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Hardy's Inequalities and Erdélyi-Kober Fractional Integrals on BMO(ρ)

K.-P. Ho

Abstract. In this paper, the Hardy's inequalities are extended to the function spaces of bounded mean oscillation associated with growth functions. We also establish the boundedness of the Erdélyi-Kober fractional integrals on the function spaces of bounded mean oscillation associated with growth functions.

Key Words and Phrases: integral operator, bounded mean oscillation, Hardy's inequality, fractional integral.

2010 Mathematics Subject Classifications: 26D10, 26D15, 42B35, 44A05, 46E30

1. Introduction

We establish the Hardy's inequality and obtain the boundedness of the Erdélyi-Kober fractional integrals on the function spaces of bounded mean oscillation associated with growth function ρ, $BMO(\rho)$ [10, 11, 12, 19].

The function spaces $BMO(\rho)$ are generalizations of the classical function space of bounded mean oscillation BMO introduced by John and Nirenberg [13]. These function spaces are also the dual spaces of the Orlicz-Hardy spaces [20, 25]. For the characterization of $BMO(\rho)$ and the use of $BMO(\rho)$ on the study of the commutator of singular integral operators, the reader is referred to [10] and [11], respectively. Furthermore, $BMO(\rho)$ also includes the Lipschitz spaces Λ_α, $0 < \alpha \leq 1$, [18, 23].

The Hardy's inequality is one of the most important inequalities in analysis. For the history and the developments of the Hardy's inequality, the reader may consult [15, 21]. For the extension of the Hardy's spaces on BMO and Hardy type spaces, the reader is referred to [1, 2, 3, 4, 5, 6, 17, 26, 27]. Motivated by the results given in [26, 27], we aim to extend the Hardy's inequality to $BMO(\rho)$.

The Hardy's inequality gives the boundedness of the Hardy operator which is a special case of the Erdélyi-Kober fractional integrals. The Erdélyi-Kober fractional integrals provide several applications on applied analysis and physics, see [14, 22]. In this paper, we also extend the mapping properties of the Erdélyi-Kober fractional integrals to $BMO(\rho)$. As BMO and Λ_α, $0 < \alpha \leq 1$ are members of $BMO(\rho)$, our results also yield the Hardy's inequality and the mapping properties of the Erdélyi-Kober fractional integrals on BMO and Λ_α, $0 < \alpha \leq 1$.

This paper is organized as follows. The definition of $BMO(\rho)$ and the dilation properties of $BMO(\rho)$ are presented in Section 2. The main result of this paper is given in Section 3.

2. Definitions

In this section, we present the definition of $BMO(\rho)$ and study the mapping properties of the dilation operators on $BMO(\rho)$.

We start with the definition of $BMO(\rho)$. Let $\mathbb{R}_+ = (0, \infty)$ and $\mathcal{I}$ denote the family of open connected intervals in $\mathbb{R}_+$. For any $I \in \mathcal{I}$, the Lebesgue measure of I is denoted by $|I|$. Let L_{loc} denote the class of locally integrable functions on $\mathbb{R}_+$.

Let $m, l \geq 0$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. We say that ρ is of upper type m if there exists a constant $C > 0$ such that

$$\rho(st) \leq Ct^m \rho(s), \quad \forall t \in (1, \infty),\ s \in \mathbb{R}_+. \tag{1}$$

It is of lower type l if there exists a constant $C > 0$ such that

$$\rho(st) \leq Ct^l \rho(s), \quad \forall t \in (0, 1),\ s \in \mathbb{R}_+. \tag{2}$$

Definition 1. *Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. If ρ is a non-decreasing function of finite upper type and $\lim_{t\to 0^{+1}} \rho(t) = 0$, then ρ is called a growth function.*

The notions of growth function, lower type and upper type had been used in [20, 25] for the study of Orlicz-Hardy spaces.

Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a growth function. We recall the definition of the function space of bounded mean oscillation associated with ρ, $BMO(\rho)$ from [10, 11, 19, 25].

Definition 2. *Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a growth function. A locally integrable function f belongs to $BMO(\rho)$ if*

$$\|f\|_{BMO(\rho)} = \sup_{I \in \mathcal{I}} \frac{1}{|I|\rho(|I|)} \int_I |f(t) - f_I| dt < \infty,$$

where $f_I = \frac{1}{|I|} \int_I f(y) dy$.

Endowed with the norm given in Definition 2, $BMO(\rho)$ becomes a Banach space provided we identify functions which differ a.e. by constant; obviously, $\|f\|_{BMO(\rho)} = 0$ for $f(x) = c = const$ a.e. in $\mathbb{R}_+$. Notice that when $\rho \equiv 1$, ρ is of upper type and lower type 0, $BMO(\rho)$ becomes the classical function space of bounded mean oscillation BMO.

When $\rho(r) = r^\alpha$, $0 < \alpha \le 1$, $BMO(\rho)$ becomes the Lipschitz space Λ_α, see [10, 18, 19, 23].

Lemma 1. *Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a growth function. If $f \in BMO(\rho)$, then $|f| \in BMO(\rho)$ and $\||f|\|_{BMO(\rho)} \le 2\|f\|_{BMO(\rho)}$.*

Proof. For any $I \in \mathcal{I}$, we have

$$\big||f(t)| - |f_I|\big| \le |f(t) - f_I|, \quad t > 0,$$

see [24, Chapter IV, Secion 1.1.3]. Thus,

$$\frac{1}{|I|}\int_I \big||f(t)| - |f_I|\big|dt \le \frac{1}{|I|}\int_I |f(t) - f_I|dt \tag{3}$$

and, hence,

$$\begin{aligned} ||f|_I - |f_I|| = \left|\frac{1}{|I|}\int_I (|f(t)| - |f_I|)dt\right| &\le \frac{1}{|I|}\int_I \big||f(t)| - |f_I|\big|dt \\ &\le \frac{1}{|I|}\int_I |f(t) - f_I|dt. \end{aligned} \tag{4}$$

Consequently, (3) and (4) give

$$\begin{aligned} &\frac{1}{|I|\rho(|I|)}\int_I ||f(t)| - |f|_I|dt \\ &\le \frac{1}{|I|\rho(|I|)}\int_I ||f(t)| - |f_I||dt + \frac{1}{|I|\rho(|I|)}\int_I ||f_I| - |f|_I|dt \\ &\le 2\frac{1}{|I|\rho(|I|)}\int_I |f(t) - f_I|dt. \end{aligned}$$

By taking supremum over $I \in \mathcal{I}$, we obtain

$$\||f|\|_{BMO(\rho)} \le 2\|f\|_{BMO(\rho)}. \quad ◀$$

The above lemma assures that whenever $f \in BMO(\rho)$, $f_+ = \max(f, 0) = (f + |f|)/2$ and $f_- = f - f_+$ belong to $BMO(\rho)$.

We now present the dilation properties of $BMO(\rho)$. For any $s > 0$ and $f \in L_{loc}$, the dilation operator $D_s f$ is defined as

$$D_s f(t) = f(t/s), \quad t \geq 0.$$

For any $I = (a, b) \in \mathcal{I}$, write $I_s = (a/s, b/s)$. Obviously, $|I_s| = |I|/s$.

Lemma 2. *Let $m, l \geq 0$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a growth function of upper type m and lower type l.*

1. *There is a constant $C > 0$ such that for any $f \in BMO(\rho)$ and $s \in (1, \infty)$, we have $D_s f \in BMO(\rho)$ and*

$$\|D_s f\|_{BMO(\rho)} \leq C s^{-l} \|f\|_{BMO(\rho)}. \tag{5}$$

2. *There is a constant $C > 0$ such that for any $f \in BMO(\rho)$ and $s \in (0, 1)$, we have $D_s f \in BMO(\rho)$ and*

$$\|D_s f\|_{BMO(\rho)} \leq C s^{-m} \|f\|_{BMO(\rho)}. \tag{6}$$

Proof. As the proof of (5) is similar to the proof of (6), for brevity, we just present the proof for (5).

Let $I \in \mathcal{I}$. By using the substitution $t = su$, we find that

$$(D_s f)_I = \frac{1}{|I|} \int_I f(t/s) dt = \frac{1}{|I|} \int_{I_s} f(u) s du = \frac{1}{|I_s|} \int_{I_s} f(u) du = f_{I_s}. \tag{7}$$

Consequently, the substitution $t = su$ gives

$$\begin{aligned} \frac{1}{|I|\rho(|I|)} \int_I |D_s f(t) - (D_s f)_I| dt &= \frac{1}{|I|\rho(|I|)} \int_{I_s} |f(u) - f_{I_s}| s du \\ &= \frac{1}{|I_s|\rho(s|I_s|)} \int_{I_s} |f(u) - f_{I_s}| du. \end{aligned} \tag{8}$$

As ρ is of lower type l and of upper type m, we find that

$$\rho(|I_s|) \leq C s^{-l} \rho(s|I_s|), \quad s \in (1, \infty) \tag{9}$$

$$\rho(|I_s|) \leq C s^{-m} \rho(s|I_s|), \quad s \in (0, 1). \tag{10}$$

Therefore, when $s \in (1, \infty)$, (8) and (9) give

$$\frac{1}{|I|\rho(|I|)} \int_I |D_s f(t) - (D_s f)_I| dt \leq C s^{-l} \frac{1}{|I_s|\rho(|I_s|)} \int_{I_s} |f(u) - f_{I_s}| du$$

$$\leq Cs^{-l}\|f\|_{BMO(\rho)}.$$

By taking supremum over $I \in \mathcal{I}$ on both sides of the above inequalities, we obtain (5).

Similarly, when $s \in (0,1)$, according to (8) and (10), we have

$$\frac{1}{|I|\rho(|I|)}\int_I |D_sf(t)-(D_sf)_I|dt \leq Cs^{-m}\frac{1}{|I_s|\rho(|I_s|)}\int_{I_s}|f(u)-f_{I_s}|du$$
$$\leq Cs^{-m}\|f\|_{BMO(\rho)}.$$

Hence,

$$\|D_sf\|_{BMO(\rho)} \leq Cs^{-m}\|f\|_{BMO(\rho)}. \quad ◀$$

3. Main results

This section contains the main results of this paper. We establish the Hardy's inequalities on $BMO(\rho)$. We also obtain the mapping properties of the Erdélyi-Kober fractional integrals on $BMO(\rho)$. These two results are consequence of a general result for the integral operators on $BMO(\rho)$.

Let $K : (0,\infty)\times(0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function. We consider the integral operator

$$Tf(t) = \int_0^\infty K(s,t)f(s)ds, \quad t \geq 0.$$

Theorem 1. *Let $m, l \geq 0$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a growth function of upper type m and lower type l. Suppose that $K : (0,\infty)\times(0,\infty) \to \mathbb{R}$ is a Lebesgue measurable function satisfying*

$$K(\lambda s, \lambda t) = \lambda^{-1}K(s,t), \quad \lambda > 0 \tag{11}$$

$$\int_0^1 |K(u,1)|u^l du + \int_1^\infty |K(u,1)|u^m du < \infty. \tag{12}$$

There exists a constant $C > 0$ such that for any $f \in BMO(\rho)$

$$\|Tf\|_{BMO(\rho)} \leq C\|f\|_{BMO(\rho)}. \tag{13}$$

Proof. In view of Lemma 1 and the fact that $f = f_+ - f_-$, we only need to consider $f \in BMO(\rho)$ with $f \geq 0$. Similarly, we can also assume that K is positive since K_+ and K_- satisfy (11) and (12) and

$$\int_0^\infty K(s,t)f(s)ds = \int_0^\infty K_+(s,t)f(s)ds - \int_0^\infty K_-(s,t)f(s)ds.$$

By using the substitution $s = ut$, from (11) we have

$$\begin{aligned} Tf(x) &= \int_0^\infty K(s,t)f(s)ds = \int_0^\infty K(ut,t)f(ut)tdu \\ &= \int_0^\infty K(u,1)f(ut)du = \int_0^\infty K(u,1)D_{1/u}f(t)du. \end{aligned}$$

Therefore, for any $I \in \mathcal{I}$,

$$\begin{aligned} (Tf)_I &= \frac{1}{|I|}\int_I \int_0^\infty K(u,1)f(ut)dudt \\ &= \int_0^\infty K(u,1)\left(\frac{1}{|I|}\int_I D_{1/u}f(t)dt\right)du \\ &= \int_0^\infty K(u,1)(D_{1/u}f)_I du. \end{aligned}$$

Since K and f are nonnegative, $(Kf)_I$ is well defined. Consequently,

$$\begin{aligned} |Tf(t) - (Tf)_I| &= \left|\int_0^\infty K(u,1)D_{1/u}f(t)du - \int_0^\infty K(u,1)(D_{1/u}f)_I du\right| \\ &\le \int_0^\infty |K(u,1)|\big|D_{1/u}f(t) - (D_{1/u}f)_I\big|du. \end{aligned}$$

By integrating over I on both sides of the above inequality, we obtain

$$\begin{aligned} &\int_I |Tf(t) - (Tf)_I|dt \\ &\le \int_I \int_0^\infty |K(u,1)|\big|D_{1/u}f(t) - (D_{1/u}f)_I\big|dudt \\ &= \int_0^\infty |K(u,1)|\left(\int_I \big|D_{1/u}f(t) - D_{1/u}(f)_I\big|dt\right)du \\ &\le \rho(|I|)|I|\int_0^\infty |K(u,1)|\|D_{1/u}f\|_{BMO(\rho)}du \\ &= \rho(|I|)|I|\Big(\int_0^1 |K(u,1)|\|D_{1/u}f\|_{BMO(\rho)}du \\ &+ \int_1^\infty |K(u,1)|\|D_{1/u}f\|_{BMO(\rho)}du\Big). \end{aligned}$$

When $u \in (0,1)$, $s = u^{-1} \in (1,\infty)$ and (5) gives

$$\|D_{1/u}f\|_{BMO(\rho)} = \|D_s f\|_{BMO(\rho)} \le Cs^{-l}\|f\|_{BMO(\rho)} = Cu^l\|f\|_{BMO(\rho)}.$$

Similarly, when $u \in (1, \infty)$, $s = u^{-1} \in (0, 1)$ and (6) yields

$$\|D_{1/u}f\|_{BMO(\rho)} = \|D_s f\|_{BMO(\rho)} \leq Cs^{-m}\|f\|_{BMO(\rho)} = Cu^m\|f\|_{BMO(\rho)}.$$

Consequently,

$$\int_I |Tf(t) - (Tf)_I|dt$$
$$\leq C\rho(|I|)|I|\|f\|_{BMO(\rho)} \left(\int_0^1 |K(u,1)|u^l du + \int_1^\infty |K(u,1)|u^m du\right)$$

for some $C > 0$. The above inequality guarantees that $(Kf)_I$ is finite and well defined. Moreover, by multiplying $\frac{1}{\rho(|I|)|I|}$ on both sides of the above inequalities, we have

$$\frac{1}{\rho(|I|)|I|}\int_I |Tf(t) - (Tf)_I|dt$$
$$\leq C\|f\|_{BMO(\rho)} \left(\int_0^1 |K(u,1)|u^l du + \int_1^\infty |K(u,1)|u^m du\right).$$

By taking supremum over $I \in \mathcal{I}$, we obtain (13). ◄

When $\rho \equiv 1$, the above result recovers the boundedness of the integral operators on BMO given in [7, Corollary 4].

As $\rho(r) = r^\alpha$, $0 < \alpha \leq 1$ is a growth function of lower type α and upper type α, Theorem 1 also gives the following result for the Lipschitz spaces.

Corollary 1. *Let $0 < \alpha \leq 1$ and $K : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function satisfying (11) and*

$$\int_0^\infty |K(u,1)|u^\alpha du < \infty.$$

There exists a constant $C > 0$ such that for any $f \in \Lambda_\alpha$

$$\|Tf\|_{\Lambda_\alpha} \leq C\|f\|_{\Lambda_\alpha}.$$

We now apply Theorem 1 to some concrete operators. The classical Hardy operator is defined as

$$Hf(t) = \frac{1}{t}\int_0^t f(s)ds.$$

For the history, development and applications of the Hardy inequality, the reader is referred to [15, 16, 21].

We now establish the Hardy inequalities on $BMO(\rho)$.

Theorem 2. *Let $m, l \geq 0$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a growth function of upper type m and lower type l. There is a constant $C > 0$ such that for any $f \in BMO(\rho)$, we have*

$$\|Hf\|_{BMO(\rho)} \leq C\|f\|_{BMO(\rho)}.$$

Proof. We find that

$$Hf(t) = \int_0^\infty K(s,t)f(s)ds,$$

where $K(s,t) = \frac{1}{t}\chi_{(s,t):s\leq t}$. It obviously satisfies (11). Moreover,

$$\int_0^1 |K(u,1)|u^l du + \int_1^\infty |K(u,1)|u^m du = \int_0^1 u^l du = \frac{1}{1+l} < \infty.$$

Thus, (12) is fulfilled. Theorem 1 yields the boundedness of $H : BMO(\rho) \to BMO(\rho)$. ◄

In particular, we have the Hardy's inequality on BMO. The validity of the Hardy's inequality on BMO is well known. For the studies of the Hardy inequality on BMO, the reader is referred to [26, 27].

We now turn to the study of the Erdélyi-Kober fractional integral operators. We use the definitions of the Erdélyi-Kober fractional integral operators from [14, (0.7)] and [22]. Let $\delta, \eta > 0$ and $\gamma \in \mathbb{R}$. For any locally integrable function f, the Erdélyi-Kober fractional integral operators are defined as

$$I_\eta^{\gamma,\delta} f(t) = \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t s^{\eta(\gamma+1)-1}(t^\eta - s^\eta)^{\delta-1} f(s)ds, \quad t \geq 0,$$
$$K_\eta^{\gamma,\delta} f(t) = \frac{t^{\eta\gamma}}{\Gamma(\delta)} \int_t^\infty (s^\eta - t^\eta)^{\delta-1} s^{-\eta(\gamma+\delta)+\eta-1} f(s)ds, \quad t \geq 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

The Erdélyi-Kober fractional integral operators have a number of applications on fractional calculus, applied mathematics and statistics, the reader is referred to [14, 22] for those applications of the Erdélyi-Kober fractional integral operators. For the mapping properties of the Erdélyi-Kober fractional integral operators on Morrey spaces, amalgam spaces and rearrangement-invariant spaces, the reader is referred to [9].

We now use Theorem 1 to study the boundedness of the Erdélyi-Kober fractional integral operators on $BMO(\rho)$.

Theorem 3. *Let $\delta, \eta > 0$ and $\gamma \in \mathbb{R}$. Let $m, l \geq 0$ and $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a growth function of upper type m and lower type l.*

1. *If $\eta(\gamma+1)+l>0$, then there is a constant $C>0$ such that for any $f\in BMO(\rho)$,*
$$\|I_\eta^{\gamma,\delta}f\|_{BMO(\rho)}\le C\|f\|_{BMO(\rho)}.$$
2. *If $-\eta\gamma+m<0$, then there is a constant $C>0$ such that for any $f\in BMO(\rho)$,*
$$\|K_\eta^{\gamma,\delta}f\|_{BMO(\rho)}\le C\|f\|_{BMO(\rho)}.$$

Proof. We have $I_\eta^{\gamma,\delta}f(t)=\int_0^\infty K_0(s,t)f(s)ds$, where
$$K_0(s,t)=\frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\chi_{(0,t)}(s)s^{\eta(\gamma+1)-1}(t^\eta-s^\eta)^{\delta-1}.$$

For any $\lambda>0$, we have
$$\begin{aligned}K_0(\lambda s,\lambda t)&=\frac{(\lambda t)^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\chi_{(0,\lambda t)}(\lambda s)(\lambda s)^{\eta(\gamma+1)-1}((\lambda t)^\eta-(\lambda s)^\eta)^{\delta-1}\\&=\lambda^{-1}K_0(s,t).\end{aligned}$$

Moreover, there exists a constant $C>0$ such that
$$\begin{aligned}&\int_0^1|K_0(u,1)|u^l du+\int_1^\infty|K_0(u,1)|u^m du\\&=\frac{1}{\Gamma(\delta)}\int_0^1 s^{\eta(\gamma+1)+l-1}(1-s^\eta)^{\delta-1}ds\\&\le C\left(\int_0^{\frac12}s^{\eta(\gamma+1)+l-1}ds+\int_{\frac12}^1(1-s^\eta)^{\delta-1}ds\right)<\infty,\end{aligned}$$
because $\eta(\gamma+1)+l>0$ and $\delta,\eta>0$. Therefore, Theorem 1 yields the boundedness of $I_\eta^{\gamma,\delta}$ on $BMO(\rho)$.

For $K_\eta^{\gamma,\delta}$, we have $K_\eta^{\gamma,\delta}f(t)=\int_0^\infty K_1(s,t)f(s)ds$, where
$$K_1(s,t)=\frac{t^{\eta\gamma}}{\Gamma(\delta)}\chi_{(t,\infty)}(s)(s^\eta-t^\eta)^{\delta-1}s^{-\eta(\gamma+\delta)+\eta-1}.$$

Obviously, for any $\lambda>0$, $K_1(\lambda s,\lambda t)=\lambda^{-1}K(s,t)$. In addition,
$$\begin{aligned}&\int_0^1|K_1(u,1)|u^l du+\int_1^\infty|K_1(u,1)|u^m du\\&=\frac{1}{\Gamma(\delta)}\int_1^\infty(s^\eta-1)^{\delta-1}s^{-\eta(\gamma+\delta)+\eta+m-1}ds\end{aligned}$$

$$\leq C\left(\int_1^2 (s^\eta - 1)^{\delta-1} ds + \int_2^\infty s^{\eta\delta-\eta-\eta(\gamma+\delta)+\eta+m-1} ds\right) < \infty,$$

because $-\eta\gamma + m < 0$ and $\delta, \eta > 0$. Theorem 1 gives the boundedness of $K_\eta^{\gamma,\delta}$ on $BMO(\rho)$. ◀

Particularly, we obtain the boundedness of the Erdélyi-Kober fractional integral operators on BMO.

Theorem 4. *Let $\delta, \eta > 0$ and $\gamma \in \mathbb{R}$.*

1. *If $\gamma > -1$, then there is a constant $C > 0$ such that for any $f \in BMO$,*

$$\|I_\eta^{\gamma,\delta} f\|_{BMO} \leq C\|f\|_{BMO}.$$

2. *If $\gamma > 0$, then there is a constant $C > 0$ such that for any $f \in BMO$,*

$$\|K_\eta^{\gamma,\delta} f\|_{BMO} \leq C\|f\|_{BMO}.$$

It is well known that the Hardy space H^1 is the pre-dual of BMO. For the studies of the Erdélyi-Kober fractional integral operators, the reader may consult [8].

We also have the corresponding result for the Lipschitz spaces.

Corollary 2. *Let $0 < \alpha \leq 1$, $\delta, \eta > 0$ and $\gamma \in \mathbb{R}$.*

1. *If $\eta(\gamma+1)+\alpha > 0$, then there is a constant $C > 0$ such that for any $f \in \Lambda_\alpha$,*

$$\|I_\eta^{\gamma,\delta} f\|_{\Lambda_\alpha} \leq C\|f\|_{\Lambda_\alpha}.$$

2. *If $-\eta\gamma + \alpha < 0$, then there is a constant $C > 0$ such that for any $f \in \Lambda_\alpha$,*

$$\|K_\eta^{\gamma,\delta} f\|_{\Lambda_\alpha} \leq C\|f\|_{\Lambda_\alpha}.$$

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References

[1] R. Bandaliyev, P. Gorka, *Hausdorff operator in Lebesgue spaces*, Math. Inequal. Appl., **22**, 2019, 657-676.

[2] K.-P. Ho, *Hardy's inequality on Hardy spaces*, Proc. Japan Acad. Ser. A Math. Sci., **92**, 2016, 125-130.

[3] K.-P. Ho, *Hardy's Inequality on Hardy-Morrey spaces with variable exponents*, Mediterr. J. Math., **14**, 2017, 79-98.

[4] K.-P. Ho, *Atomic decompositions and Hardy's inequality on weak Hardy-Morrey spaces*, Sci. China Math., **60**, 2017, 449-468.

[5] K.-P. Ho, *Discrete Hardy's inequality with* $0 < p \leq 1$, J. King Saud Univ. Sci., **30**, 2018, 489-492.

[6] K.-P. Ho, *Hardy's inequality on Hardy-Morrey spaces*, Georgian Math. J., **26**, 2019, 40-414.

[7] K.-P. Ho, *Integral operators on BMO and Campanato spaces*, Indag. Math., **30**, 2019, 1023-1035.

[8] K.-P. Ho, *Erdélyi-Kober fractional Integrals on Hardy space and BMO*, Proyecciones, **39**, 2020, 663-677.

[9] K.-P. Ho, *Erdélyi-Kober fractional integral operators on ball Banach function spaces*, Rend. Semin. Mat. Univ. Padova (to appear).

[10] S. Janson, *On functions with conditions on the mean oscillation*, Ark. Mat., **14**, 1976, 189–196.

[11] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat., **16**, 1978, 263–270.

[12] S. Janson, *Lipschitz spaces and bounded mean oscillation*, Duke Math. J., **47**, 1980, 959-982.

[13] F. John, L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14**, 1961, 415-426.

[14] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Chapman and Hall/CRC, 1993.

[15] A. Kufner, L. Maligranda, L.-E. Persson, *The Prehistory of the Hardy Inequality*, Amer. Math. Monthly, **113**, 715-732, 2006.

[16] A. Kufner, L.-E. Persson, N. Samko, *Weighted Inequalities of Hardy type*, World Scientific Publishing Company, 2017.

[17] E. Liflyand, F. Moricz, *The Hausdorff operator is bounded on real H^1 space*, Proc. Amer. Math. Soc., **128**, 2000, 1391-1396.

[18] N.G. Meyers, *Mean oscillation over cubes and Hö1der continuity*, Proc. Amer. Math. Soc., **15**, 1964, 717–721.

[19] E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math., **5**, 2001, 587-602.

[20] E. Nakai, Y. Sawano, *Orlicz-Hardy spaces and their duals*, Sci. China Math., **57**, 2014, 903-962.

[21] B. Opic, A. Kufner, *Hardy-type inequalities*, Pitman Reserach Notes in Math. Series 219, Longman Sci. and Tech, Harlow, 1990.

[22] I. Sneddon, *The use in mathematical physics of Erdélyi-Kober operators and of some of their generalizations*, In: Ross B. (eds) Fractional Calculus and Its Applications. Lecture Notes in Mathematics, **457**, Springer, Berlin, Heidelberg.

[23] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuola Norm. Sup. Pisa, **19**, 1965, 593–608.

[24] E. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[25] Viviani, B. An atomic decomposition of the predual of $BMO(\rho)$, Rev. Mat. Iberoamericana, **3**, 1987, 401-425.

[26] J. Xiao, *A reverse BMO-Hardy inequality*, Real Anal. Exchange, **25**, 1999, 673-678.

[27] J. Xiao, *L^p and BMO bounds of weighted Hardy-Littlewood averages*, J. Math. Anal. Appl., **262**, 2001, 660-666.

Kwok-Pun Ho
Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, Hong Kong, China
E-mail: vkpho@eduhk.hk

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