 that

$$L^2(K,\mathbb{C}^M)=\oplus_{m\in\mathbb{Z}}W_m.$$

As is the case in the standard situation (see [6, 10, 14, 15]), the main purpose of VNUMRA is to construct orthonormal basis of $L^2(K,\mathbb{C}^M)$ given by appropriate translates and dilates of a finite collection of functions, called the associated wavelets.

Definition 4. *A collection* $\{\mathbf{\Psi}_\ell\}_{\ell=1,2,\dots,qN-1}$ *of functions in* V_1 *will be called a set of wavelets associated with a given VNUMRA if the family of functions* $\{\mathbf{\Psi}_\ell(x-\lambda)\}_{\ell=1,2,\dots,qN-1,\lambda\in\Lambda}$ *is an orthonormal system of* W_0.

On taking the Fourier transform on both sides of (15), we have

$$\hat{\mathbf{\Phi}}(\mathfrak{p}^{-1}N\zeta)=G(\zeta)\hat{\mathbf{\Phi}}(\zeta),\quad \zeta\in K, \tag{17}$$

where

$$G(\zeta)=\frac{1}{qN}\sum_{\lambda\in\Lambda}G_\lambda\overline{\chi_\lambda(\zeta)}. \tag{18}$$

Since $\Lambda=\{0,u(r)/N\}+\mathcal{Z}$, we can write

$$G(\zeta)=G^1_\lambda+G^2_\lambda\overline{\chi\left(\frac{r}{N}\zeta\right)}, \tag{19}$$

where $\{G^1_\lambda\}$ and $\{G^2_\lambda\}$ are $M\times M$ constant symmetric matrix sequences, (for details see [12]). Then

$$\begin{aligned}\hat{\mathbf{\Phi}}(\zeta)&=G\left(\frac{\mathfrak{p}\zeta}{N}\right)\hat{\mathbf{\Phi}}\left(\frac{\mathfrak{p}\zeta}{N}\right)\\&=G\left(\frac{\mathfrak{p}\zeta}{N}\right)G\left(\left(\frac{\mathfrak{p}}{N}\right)^2\zeta\right)G\left(\left(\frac{\mathfrak{p}}{N}\right)^3\zeta\right)\cdots\hat{\mathbf{\Phi}}(0)\\&=\prod_{m=1}^{\infty}G\left(\left(\frac{\mathfrak{p}}{N}\right)^m\right)\hat{\mathbf{\Phi}}(0).\end{aligned} \tag{20}$$

Equation (20) implies that

$$G(0)=I_M\quad or\quad \sum_{\lambda\in\Lambda}G_\lambda=I_M, \tag{21}$$

where I_M denotes the identity matrix of order $M\times M$.

$$W_j = clos_{L^2(K,\mathbb{C}^M)}\left(span\{\mathbf{\Psi}_k\left((\mathfrak{p}^{-1}N)^j x-\lambda\right), \lambda\in\Lambda, \quad k=1,2,...,qN-1\}\right), j\in\mathbb{Z}. \tag{22}$$

Since $\mathbf{\Psi}_k(x)\in W_0\subset V_1$, there exists a uniquely supported sequence $\{G_{\lambda,k}\}_{\lambda\in\Lambda,k=1,2,...qN-1}$ of $M\times M$ constant matrices such that

$$\mathbf{\Psi}_k(x)=\sum_{\lambda\in\Lambda}G_{\lambda,k}\mathbf{\Psi}((\mathfrak{p}^{-1}N)x-\lambda). \tag{23}$$

On taking the Fourier transform on both sides of (23), we have

$$\hat{\mathbf{\Psi}}_k((\mathfrak{p}^{-1}N)\zeta)=H_k(\zeta)\hat{\varphi}(\zeta), \tag{24}$$

where

$$H_k(\zeta)=\frac{1}{qN}\sum_{\lambda\in\Lambda}G_{\lambda,k}\overline{\chi(\lambda\zeta)}. \tag{25}$$

Since $\Lambda=\{0,u(r)/N\}+\mathcal{Z}$, we can write

$$H_k(\zeta)=G^1_{\lambda,k}+G^2_{\lambda,k}\overline{\chi\left(\frac{r}{N}\right)\zeta}, \tag{26}$$

where $\{G^1_{\lambda,k}\}$ $\quad and \quad$ $\{G^2_{\lambda,k}\}$ are $M\times M$ constant symmetric matrix sequences.

Lemma 1. *Consider a VNUMRA as in Definition 3. Let* $\mathbf{\Psi}_0=\mathbf{\Phi}, H_0(\cdot)=G(\cdot)$ *and suppose that there exists* $qN-1$ *functions* $\mathbf{\Psi}_k, k=1,2,...,qN-1$ *in* V_1. *Then the family of functions* $\{\mathbf{\Psi}_k(x-\lambda)\}_{\lambda\in\Lambda,k=0,1,2,...,qN-1}$ *will form an orthonormal system for* V_1 *iff for* $k,l\in\{0,1,2,...,qN-1\}$

$$\sum_{r=0}^{qN-1}H_k\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}u(r))\right)\overline{H_\ell\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}u(r))\right)}=\delta_{k,\ell}I_M. \tag{27}$$

For the proof of the lemma, we refer to [12].

By the orthonormality of $\mathbf{\Psi}_k(x)\in L^2(K,\mathbb{C}^M)$, $\quad k=0,1,2,...,qN-1$ (or the orthonormality of VNUMRA V_j), as proved in [12] we have the following conditions:

$$\sum_{s=0}^{qN-1}F_{k,\ell}\left(\zeta+\mathfrak{p}u(s)\right)=q\delta_{k,\ell}I_M. \tag{28}$$

In [12], the following result on the existence of a vector-valued wavelet function was proved:

Theorem 1. *Suppose* $\{\mathbf{\Psi}_k(t-\lambda)_{\lambda\in\Lambda,\ \ k=0,1,\ldots,qN-1}$ *is the system as defined in Lemma 1 and orthonormal in* V_1. *Then this system is complete in* $W_0 \equiv V_1 \ominus V_0$.

If $\mathbf{\Psi}_0, \mathbf{\Psi}_1, \ldots, \mathbf{\Psi}_{qN-1} \in V_1$ *are as in Lemma 1 , one can obtain from them an orthonormal basis for* $L^2(K, \mathbb{C}^M)$ *by following the standard procedure for construction of wavelets from a given MRA [6, 10, 14, 15]. It can be easily checked that for every* $m \in \mathbb{Z}$, *the collection* $\{(qN)^{m/2}\mathbf{\Psi}_k((\mathfrak{p}^{-1}N)^m x-\lambda)\}_{\lambda\in\Lambda,\ \ k=0,1,\ldots,qN-1}$ *is a complete orthonormal system for* V_{m+1}. *Given a VNUMRA, since* W_m *is the orthogonal complement of* V_m *in* $V_{m+1}, m \in \mathbb{Z}$ *and*

$$L^2(K, \mathbb{C}^M) = \oplus_{m\in\mathbb{Z}} W_m,$$

where $\oplus$ *denotes the orthogonal direct sum with the inner product of* $L^2(K, \mathbb{C}^M)$. *From this it follows immediately that the collection* $\{(qN)^{m/2}\mathbf{\Psi}_k((\mathfrak{p}^{-1}N)^m x - \lambda)\}_{\lambda\in\Lambda,\ m\in\mathbb{Z},\ k=0,1,\ldots,qN-1}$ *forms a complete orthonormal system for* $L^2(K, \mathbb{C}^M)$. *When N = 1, we recover the usual construction of vector-valued wavelets from vector-valued multiresolution analysis.*

4. Vector-valued nonuniform wavelet packets and their properties

In this section, we will define the vector-valued nonuniform wavelet packets (VNUWP) and investigate their properties. Let

$$\Gamma^0(x) = \varphi(x), \Gamma^k(x) = \psi_k(x), \Theta_\lambda^{(0)} = G_\lambda,$$

$$\Theta_\lambda^{(k)} = G_{\lambda,k}, \lambda \in \Lambda, k = 1, 2, \ldots, qN-1.$$

Definition 5. *The family of vector-valued nonuniform functions* $\{\Gamma^{(\mathfrak{p}^{-1}N)n+u(k)}(x)$, $n \in \mathbb{N}_0,\ \ k = 0, 1, \ldots, qN-1\}$ *is called a vector-valued nonuniform wavelet packet w.r.t the orthogonal vector-valued scaling function* $\Gamma^0(x)$, *where*

$$\Gamma^{(\mathfrak{p}^{-1}N)n+u(k)}(x) = \sum_{\lambda\in\Lambda} \Theta_\lambda^{(k)} \Gamma^n((\mathfrak{p}^{-1}N)x - \lambda), \quad k = 0, 1, \ldots, qN-1. \tag{29}$$

By taking the Fourier transform on both sides, we get

$$\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(k)}(qN\zeta) = \Theta^{(k)}(\zeta)\hat{\Gamma}^n(\zeta), \quad k = 0, 1, \ldots, qN-1, \tag{30}$$

where

$$\Theta^{(k)}(\zeta) = \sum_{\lambda\in\Lambda} \Theta_\lambda^{(k)} \overline{\chi(\lambda\zeta)}, \quad k = 0, 1, \ldots, qN-1. \tag{31}$$

Since $\Lambda = \{0, u(r)/N\} + \mathcal{Z}$, we can write

$$\Theta^{(k)}(\zeta) = K_\lambda^{(k)} + S_\lambda^{(k)}\overline{\chi(r/N, \zeta)}, \tag{32}$$

where $\{K_\lambda^{(k)}\}$ *and* $\{S_\lambda^{(k)}\}$ *are* $M \times M$ constant symmetric matrix sequences.

Thus we have

$$\Theta^{(0)}(\zeta) = G(\zeta), \Theta^{(k)}(\zeta) = H_k(\zeta). \tag{33}$$

Then (27) can be written as

$$\begin{aligned}
&\Theta^{(k)}(\zeta)\Theta^{(l)}(\zeta)^\star + \Theta^{(k)}\left(\zeta + \frac{\mathfrak{p}u(1)}{\mathfrak{p}^{-1}N}\right)\overline{\Theta^{(l)}\left(\zeta + \frac{\mathfrak{p}u(1)}{\mathfrak{p}^{-1}N}\right)}\\
&\quad + \Theta^{(k)}\left(\zeta + \frac{\mathfrak{p}u(2)}{\mathfrak{p}^{-1}N}\right)\overline{\Theta^{(l)}\left(\zeta + \frac{\mathfrak{p}u(2)}{\mathfrak{p}^{-1}N}\right)}\\
&\quad + ... + \Theta^{(k)}\left(\zeta + \frac{\mathfrak{p}u(qN-1)}{\mathfrak{p}^{-1}N}\right)\overline{\Theta^{(l)}\left(\zeta + \frac{\mathfrak{p}u(qN-1)}{\mathfrak{p}^{-1}N}\right)}\\
&= \delta_{k,l}I_M, \quad \zeta \in K, \quad k, l \in 0, 1, 2, ..., qN-1.
\end{aligned} \tag{34}$$

(28) can be written as

$$\sum_{p=0}^{qN-1} \alpha^p w^n\left(\zeta + u(p)\right) = 0, \quad where \quad \alpha = \overline{\chi(r/N)}, \tag{35}$$

and $w^n(\zeta) = \sum_{j\in\mathbb{Z}} \hat{\Gamma}^n(\zeta + Nj)\hat{\Gamma}^n(\zeta + Nj)^\star$. Now we will investigate the properties of the vector-valued nonuniform wavelet packets.

Theorem 2. *If* $\{\Gamma^n(x), n \in \mathbb{N}_0\}$ *is a vector-valued nonuniform wavelet packet with respect to orthogonal vector-valued nonuniform scaling function* $\varphi(x)$, *then* $\forall n \in \mathbb{N}_0$, *we have*

$$\langle \Gamma^n(\cdot - \lambda), \Gamma^n(\cdot - \sigma)\rangle = \delta_{\lambda,\sigma}I_M, \quad \lambda, \sigma \in \Lambda. \tag{36}$$

Proof. We will prove the result by induction on n.

If n = 0, then (36) follows directly from hypothesis. Suppose $0 \le n < (qN)^l$ for some integer l. Then, for some $(qN)^{l-1} \le \left[\frac{n}{qN}\right] < \Theta^l$, where $[x]$ denotes the greatest integer of x and order $n = qN\left[\frac{n}{qN}\right] + k, \quad k = 0, 1, 2, ..., qN-1$.

Therefore by induction, we have

$$\left\langle \Gamma^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\cdot - \lambda), \Gamma^{\left[\frac{n}{qN}\right]}(\cdot - \sigma)\right\rangle = \delta_{\lambda,\sigma}I_M. \tag{37}$$

We obtain

$$\langle \Gamma^n(\cdot - \lambda), \Gamma^n(\cdot - \sigma)\rangle = \int_K \overline{\chi\left((\lambda-\sigma),\zeta\right)}\hat{\Gamma}^n(\zeta)\hat{\Gamma}^n(\zeta)^{\star} d\zeta$$

$$\int_{N\mathfrak{D}} \overline{\chi\left((\lambda-\sigma),\zeta\right)} \sum_{j\in\mathbb{Z}} \hat{\Gamma}^n(\zeta+Nj)\hat{\Gamma}^n(\zeta+Nj)^{\star} d\zeta.$$

Let

$$w^n(\zeta) = \sum_{j\in\mathbb{Z}} \hat{\Gamma}^n(\zeta+Nj)\hat{\Gamma}^n(\zeta+Nj)^{\star}.$$

Then, using (30) and (37), we obtain

$$\begin{aligned}
& w^n(qN\zeta) \\
&= \sum_{j\in\mathbb{Z}} \hat{\Gamma}^n(\mathfrak{p}^{-1}N(\zeta+u(j)))\overline{\hat{\Gamma}^n(\mathfrak{p}^{-1}N(\zeta+u(j)))} \\
&= \sum_{j\in\mathbb{Z}} \Theta^{(k)}(\zeta+u(j))\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}((\zeta+u(j)))\overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}((\zeta+u(j)))^{\star}\Theta^{(k)}(\zeta+U(j))} \\
&= \sum_{j=n.qN} \Theta^{(k)}(\zeta+nN)\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN)\overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN)^{\star}\Theta^{(k)}(\zeta+nN)} \\
&\quad + \sum_{j=n.qN+1} \Theta^{(k)}\left(\zeta+nN+u(1)\right)\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(1)\right) \\
&\quad \times \overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(1)\right)}\;\overline{\Theta^{(k)}\left(\zeta+nN+u(1)\right)} \\
&\quad + \sum_{j=n.qN+2} \Theta^{(k)}\left(\zeta+nN+u(2)\right)\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(2)\right) \\
&\quad \times \overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(2)\right)}\;\overline{\Theta^{(k)}\left(\zeta+nN+u(2)\right)} \\
&\cdots \\
&\quad + \sum_{j=n.qN+(qN-1)} \Theta^{(k)}\left(\zeta+nN+u(\mathfrak{p}^{-1}N-1)\right)\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(\mathfrak{p}^{-1}N-1)\right) \\
&\quad \times \overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(\mathfrak{p}^{-1}N-1)\right)}\;\overline{\Theta^{(k)}\left(\zeta+nN+u(\mathfrak{p}^{-1}N-1)\right)} \\
&= \Theta^{(k)}(\zeta)\left[\sum_{j=n.qN} \hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN)\overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN)}\right]\overline{\Theta^{(k)}(\zeta)} \\
&\quad + \Theta^{(k)}\left(\zeta+u(1)\right)
\end{aligned}$$

$$\times\left[\sum_{j=n.qN+1}\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(1)\right)\overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}\left(\zeta+nN+u(1)\right)}\right]$$
$$\times\overline{\Theta^{(k)}\left(\zeta+u(1)\right)}$$
$$+\Theta^{(k)}\left(\zeta+u(2)\right)$$
$$\times\left[\sum_{j=n.qN+2}\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN+u(2))\overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN+u(2))}\right]$$
$$\times\overline{\Theta^{(k)}\left(\zeta+u(2)\right)}$$
$$+\cdots$$
$$\Theta^{(k)}\left(\zeta+u(\mathfrak{p}^{-1}N-1)\right)\left[\sum_{j=n.qN+(qN-1)}\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN+u(\mathfrak{p}^{-1}N-1))\right.$$
$$\left.\overline{\hat{\Gamma}^{\left[\frac{n}{\mathfrak{p}^{-1}N}\right]}(\zeta+nN+u(\mathfrak{p}^{-1}N-1))}\right]\overline{\Theta^{(k)}\left(\zeta+u(\mathfrak{p}^{-1}N-1)\right)}$$
$$\left[\Theta^{(k)}(\zeta)\overline{\Theta^{(k)}(\zeta)}+\Theta^{(k)}\left(\zeta+u(1)\right)\overline{\Theta^{(k)}\left(\zeta+u(1)\right)}\right.$$
$$+\Theta^{(k)}\left(\zeta+u(2)\right)\overline{\Theta^{(k)}\left(\zeta+u(2)\right)}+\cdots$$
$$\left.+\Theta^{(k)}\left(\zeta+u(qN-1)\right)\overline{\Theta^{(k)}\left(\zeta+u(qN-1)\right)}\right]$$
$$q\sum_{j=0}^{qN-1}\Theta^{(k)}(\zeta+u(j))\overline{\Theta^{(k)}(\zeta+u(j))}.$$

Also

$$\sum_{p=0}^{qN-1}w^n(\zeta+u(p))=\sum_{j\in\mathbb{Z}}\hat{\Gamma}^n(\zeta+u(j))\overline{\hat{\Gamma}^n(\zeta+u(j))}$$
$$=q\sum_{j=0}^{qN-1}\Theta^{(k)}\left(\frac{\mathfrak{p}}{N}\left(\zeta+\mathfrak{p}u(j)\right)\right)\overline{\Theta^{(k)}\left(\frac{\mathfrak{p}}{N}\left(\zeta+\mathfrak{p}u(j)\right)\right)}.$$

If $\lambda=u(m_1),\quad \sigma=u(m_2),\quad where\quad m_1,m_2\in\mathbb{Z}$, using (34) we have

$$\langle\Gamma^n(\cdot-\lambda),\Gamma^n(\cdot-\sigma)\rangle$$
$$=\int_K\overline{\chi\left(u(m_1-m_2),\zeta\right)}\hat{\Gamma}^n(\zeta)\overline{\hat{\Gamma}^n(\zeta)}d\zeta$$
$$=\int_{N\mathfrak{D}}\overline{\chi\left(u(m_1-m_2),\zeta\right)}\sum_{j\in\mathbb{Z}}\hat{\Gamma}^n(\zeta+Nj)\overline{\hat{\Gamma}^n(\zeta+Nj)}d\zeta$$

$$= \int_{N\mathfrak{D}} \overline{\chi\left(u(m_1-m_2),\zeta\right)} w^n(\zeta) d\zeta$$

$$= \int_{\mathfrak{D}} \overline{\chi\left(u(m_1-m_2),\zeta\right)} \left[\sum_{p=0}^{qN-1} w^n(\zeta+u(p))\right] d\zeta$$

$$= q\int_{\mathfrak{D}} \overline{\chi\left(u(m_1-m_2),\zeta\right)} \sum_{j=0}^{qN-1} \Theta^{(k)}\left(\frac{\mathfrak{p}}{N}\left(\zeta+\mathfrak{p}u(j)\right)\right) \overline{\Theta^{(k)}\left(\frac{\mathfrak{p}}{N}\left(\zeta+\mathfrak{p}u(j)\right)\right)} d\zeta$$

$$= \delta_{m_1,m_2} I_M$$

$$= \delta_{\lambda,\sigma} I_M.$$

When $\lambda = u(m_1), \sigma = u(m_2) + r/N$, where $m_1, m_2 \in \mathbb{Z}$, we obtain using (35)

$$\langle \Gamma^n(. - \lambda), \Gamma^n(. - \sigma)\rangle$$

$$= \int_{N\mathfrak{D}} \overline{\chi\left(u(m_1-m_2),\zeta\right)} \overline{\chi\left(\frac{r}{N},\zeta\right)} w^n(\zeta) d\zeta$$

$$= \int_{\mathfrak{D}} \overline{\chi\left(u(m_1-m_2),\zeta\right)} \overline{\chi\left(\frac{r}{N},\zeta\right)} \left(\sum_{p=0}^{qN-1} \overline{\chi\left(\frac{r}{N},\zeta\right)} w^n(\zeta+u(p))\right) d\zeta$$

$$= 0.$$

◄

Theorem 3. *For any* $n_1, n_2 \in \mathbb{N}_0$ *and* $\lambda, \sigma \in \Lambda$, *we have*

$$\langle \Gamma^{n_1}(\cdot - \lambda), \Gamma^{n_2}(\cdot - \sigma)\rangle = \delta_{n_1,n_2}\delta_{\lambda,\sigma} I_M,$$

where $\{\Gamma^n(x) : n \in \mathbb{N}_0\}$ *is VNUWP with respect to orthogonal vector-valued scaling function* $\varphi(x)$.

Proof. If $n_1 = n_2$, then the result follows by Theorem 2. If $n_1 \neq n_2$, without loss of generality we can assume that $n_1 > n_2$.

Write

$$n_1 = qN\left[\frac{n_1}{qN}\right] + k, \quad n_2 = qN\left[\frac{n_2}{qN}\right] + l,$$

where $k, l \in \{0, 1, 2, ..., qN-1\}$.

Case (i) If $\left[\frac{n_1}{qN}\right] = \left[\frac{n_2}{qN}\right]$, then $k \neq l$.

$$\langle \Gamma^{n_1}(\cdot - \lambda), \Gamma^{n_2}(\cdot - \sigma)\rangle$$

$$= \int_K \overline{\chi((\lambda-\sigma),\zeta)}\,\overline{\hat{\Gamma}^{n_1}(\zeta)\hat{\Gamma}^{n_2}(\zeta)}d\zeta$$
$$= \int_{N\mathfrak{D}} \overline{\chi((\lambda-\sigma),\zeta)} \sum_{j\in\mathbb{Z}} \hat{\Gamma}^{n_1}(\zeta+Nj)\overline{\hat{\Gamma}^{n_2}(\zeta+Nj)}d\zeta$$
$$= \int_{N\mathfrak{D}} \overline{\chi((\lambda-\sigma),\zeta)}v(\zeta)d\zeta,$$

where $v(\zeta)=\sum_{j\in\mathbb{Z}} \hat{\Gamma}^{n_1}(\zeta+Nj)\overline{\hat{\Gamma}^{n_2}(\zeta+Nj)}$. Therefore,

$$\begin{aligned} v(qN\zeta) &= \sum_{j\in\mathbb{Z}} \hat{\Gamma}^{n_1}\left(\mathfrak{p}^{-1}N(\zeta+u(j))\right)\overline{\hat{\Gamma}^{n_2}\left(\mathfrak{p}^{-1}N(\zeta+u(j))\right)} \\ &= \sum_{j\in\mathbb{Z}} \Theta^k(\zeta+u(j))\hat{\Gamma}^{\left[\frac{n_1}{\mathfrak{p}^{-1}N}\right]}(\zeta+u(j))\overline{\hat{\Gamma}^{\left[\frac{n_2}{\mathfrak{p}^{-1}N}\right]}(\zeta+u(j))}\;\overline{\Theta^l(\zeta+u(j))}. \end{aligned}$$

On proceeding as in Theorem 2, we obtain

$$v(qN\zeta)=\sum_{j=0}^{qN-1} \Theta^k(\zeta+u(j))\overline{\Theta^l(\zeta+u(j))}.$$

Also

$$\begin{aligned} \sum_{p=0}^{qN-1} v(\zeta+u(p)) &= \sum_{j\in\mathbb{Z}} \hat{\Gamma}^{n_1}(\zeta+u(j))\overline{\hat{\Gamma}^{n_2}(\zeta+u(j))} \\ &= q\sum_{j=0}^{qN-1} \Theta^{(k)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}u(j))\right)\overline{\Theta^{(k)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}u(j))\right)}. \end{aligned}$$

If $\lambda=u(m_1)\quad and\quad \sigma=u(m_2),\quad where\quad m_1,m_2\in\mathbb{Z}$, we obtain

$$\begin{aligned} &\left\langle \Gamma^{n_1}(\cdot-\lambda),\Gamma^{n_2}(\cdot-\sigma)\right\rangle \\ &= \int_{N\mathfrak{D}} \overline{\chi(u(m_1-m_2),\zeta)}v(\zeta)d\zeta \\ &= \int_{\mathfrak{D}} \overline{\chi(u(m_1-m_2),\zeta)}\left[\sum_{p=0}^{qN-1} v(\zeta+u(p))\right]d\zeta \\ &= q\int_{\mathfrak{D}} \overline{\chi(u(m_1-m_2),\zeta)}\sum_{j=0}^{qN-1} \Theta^{(k)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}u(j))\right)\overline{\Theta^{(l)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}u(j))\right)}d\zeta \end{aligned}$$

$$= \delta_{m_1,m_2}\delta_{k,l}I_M$$
$$= \delta_{\lambda,\sigma}\delta_{k,l}I_M.$$

If $\lambda = u(m_1) + r/N$ and $\sigma = u(m_2)$, where $m_1, m_2 \in \mathbb{Z}$, we obtain using (35)

$$\begin{aligned}
&\langle \Gamma^{n_1}(. - \lambda), \Gamma^{n_2}(. - \sigma)\rangle \\
&= \int_{N\mathfrak{D}} \overline{\chi\left(u(m_1 - m_2), \zeta\right)} \overline{\chi\left(\frac{r}{N}, \zeta\right)\chi\left(\frac{r}{N}, \zeta\right)} v(\zeta) d\zeta \\
&= \int_{\mathfrak{D}} \overline{\chi\left(u(m_1 - m_2), \zeta\right)} \overline{\chi\left(\frac{r}{N}, \zeta\right)} \left[\sum_{p=0}^{qN-1} \overline{\chi\left(\frac{r}{N}, \zeta\right)} v(\zeta + u(p))\right] d\zeta \\
&= 0.
\end{aligned}$$

Case (ii) If $\left[\frac{n_1}{qN}\right] = \left[\frac{n_2}{qN}\right]$, then take $\left[\frac{n_1}{qN}\right] = qN\left[\frac{[n_1/qN]}{qN}\right] + u(k_1)$ and $\left[\frac{n_2}{qN}\right] = qN\left[\frac{[n_2/qN]}{qN}\right] + (l_1)$, where $u(k_1), l_1 \in \{0, 1, 2, ..., qN - 1\}$.

Let $\left[\frac{n_1}{qN}\right] = qNp_1 + u(k_1)$ and $\left[\frac{n_2}{qN}\right] = qN\Theta_1 + u(l_1)$, where $p_1 = \left[\frac{[n_1/qN]}{qN}\right]$ and $\Theta_1 = \left[\frac{[n_2/qN]}{qN}\right]$.

If $p_1 = \Theta_1$, then the result follows immediately from Case (i).

If $p_1 \neq \Theta_1$, then take

$p_1 = qN\left[\frac{[p_1/qN]}{qN}\right] + u(k_2) = qNp_2 + u(k_2)$ and $\Theta_1 = qN\left[\frac{[\Theta_1/qN]}{qN}\right] + u(l_2) = qN\Theta_2 + u(l_2)$, where $k_2, l_2 \in \{0, 1, 2, ..., qN - 1\}$.

If $p_2 = \Theta_2$, then the result follows from Case (i).

If $p_2 \neq \Theta_2$, then apply the above procedure. After performing a finite number of steps, we have $p_{m-1} = qNp_m + u(k_m)$ *and* $\Theta_{m-1} = qN\Theta_m + u(l_m)$, where $k_m, l_m \in \{0, 1, 2, ...qN - 1\}$ and $p_m, \Theta_m \in \{0, 1, 2, ...qN - 1\}$.

Case I If $p_m = \Theta_m$.

Case II If $p_m \neq \Theta_m$.

For Case I the result follows from Case (i).

For Case II, we have

$$\begin{aligned}
\langle \Gamma^{n_1}(\cdot - \lambda), \Gamma^{n_2}(\cdot - \sigma)\rangle &= \int_K \overline{\chi\left((\lambda - \sigma), \zeta\right)} \hat{\Gamma}^{n_1}(\zeta) \overline{\hat{\Gamma}^{n_2}(\zeta)} d\zeta \\
&= \int_K \overline{\chi\left((\lambda - \sigma), \zeta\right)} \Theta^{(k_1)}\left(\frac{\mathfrak{p}}{N}\zeta\right) \hat{\Gamma}^{\left[\frac{n_1}{\mathfrak{p}^{-1}N}\right]}\left(\frac{\mathfrak{p}}{N}\zeta\right) \\
&\overline{\hat{\Gamma}^{\left[\frac{n_2}{\mathfrak{p}^{-1}N}\right]}\left(\frac{\mathfrak{p}}{N}\zeta\right)} \overline{\Theta^{(l_1)}\left(\frac{\mathfrak{p}}{N}\zeta\right)} d\zeta
\end{aligned}$$

$$
\begin{aligned}
&= \int_K \overline{\chi\left((\lambda-\sigma),\zeta\right)}\Theta^{(k_1)}\left(\frac{\mathfrak{p}}{N}\zeta\right)\Theta^{(k_2)}\left(\left(\frac{\mathfrak{p}}{N}\right)^2\zeta\right) \\
&\cdots\Theta^{(k_m)}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta\right)\hat{\Gamma}^{p_m}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta\right) \\
&\overline{\hat{\Gamma}^{\Theta_m}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta\right)}\,\overline{\Theta^{(l_m)}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta\right)}\cdots \\
&\overline{\Theta^{(l_2)}\left(\left(\frac{\mathfrak{p}}{N}\right)^2\zeta\right)}\,\overline{\Theta^{(l_1)}\left(\frac{\zeta}{(qN)}\right)}d\zeta \\
&= \int_K \overline{\chi\left((\lambda-\sigma),\zeta\right)} \\
&\left[\prod_{n=1}^{m}\Theta^{(k_n)}\left(\left(\frac{\mathfrak{p}}{N}\right)^n\zeta\right)\right]\hat{\Gamma}^{p_m}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta\right) \\
&\overline{\hat{\Gamma}^{\Theta_m}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta\right)}\,\overline{\left[\prod_{n=1}^{m}\Theta^{(l_n)}\left(\left(\frac{\mathfrak{p}}{N}\right)^n\zeta\right)\right]}d\zeta \\
&= \int_{N\mathfrak{D}} \overline{\chi\left((\lambda-\sigma),\zeta\right)}\left[\prod_{n=1}^{m}\Theta^{(k_n)}\left(\left(\frac{\mathfrak{p}}{N}\right)^n\zeta\right)\right] \\
&\left[\sum_{j\in\mathbb{Z}}\hat{\Gamma}^{p_m}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta+Nj\right)\overline{\hat{\Gamma}^{\Theta_m}\left(\left(\frac{\mathfrak{p}}{N}\right)^m\zeta+Nj\right)}\right] \\
&\overline{\left[\prod_{n=1}^{m}\Theta^{(l_n)}\left(\left(\frac{\mathfrak{p}}{N}\right)^n\zeta\right)\right]}d\zeta \\
&= 0,
\end{aligned}
$$

which completes the proof. ◄

Corollary 1. *If $\{\Gamma^n(x), n\in\mathbb{N}_0\}$ is a vector-valued nonuniform wavelet packet with respect to orthogonal vector-valued nonuniform scaling function $\varphi(x)$, then $\forall n\in\mathbb{N}_0,\quad$ and $\quad k,l\in\{0,1,...,qN-1\}$, we have*

$$\langle\Gamma^{(\mathfrak{p}^{-1}N)n+u(k)}(\cdot-\lambda),\Gamma^{(\mathfrak{p}^{-1}N)n+u(l)}(\cdot-\sigma)\rangle=\delta_{\lambda,\sigma}\delta_{k,l}I_M,\quad \lambda,\sigma\in\Lambda.$$

Proof. We have

$$
\begin{aligned}
&\langle\Gamma^{(\mathfrak{p}^{-1}N)n+u(k)}(\cdot-\lambda),\Gamma^{(\mathfrak{p}^{-1}N)n+u(l)}(\cdot-\sigma)\rangle \\
&= \int_K \overline{\chi\left((\lambda-\sigma),\zeta\right)}\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(k)}(\zeta)\overline{\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(l)}(\zeta)}d\zeta
\end{aligned}
$$

$$= \int_{N\mathfrak{D}} \overline{\chi\left((\lambda-\sigma),\zeta\right)} \sum_{j\in\mathbb{Z}} \hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(k)}(\zeta+Nj)\overline{\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(l)}(\zeta+Nj)}d\zeta.$$

Let

$$v^n(\zeta) = \sum_{j\in\mathbb{Z}} \hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(k)}(\zeta+Nj)\overline{\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(l)}(\zeta+Nj)}.$$

Therefore, on solving the equation as in Theorem 2, we have

$$\begin{aligned} v^n(qN\zeta) &= \sum_{j\in\mathbb{Z}} \hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(k)}\left(\mathfrak{p}^{-1}N(\zeta+u(j))\right)\overline{\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(l)}\left(\mathfrak{p}^{-1}N(\zeta+u(j))\right)} \\ &= \sum_{j\in\mathbb{Z}} \Theta^{(k)}(\zeta+u(j))\hat{\Gamma}^n(\zeta+u(j))\overline{\hat{\Gamma}^n(\zeta+u(j))}\ \overline{\Theta^{(l)}(\zeta+u(j))} \\ &= q\sum_{j=0}^{qN-1} \overline{\Theta^{(k)}(\zeta+u(j))\Theta^{(l)}(\zeta+u(j))}. \end{aligned}$$

Also

$$\sum_{j=0}^{qN-1} v^v(\zeta+u(p)) = \sum_{j\in\mathbb{Z}} \hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(k)}(\zeta+u(j))\overline{\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(l)}(\zeta+u(j))}.$$

Therefore, we have

$$\sum_{j\in\mathbb{Z}} \hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(k)}(\zeta+u(j))\overline{\hat{\Gamma}^{(\mathfrak{p}^{-1}N)n+u(l)}(\zeta+u(j))}$$

$$= q\sum_{j=0}^{qN-1} \Theta^{(k)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}N)\right)\overline{\Theta^{(l)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}N)\right)}.$$

When $\lambda = u(m_1)$ *and* $\sigma = u(m_2)$, *where* $m_1, m_2 \in \mathbb{Z}$, using (34) we obtain

$$\begin{aligned} &\langle \Gamma^{(\mathfrak{p}^{-1}N)n+u(k)}(.-\lambda), \Gamma^{(\mathfrak{p}^{-1}N)n+u(l)}(.-\sigma)\rangle \\ &= \int_{N\mathfrak{D}} \overline{\chi\left((\lambda-\sigma),\zeta\right)} v^n(\zeta)d\zeta \\ &= \int_{N\mathfrak{D}} \overline{\chi\left((\lambda-\sigma),\zeta\right)} \left[\sum_{p=0}^{qN-1} v^n(\zeta+u(p))\right]d\zeta \\ &= q\int_{\mathfrak{D}} \overline{\chi\left((\lambda-\sigma),\zeta\right)} \sum_{j=0}^{qN-1} \Theta^{(k)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}N)\right)\overline{\Theta^{(l)}\left(\frac{\mathfrak{p}}{N}(\zeta+\mathfrak{p}N)\right)}d\zeta \end{aligned}$$

$$= q \int_{\mathfrak{D}} \overline{\chi((\lambda - \sigma), \zeta)} \delta_{k,l} I_M d\zeta$$
$$= \delta m_1, m_2 \delta_{k,l} I_M = \delta_{\lambda,\sigma} \delta_{k,l} I_M.$$

When $\lambda = u(m_1), \sigma = u(m_2) + r/N, \quad where \quad m_1, m_2 \in \mathbb{Z}$, we obtain using (35)

$$\langle \Gamma^{(\mathfrak{p}^{-1}N)n+u(k)}(. - \lambda), \Gamma^{(\mathfrak{p}^{-1}N)n+u(l)}(. - \sigma) \rangle$$
$$= \int_{N\mathfrak{D}} \overline{\chi((\lambda - \sigma), \zeta)} \overline{\chi\left(\frac{r}{N}, \zeta\right)} v^n(\zeta) d\zeta$$
$$= \int_{\mathfrak{D}} \overline{\chi((\lambda - \sigma), \zeta)} \overline{\chi\left(\frac{r}{N}, \zeta\right)} \left(\sum_{p=0}^{qN-1} \overline{\chi\left(\frac{r}{N}, \zeta\right)} v^n(\zeta + u(p)) \right) d\zeta$$
$$= 0,$$

which completes the proof. ◄

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