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Existence of Weak Solutions for a Nonlocal Singular Elliptic Problem

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Abstract. In this article, a nonlocal singular elliptic problem is considered. The existence of an unbounded sequence of weak solutions is proved and, under different conditions, strong convergence of a sequence of weak solutions is studied. The main tools are variational methods and critical point theory.

Key Words and Phrases: infinitely many solutions, singular problem, nonlocal problem, Kirchhoff–type problem.

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1. Introduction

The study of partial differential equations (PDEs) started in the 18th Century in the work of Euler, D'Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science.

The area of PDEs has been growing steadily since the middle of the 19th Century. After Poincaré's prophetic paper [18] in 1890 one can say that on one side PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics (see, for example, [3, 4, 5, 6, 7, 8, 9, 10, 11, 19, 20, 21, 22, 23, 24, 25, 26, 27]) and on the other side there are the potential applications which have often turned out to be quite revolutionary of PDEs as an instrument in the development of other branches of mathematics (see [2]).

The Laplace equation as one of the PDEs famous problem was first studied by Laplace in his work on gravitational potential fields around 1780. Since then

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this problem is noticeable and investigated by many researchers (see [14, 16] for recent articles).

In 1883, Kirchhoff [13] presented a model given by

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right| dx \right) \frac{\partial^2 u}{\partial x^2} = 0.$$

This equation is an extension of the classical D'Alembert's wave equation by taking into account the effects of the changes in the length of the string during the vibrations. The parameters in this equation have the following meanings: h is the cross-section area, E is the Young modulus, ρ is the mass density, L is the length of the string, and ρ_0 is the initial tension. After that, many people studied the elliptic problems involving Kirchhoff–type operators. For instance, in [7] the authors study the weak solutions for the following Kirchhoff–type problem:

$$\begin{cases} T(u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $T(u) = K\left(\int_\Omega [\Phi(|\nabla u|) + \Phi(|u|)]dx\right)(-div(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u)$, and $K : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function such that there exists a positive number m with $K(t) \geq m$ for all $t \geq 0$. If $\Phi(t) = t^p$, then this problem becomes the well-known p-Kirchhoff-type problem.

Since these equations contain an integral over Ω, they are called nonlocal problems as they are no longer a pointwise identity. We can refer to [15, 17, 29, 31] for some Kirchhoff type problems.

On the other hand, recently many authors have paid attention to the existence and multiplicity of solutions for the singular boundary value problems. For example, Ferrara et al. [4] consider the following problem with singular term:

$$\begin{cases} -\Delta_p u = \mu \frac{|u|^{p-2}u}{|x|^p} + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ denotes the usual p-Laplacian. Khodabakhshi et al. [12] prove the existence of at least three weak solutions for the following singular problem:

$$\begin{cases} -\Delta_p u + \frac{|u|^{p-2}u}{|x|^p} = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Makvand et al. [15] study the existence of at least one weak solution for the following fourth-order singular problem:

$$\begin{cases} \Delta_p^2 u = \mu \frac{|u|^{p-2}u}{|x|^{2p}} + \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ stands for the p-biharmonic operator (see [30]).

Motivated by such facts, we study the existence of weak solutions for the following Kirchhoff-type problem involving a singular term:

$$\begin{cases} -K\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u + a(x)\frac{|u|^{q-2}u}{|x|^q} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $K : R^+ \to R$ is a continuous function such that there exist two positive numbers k_0, k_1 with $k_0 \leq K(t) \leq k_1$ for all $t \geq 0$, and $a \in L^\infty(\Omega)$ with $\mathrm{ess}_\Omega \inf a(x) > 0$. Moreover, $1 < q < N < p$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. As we see, in this article the exponent in the singular term is different from p in the p-Laplacian operator, and the problem is studied where $N < p$. Therefore we have the compact embedding $X := W_0^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$.

Definition 1. *The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be an L^1-Carathéodory function if*

(A_1) *the function $x \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$;*

(A_2) *the function $t \mapsto f(x,t)$ is continuous for a.e. $x \in \Omega$;*

(A_3) *for every $h > 0$ there exists a function $\ell_h \in L^1(\Omega)$ such that*

$$\sup_{|t|\leq h} |f(x,t)| \leq \ell_h(x),$$

for a.e. $x \in \Omega$.

Here we recall some preliminaries which are needed in the sequel. Let $X := W_0^{1,p}(\Omega)$, endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}},$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, containing the origin and with smooth boundary $\partial\Omega$. Moreover, let $\|\cdot\|_1$ denote the norm of $L^1(\Omega)$; i.e.,

$$\|u\|_1 := \int_\Omega |u(x)|dx$$

and $\|\cdot\|_\infty$ be the usual norm in $L^\infty(\Omega)$. One of the main tools we use in this article is the classical Hardy inequality which is

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}dx \leq \frac{1}{H}\int_\Omega |\nabla u(x)|^q dx \qquad \text{for all } u \in X, \tag{2}$$

where $H := \left(\frac{N-q}{q}\right)^q$.

Let $F(x,t) := \int_0^t f(x,s)ds$, for every (x,t) in $\Omega \times \mathbb{R}$ and $\hat{K}(t) := \int_0^t K(s)ds$, for all $t \geq 0$.

Then we define the following functionals for every $u \in X$:

- $\Phi(u) := \frac{1}{p}\hat{K}(\|u\|^p) + \frac{1}{q}\int_\Omega \frac{|u(x)|^q}{|x|^q}a(x)dx$,
- $\Psi(u) := \int_\Omega F(x,u(x))dx$,
- $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$.

From Hardy inequality, one has

$$\frac{k_0}{p}\|u\|^p \leq \Phi(u) \leq \frac{k_1}{p}\|u\|^p + \frac{\|a\|_\infty}{qH}\int_\Omega |\nabla u(x)|^q dx \tag{3}$$

for every $u \in X$. So Φ is well defined, coercive and Gâteaux differentiable and its Gâteaux derivative is given by

$$\begin{aligned}\Phi'(u)(v) = \quad & K\left(\int_\Omega |\nabla u(x)|^p dx\right)\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx \\ & + \int \frac{|u(x)|^{q-2}u(x)v(x)}{|x|^q}a(x)dx\end{aligned}$$

for every $v \in X$. Also Φ is strongly continuous.

Moreover, by standard arguments, Ψ is well defined and continuously Gâteaux differentiable functional and its derivative

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$$

is compact operator for every $v \in X$.

Definition 2. *It is said that a function $u : \Omega \to \mathbb{R}$ is a weak solution of* (1) *if $u \in X$ and*

$$\begin{aligned}K\left(\int_\Omega |\nabla u|^p dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \quad & \int_\Omega \frac{|u|^{q-2}uv}{|x|^q}a(x)dx \\ & -\lambda\int_\Omega f(x,u)vdx = 0,\end{aligned}$$

for every $v \in X$ and fixed real parameter λ, which shows that the critical points of I_λ are exactly the weak solutions of (1).

Here is our main tool to study the weak solutions for (1) [1, Theorem 2.1], which is a version of Ricceri's critical points principle [28, Theorem 2.5].

Theorem 1. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{\Phi(u)<r} \frac{\sup_{\Phi(v)<r} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \textit{and} \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) if $\gamma < +\infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternatives hold: either

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) if $\delta < +\infty$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternatives hold: either

(c_1) there is a global minimum of Φ which is a local minimum of I_λ, or

(c_2) there is a sequence $\{u_n\}$ for pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with

$$\lim_{n\to+\infty} \Phi(u_n) = \inf_{u\in X} \Phi(u).$$

2. Multiple solutions

Let $D > 0$ such that $B(x_0, D) \subseteq \Omega$ and $\overline{B(x_0, D)}$ not containing the origin, for fixed $x_0 \in D$, where $B(x_0, D)$ denotes the ball with center at x_0 and radius D. As $p > N$, from the compact embedding $X \hookrightarrow C(\bar{\Omega})$, there exists a constant c such that

$$\max_{x\in\bar{\Omega}} |u(x)| \leq c\|u\|, \qquad \text{for all } u \in X. \tag{4}$$

Put

$$\alpha := \frac{k_1}{p}\left(\left(\frac{2}{D}\right)^p \omega\left(D^N - \left(\frac{D}{2}\right)^N\right)\right), \tag{5}$$

and

$$\beta := \frac{\|a\|_\infty}{Hq}\left(\left(\frac{2}{D}\right)^q \omega\left(D^N - \left(\frac{D}{2}\right)^N\right)\right), \tag{6}$$

where $\omega := \frac{\Gamma\frac{N}{2}}{\Gamma(1+\frac{N}{2})}$, and Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz, \qquad \text{for all } t > 0.$$

Moreover, let

$$A := \liminf_{\xi\to+\infty} \frac{\|\ell_\xi\|_1}{\xi^{p-1}},$$

and

$$B := \limsup_{\xi\to+\infty} \frac{\int_{B(x_0,\frac{D}{2})} F(x,\xi)dx}{\xi^p},$$

where $\ell_\xi \in L^1(\Omega)$ satisfies condition (A_3) on $f(x,t)$ for every $\xi > 0$.

Here we present a theorem which shows that there exists an unbounded sequence of weak solutions for (1).

Theorem 2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1–Carathéodory function such that*

(F_1) $F(x,t) \geq 0$ for every $(x,t) \in \Omega \times \mathbb{R}^+$;

(F_2) $A < \frac{k_0}{p\alpha c^p}B$, where c and α are given by (4) and (5).

Then for every $\lambda \in \Lambda := \left]\frac{\alpha}{B}, \frac{k_0}{pAc^p}\right[$, the problem (1) admits an unbounded sequence of weak solutions in X.

Proof. Let X, Φ and Ψ be as in Section 1, Φ and Ψ satisfy the assumptions in Theorem 1.

From (3) and (4), we get

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) &= \{u \in X;\ \Phi(u) < r\} \\ &\subset \left\{u \in X;\ \frac{k_0}{p}\|u\|^p < r\right\} \\ &\subset \left\{u \in X;\ \max_{x\in\bar{\Omega}} |u(x)| < c\left(\frac{pr}{k_0}\right)^{\frac{1}{p}}\right\}.\end{aligned} \tag{7}$$

Take $\{\xi_n\} \subset \mathbb{R}^+$ such that $\xi_n \to +\infty$ as $n \to +\infty$, and

$$\lim_{n\to+\infty} \frac{\|\ell_{\xi_n}\|_1}{\xi_n^{p-1}} = A.$$

Set $r_n := \frac{k_0 \xi_n^p}{pc^p}$, for all $n \in \mathbb{N}$. Then from (7), one has

$$\max_{x\in\bar{\Omega}} |u(x)| \leq c\left(\frac{pr_n}{k_0}\right)^{\frac{1}{p}} = \xi_n,$$

for all $u \in X$ with $\Phi(u) < r_n$.

Now, since $\Phi(0) = \Psi(0) = 0$, we get

$$\begin{aligned}\varphi(r_n) &= \inf_{\Phi(v)<r_n} \frac{\left(\sup_{\Phi(u)<r_n} \Psi(u)\right) - \Psi(v)}{r_n - \Phi(v)} \\ &\leq \frac{\sup_{\Phi(u)<r_n} \Psi(u)}{r_n} \\ &\leq \frac{\sup_{|u(x)|<\xi_n} \int_\Omega F(x,u(x))dx}{r_n} \\ &\leq \frac{\xi_n \|\ell_{\xi_n}\|_1}{\frac{k_0\xi_n^p}{pc^p}} = \frac{pc^p \|\ell_{\xi_n}\|_1}{k_0 \xi_n^{p-1}}.\end{aligned}$$

Therefore,

$$\gamma = \liminf_{r\to+\infty} \varphi(r) \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \frac{pc^p}{k_0} \liminf_{n\to+\infty} \frac{\|\ell_{\xi_n}\|_1}{\xi_n^{p-1}} = \frac{pc^p}{k_0} A < +\infty,$$

and $\lambda < \frac{1}{\gamma}$ as $\lambda \in]\frac{\alpha}{B}, \frac{k_0}{pAc^p}[$.

Now, in order to show that I_λ is unbounded from below, let $\{t_n\}$ be a sequence of real numbers with $\lim_{n\to+\infty} t_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_{B(x_0,\frac{D}{2})} F(x,t_n)dx}{t_n^p} = B, \tag{8}$$

Moreover, define $\{w_n\} \subseteq X$ as follows:

$$w_n(x) := \begin{cases} 0 & x \in \Omega \setminus B(x_0, D), \\ t_n & x \in B(x_0, \frac{D}{2}), \\ \frac{2t_n}{D}(D - |x - x_0|) & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}). \end{cases}$$

Using Hardy inequality we have,

$$\begin{aligned}\Phi(w_n) &\le \frac{k_1}{p}\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})}|\nabla w_n|^p dx + \frac{\|a\|_\infty}{qH}\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})}|\nabla w_n|^q dx\\ &\le t_n^p\alpha + t_n^q\beta.\end{aligned}$$

On the other hand, from (F_1) we have

$$\Psi(w_n) = \int_\Omega F(x, w_n(x))dx \ge \int_{B(x_0,\frac{D}{2})} F(x, t_n)dx$$

for every $n \ge 1$.

Here,

$$I_\lambda(w_n) \le \alpha t_n^p + \beta t_n^q - \lambda \int_{B(x_0,\frac{D}{2})} F(x, t_n)dx.$$

If $B < +\infty$, let $\varepsilon \in]\frac{\alpha}{\lambda B}, 1[$. Then from (8), there exists N_ε such that

$$\int_{B(x_0,\frac{D}{2})} F(x, t_n)dx > \varepsilon B t_n^p, \qquad \text{for all } n > N_\varepsilon.$$

Therefore,

$$I_\lambda(w_n) < \alpha t_n^p + \beta t_n^q - \lambda\varepsilon B t_n^p = (\alpha - \lambda\varepsilon B)t_n^p + \beta t_n^q,$$

for every $n > N_\varepsilon$. So

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty,$$

as $q < p$ and $(\alpha - \lambda\varepsilon B) < 0$. If $B = +\infty$, let $L > \frac{\alpha}{\lambda}$. Then from (8), there exists N_L such that

$$\int_{B(x_0,\frac{D}{2})} F(x, t_n)dx > L t_n^p, \qquad \text{for all } n > N_L.$$

Therefore, we have

$$\begin{aligned} I_\lambda(w_n) &< \alpha t_n^p + \beta t_n^q - \lambda L t_n^p\\ &= (\alpha - \lambda L)t_n^p + \beta t_n^q \qquad \text{for all } n > N_L.\end{aligned}$$

Since L is arbitrary in this case, we also get

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty,$$

as $q < p$ and $(\alpha - \lambda L) < 0$. Thus, the result is obtained owing to Theorem 1(b). ◀

Now, we present another result which implies the problem (1) has a sequence of weak solutions which converges to zero in X. To this end, set

$$A' := \liminf_{\xi \to 0^+} \frac{\|\ell_\xi\|_1}{\xi^{p-1}}, \quad B' := \limsup_{\xi \to 0^+} \frac{\int_{B(x_0, \frac{D}{2})} F(x, \xi) dx}{\xi^p}.$$

Arguing as in the previous theorem and using Theorem 1 (c), we obtain the following result.

Theorem 3. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1–Carathéodory function such that*

(F_1') $F(x,t) \geq 0$; for every $(x,t) \in \Omega \times \mathbb{R}^+$;

(F_2') $A' < \frac{k_0}{p\alpha c^p} B'$, where α and c are as in Theorem 2.

Then for every $\lambda \in \Lambda' := \left] \frac{\alpha}{B'}, \frac{k_0}{pA'c^p} \right[$, the problem (1) admits a sequence of weak solutions, which converges strongly to zero in X.

Proof. Considering Φ, Ψ and I_λ as in Section 1, for fixed $\lambda \in \Lambda'$, first we show that $\lambda < \frac{1}{\delta}$. We know that Φ and Ψ satisfy the assumptions needed in Theorem 1.

Let $\{t_n\}$ be a sequence of positive numbers such that $\lim_{n \to +\infty} t_n = 0$ and

$$A' = \lim_{n \to +\infty} \frac{\|\ell_{t_n}\|_1}{t_n^{p-1}}.$$

As we know $\inf_X \Phi = 0$, therefore

$$\delta := \liminf_{r \to 0^+} \varphi(r).$$

Set $r_n := \frac{k_0 t_n^p}{pc^p}$, for every $n \in \mathbb{N}$. Then from (3) and (4) we have

$$\|u\|_\infty := \max_{x \in \bar{\Omega}} |u(x)| \leq c \left(\frac{pr_n}{k_0} \right)^{\frac{1}{p}} = t_n,$$

for all $u \in X$ with $\Phi(u) < r_n$. Thus,

$$\varphi(r_n) \leq \frac{\sup_{\Phi(u) < r_n} \Psi(u)}{r_n} \leq \frac{pc^p \|\ell_{t_n}\|_1}{k_0 t_n^{p-1}}.$$

Therefore

$$\delta \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \frac{pc^p}{k_0} \liminf_{n\to+\infty} \frac{\|\ell_{t_n}\|_1}{t_n^{p-1}} = \frac{pc^p}{k_0} A' < +\infty,$$

and $\lambda < \frac{1}{\delta}$, so $\Lambda' \subset]0, \frac{1}{\delta}[$. Let's we show that zero is not a local minimum of I_λ.

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0$ as $n \to +\infty$, and

$$B' = \lim_{n\to+\infty} \frac{\int_{B(x_0, \frac{D}{2})} F(x, \xi_n)dx}{\xi_n^p}.$$

Moreover, set $\{w_n\} \subset X$, similar to the proof of Theorem 2, but defined by $\{\xi_n\}$ above instead of $\{t_n\}$, so we obtain that $I_\lambda(w_n) < 0$ for n large enough, hence zero is not a local minimum of I_λ since

$$\lim_{n\to+\infty} I_\lambda(w_n) < I_\lambda(0) = 0.$$

Therefore, by Theorem 1 (c), there exists a sequence $\{u_n\}$ of critical points of I_λ in X such that

$$\lim_{n\to+\infty} \Phi(u_n) = 0,$$

and this implies that $\{u_n\}$ converges strongly to zero in X due to (3). ◀

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