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# Ky Fan Inequalities for Bivariate Means

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**Abstract.** In this note we give necessary and sufficient conditions for bivariate, homogeneous, symmetric means $M$ and $N$ to satisfy Ky Fan inequalities

$$\frac{M}{M'} < \frac{N}{N'} \qquad \text{and} \qquad \frac{1}{M} - \frac{1}{M'} < \frac{1}{N} - \frac{1}{N'}.$$

**Key Words and Phrases**: Ky Fan inequality, mean, homogeneity.
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## 1. Introduction

In mathematics there are many inequalities named after Professor Ky Fan. We find them in matrix and operator theory, Ky Fan-Taussky-Todd inequality generalizes the Wirtinger inequality, Ky Fan minimax inequality is often used in game theory, mathematical economics or control theory. For an overview of Ky Fan results see [8]. In this note we investigate the Ky Fan arithmetic-geometric mean inequality and its generalizations.

Denote by $\mathsf{G}$ and $\mathsf{A}$ the geometric and arithmetic means of nonnegative numbers $x_i,\ i = 1, \dots, n$

$$\mathsf{G}(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i^{1/n}, \quad \mathsf{A}(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

The classical Ky Fan inequality states that if $0 < x_i \le 1/2$, then

$$\frac{\mathsf{G}(x_1, \dots, x_n)}{\mathsf{G}(1 - x_1, \dots, 1 - x_n)} \le \frac{\mathsf{A}(x_1, \dots, x_n)}{\mathsf{A}(1 - x_1, \dots, 1 - x_n)}.$$

This result has an interesting history: it has been published in [2] with reference to an unpublished result of Ky Fan. In fact, in 1959 Ky Fan published the following problem in [4] (solved by T. Stanisz in [5]).

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**Problem 1.** *Let $f$ be a measurable function such that $0 < f(x) \le 1/2$ for $x \in [0,1]$. Prove*

$$\frac{\exp\int_0^1 \log f(x)dx}{\int_0^1 f(x)dx} \le \frac{\exp\int_0^1 \log[1-f(x)]dx}{\int_0^1 [1-f(x)]dx}.$$

Clearly, from this inequality we easily obtain the Ky Fan inequality by taking $f(x) = x_i$ on the sets of measure $1/n$.

The inequality has been generalized in different directions (see e.g. [1, 3]). Especially interesting are the extensions where other bivariate means are involved.

In the papers of Neuman and Sándor [6, 7] we find the following sequence of inequalities:

$$\frac{\mathsf{G}}{\mathsf{G}'} \le \frac{\mathsf{L}}{\mathsf{L}'} \le \frac{\mathsf{P}}{\mathsf{P}'} \le \frac{\mathsf{A}}{\mathsf{A}'} \le \frac{\mathsf{NS}}{\mathsf{NS}'} \le \frac{\mathsf{T}}{\mathsf{T}'}, \tag{1}$$

$$\frac{1}{\mathsf{L}'} - \frac{1}{\mathsf{L}} \le \frac{1}{\mathsf{I}'} - \frac{1}{\mathsf{I}} \le \frac{1}{\mathsf{A}'} - \frac{1}{\mathsf{A}} \tag{2}$$

(here $\mathsf{L}, \mathsf{P}, \mathsf{NS}, \mathsf{T}, \mathsf{I}$ stand for the logarithmic, first Seiffert, Neuman-Sándor, second Seiffert and identric means of arguments $0 < x, y \le 1/2$, and prime denotes the same mean with arguments $1-x$ and $1-y$).

The aim of this note is to establish the conditions for two symmetric, homogeneous means $M, N$ under which the Ky Fan inequalities

$$\frac{M(x,y)}{M(1-x,1-y)} \le \frac{N(x,y)}{N(1-x,1-y)} \tag{3}$$

and

$$\frac{1}{M(1-x,1-y)} - \frac{1}{M(x,y)} \le \frac{1}{N(1-x,1-y)} - \frac{1}{N(x,y)} \tag{4}$$

hold, provided $0 < x, y \le 1/2$. We shall use the concept of Seiffert functions, that are in one-to-one correspondence with the symmetric and homogeneous means.

## 2. Definition and notation

A function $M : \mathbb{R}_+^2 \to \mathbb{R}$ is called a *mean* if for all $x, y > 0$

$$\min(x,y) \le M(x,y) \le \max(x,y).$$

Mean is *symmetric* if $M(x,y) = M(y,x)$ for all $x, y > 0$ and *homogeneous* if for all $\lambda > 0$

$$M(\lambda x, \lambda y) = \lambda M(x,y).$$

A function $m : (0,1) \to \mathbb{R}$ satisfying

$$\frac{z}{1+z} \le m(z) \le \frac{z}{1-z}$$

is called a *Seiffert function.*

Let us recall a result from [13].

**Theorem 1.** *The formula*

$$M(x,y) = \frac{|x-y|}{2m\left(\frac{|x-y|}{x+y}\right)}$$

*establishes a one-to-one correspondence between the set of symmetric, homogeneous means and the set of Seiffert functions.*

**Remark 1.** *If there is no risk of ambiguity we shall skip the argument of means. For* $0 < x, y \le 1/2$ *and an arbitrary mean* $M$ *we shall write* $x' = 1-x, y' = 1-y$ *and denote* $M(x', y')$ *by* $M'$.

## 3. Ky Fan inequalities

Let us introduce first a class of functions.

**Definition 1.** *A function* $f : [0,1) \to \mathbb{R}$ *is a Ky Fan function if for every* $a \in (0,1]$

$$f(a) \ge \sup_{t \in \left[0, \frac{a}{1+2a}\right]} f(t).$$

**Remark 2.** *Note that nondecreasing functions are obviously Ky Fan functions, because* $\frac{a}{1+2a} \le a$*, but the class is much broader. If* $f$ *is nondecreasing in* $[0, 1/3]$ *and* $f(x) \ge f(1/3)$ *for* $x > 1/3$*, then* $f$ *is also a Ky Fan function.*

Now we prove the theorem that is the key tool in our investigations.

**Theorem 2.** *For a function* $f : [0,1] \to \mathbb{R}$ *the following conditions are equivalent:*

*i)* $f$ *is a Ky Fan function;*

*ii) for all* $x, y \in (0, 1/2]$

$$f\left(\frac{|x'-y'|}{x'+y'}\right) \le f\left(\frac{|x-y|}{x+y}\right). \tag{5}$$

*Proof.* Fix $a \in [0,1)$ and assume without loss of generality that $x \geq y$. Consider the level set

$$I_a = \left\{(x,y) : \frac{x-y}{x+y} = a\right\} = \left\{\left(x, \frac{1-a}{1+a}x\right) : 0 < x \leq 1/2\right\}.$$

For $(x,y)$ in $I_a$ we have

$$\frac{|x'-y'|}{x'+y'} = \frac{ax}{1+a-x},$$

which means that for fixed $\frac{|x-y|}{x+y} = a$ the expression $\frac{|x'-y'|}{x'+y'}$ assumes all values between 0 and $\frac{a}{1+2a}$. So if $f$ is a Ky Fan function, then the inequality in ii) holds for all $x, y$, and *vice versa* , if the inequality ii) holds, then for fixed $a$ the value of $f(a)$ must be not lesser that the greatest of $f(u)$, where $0 \leq u \leq \frac{a}{1+2a}$, which shows that $f$ is a Ky Fan function. ◀

The consequence of Theorem 2 is the following.

**Theorem 3.** *Let $M, N$ be two symmetric, homogeneous means and let $m, n$ be their Seiffert functions. The Ky Fan inequality*

$$\frac{M(x,y)}{M(1-x,1-y)} \leq \frac{N(x,y)}{N(1-x,1-y)}$$

*holds for all $0 < x, y \leq 1/2$ if and only if the function $\frac{m}{n}$ is a Ky Fan function.*

*Proof.* By Theorem 1 the inequality

$$\frac{M(x,y)}{M(1-x,1-y)} \leq \frac{N(x,y)}{N(1-x,1-y)}$$

can be written as

$$\frac{m\left(\frac{|x-y|}{2-x-y}\right)}{m\left(\frac{|x-y|}{x+y}\right)} \leq \frac{n\left(\frac{|x-y|}{2-x-y}\right)}{n\left(\frac{|x-y|}{x+y}\right)}$$

or

$$\frac{m}{n}\left(\frac{|x-y|}{2-x-y}\right) \leq \frac{m}{n}\left(\frac{|x-y|}{x+y}\right),$$

so by Theorem 2 the proof is complete. ◀

As we shall see in the examples below the ratio of two Seiffert means is in general monotone, but at the end of this section we will construct two means satisfying the Ky Fan inequality with nonmonotonic ratio of Seiffert functions.

The next corollary gives a useful criterion to verify if the Ky Fan inequality holds.

**Corollary 1.** *Under the assumptions of Theorem 3, if the function* $q(t) = \frac{M(1,t)}{N(1,t)}$ *increases in* $(0,1)$*, then for all* $0 < x, y \leq 1/2$ *the Ky Fan inequality*

$$\frac{M(x,y)}{M(1-x,1-y)} \leq \frac{N(x,y)}{N(1-x,1-y)}$$

*holds.*

*Proof.* Let us recall the formula connecting the mean and its Seiffert function ([13]):

$$m(z) = \frac{z}{M(1+z,1-z)}. \tag{6}$$

This gives:

$$\frac{n}{m}(z) = \frac{M(1+z,1-z)}{N(1+z,1-z)} = \frac{M\left(1, \frac{1-z}{1+z}\right)}{N\left(1, \frac{1-z}{1+z}\right)} = \frac{M(1,s)}{N(1,s)} = q(s).$$

Here $s = \frac{1-z}{1+z}$ decreases from 1 to 0 as $z$ travels in the opposite direction, so $n/m$ decreases if and only if $q$ increases. Therefore, if $q$ increases, then $m/n$ increases, so by Remark 2 is a Ky Fan function, and by Theorem 2 the Ky Fan inequality for means holds. ◀

Let us illustrate our results with some examples:

**Example 1.** *This result was proved by Chan, Goldberg and Gonek in [9]: if*

$$\mathsf{A}_r(x,y) = \begin{cases} \left(\frac{x^r+y^r}{2}\right)^{1/r} & r \neq 0, \\ \sqrt{xy} & r = 0 \end{cases}$$

*is a power mean of order* $r$*, then for* $r < s$

$$\frac{\mathsf{A}_r}{\mathsf{A}'_r} \leq \frac{\mathsf{A}_s}{\mathsf{A}'_s}. \tag{7}$$

*Indeed, for* $0 < t < 1$ *and* $rs \neq 0$

$$\operatorname{sgn} \frac{d}{dt} \log \frac{\mathsf{A}_r(1,t)}{\mathsf{A}_s(1,t)} = \operatorname{sgn} \frac{t^r - t^s}{t(t^r+1)(t^s+1)} = \operatorname{sgn}(s-r),$$

*else if* $s = 0$*, then*

$$\operatorname{sgn} \frac{d}{dt} \log \frac{\mathsf{A}_r(1,t)}{\mathsf{A}_0(1,t)} = \operatorname{sgn} \frac{t^r - 1}{2t(t^r+1)} = -\operatorname{sgn} r,$$

*so the inequality* (7) *is true by Corollary 1.*

**Example 2.** *The last term in the chain of inequalities* (1) *contains the second Seiffert mean*

$$\mathsf{T}(x,y) = \frac{|x-y|}{2\arctan\frac{|x-y|}{x+y}}.$$

*Let* $\mathsf{Q}(x,y) = \sqrt{\frac{x^2+y^2}{2}}$ *be the quadratic (or root mean square).*
*We have* $\mathsf{t}(z) = \arctan z$ *and* $\mathsf{q}(z) = \frac{z}{\sqrt{1+z^2}}$ *and*

$$\frac{d}{dz}\frac{\mathsf{t}}{\mathsf{q}}(z) = \frac{z - \arctan z}{z^2\sqrt{1+z^2}} > 0,$$

*so by Theorem 3 we can extend the chain* (1)*:*

$$\frac{\mathsf{T}}{\mathsf{T}'} < \frac{\mathsf{Q}}{\mathsf{Q}'}.$$

**Example 3.** *Consider now the Heronian mean*

$$\mathsf{He}(x,y) = \frac{x+\sqrt{xy}+y}{3}.$$

*For* $0 < t < 1$ *we have*

$$q(t) = \frac{\mathsf{He}(1,t)}{\mathsf{A}_{2/3}(1,t)} = \frac{2\sqrt{2}}{3}\frac{t+\sqrt{t}+1}{(t^{2/3}+1)^{3/2}}$$

*and*

$$\frac{dq}{dt}(t) = \frac{\sqrt{2}}{3}\frac{(1-t^{1/6})^3(1+t^{1/6})}{(1+t^{2/3})^{5/2}t^{1/2}} > 0.$$

*We also see that the quotient*

$$\begin{aligned}\frac{\mathsf{He}(1,t)}{\mathsf{A}_{1/2}(1,t)} &= \frac{4}{3}\frac{t+\sqrt{t}+1}{t+2\sqrt{t}+1} = \frac{4}{3}\left[1 - \frac{\sqrt{t}}{t+2\sqrt{t}+1}\right] \\ &= \frac{4}{3}\left[1 - \frac{1}{\sqrt{t}+2+\frac{1}{\sqrt{t}}}\right]\end{aligned}$$

*decreases in* $(0,1)$*. Therefore by Corollary 1 we have*

$$\frac{\mathsf{A}_{1/2}}{\mathsf{A}'_{1/2}} < \frac{\mathsf{He}}{\mathsf{He}'} < \frac{\mathsf{A}_{2/3}}{\mathsf{A}'_{2/3}}.$$

**Example 4.** *We know ([11]) that the logarithmic mean* $\mathsf{L}(x,y) = \frac{x-y}{\log x - \log y}$ *satisfies the inequality* $\mathsf{L} < \mathsf{A}_{1/3}$. *Consider*

$$q(t) = \frac{\mathsf{A}_{1/3}(1,t)}{\mathsf{L}(1,t)} = \frac{(t^{1/3}+1)^3}{8(t-1)} \log t.$$

*Its derivative equals*

$$\frac{dq}{dt}(t) = \frac{(t^{1/3}+1)^2[t^{4/3}+t-t^{1/3}-1-(t+t^{1/3})\log t]}{8t(t-1)^2}.$$

*To evaluate the sign of the expression in square brackets substitute* $t = s^3$ *and calculate its Taylor series at* $s_0 = 1$*:*

$$\begin{aligned} t^{4/3}+t-t^{1/3}-1-(t+t^{1/3})\log t &= s^4+s^3-s-1-3(s^2+1)s\log s \\ &= -\sum_{n=5}^{\infty} \frac{n^2-5n+12}{n(n-1)(n-2)(n-3)}(1-s)^n < 0, \end{aligned}$$

*which proves the inequality*

$$\frac{\mathsf{L}}{\mathsf{L}'} < \frac{\mathsf{A}_{1/3}}{\mathsf{A}'_{1/3}}.$$

**Example 5.** *There is another mean similar to the Seiffert means that lies between the arithmetic and the first Seiffert mean:*

$$\mathsf{P}(x,y) = \frac{|x-y|}{2\arcsin\left(\frac{|x-y|}{x+y}\right)} \le \frac{|x-y|}{2\sinh\left(\frac{|x-y|}{x+y}\right)} = \mathsf{S}_{\sinh}(x,y) \le \frac{x+y}{2} = \mathsf{A}(x,y).$$

*Their respective Seiffert functions are* $\arcsin, \sinh$ *and* $\mathrm{id}$. *The hyperbolic sine is convex for positive arguments, therefore* $\frac{\sinh z}{z}$ *increases as a divided difference of a convex function. So by Theorem 3 we conclude*

$$\frac{\mathsf{S}_{\sinh}}{\mathsf{S}'_{\sinh}} \le \frac{\mathsf{A}}{\mathsf{A}'}.$$

**Example 6.** *It is known [10] that* $\mathsf{A}_{1/2} < \mathsf{P}$. *We have* $\mathsf{a}_{1/2}(z) = \frac{2z}{1+\sqrt{1-z^2}}$. *To check the monotonicity of* $\mathsf{p}/\mathsf{a}_{1/2}$ *we substitute* $z = \sin t$ *to get* $\frac{t}{2\sin t}(1+\cos t)$. *Its derivative* $\frac{t-\sin t}{2\cos t - 2}$ *is negative, so*

$$\frac{\mathsf{A}_{1/2}}{\mathsf{A}'_{1/2}} < \frac{\mathsf{P}}{\mathsf{P}'}.$$

**Example 7.** *The tangent is also a Seiffert function. It satisfies the inequality* $\tan z < \operatorname{artanh} z$ *for* $0 < z < 1$*, so its means satisfy*

$$\mathsf{L}(x,y) = \frac{|x-y|}{2\operatorname{artanh}\left(\frac{|x-y|}{x+y}\right)} < \frac{|x-y|}{2\tan\left(\frac{|x-y|}{x+y}\right)} = \mathsf{S}_{\tan}(x,y).$$

*Let us investigate the quotient of the Seiffert functions:*

$$\frac{d}{dz}\frac{\operatorname{artanh} z}{\tan z} = \frac{\frac{1}{2}\sin 2z - (1-z^2)\operatorname{artanh} z}{(1-z^2)\sin^2 z}.$$

*Using Taylor expansion we obtain*

$$\begin{aligned}\frac{1}{2}\sin 2z - (1-z^2)\operatorname{artanh} z &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2z)^{2n+1}}{(2n+1)!} - (1-z^2)\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}\\ &= \sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n-1)}\left[2 + \frac{(-1)^n 4^n(2n-1)}{(2n)!}\right] z^{2n+1} > 0,\end{aligned}$$

*because all the coefficients in square brackets are nonnegative. Thus by Theorem 3*

$$\frac{\mathsf{L}}{\mathsf{L}'} < \frac{\mathsf{S}_{\tan}}{\mathsf{S}'_{\tan}}.$$

At the end of this section we give an example of two means that satisfy the Ky Fan inequality, but the ratio of their Seiffert functions is not monotone.

**Example 8.** *Let*

$$V(x,y) = \begin{cases} \frac{x+y}{2} & \text{if} \quad \frac{1}{3}x \le y \le 3x,\\ \frac{|x-y|}{2} & \text{otherwise.}\end{cases}$$

*Its Seiffert function is given by*

$$v(z) = \begin{cases} z & \text{if} \quad z \le 1/2,\\ 1 & \text{otherwise.}\end{cases}$$

*It is easy to verify that* $v(z)/z$ *is a Ky Fan function, so the inequality* $V/V' \le \mathsf{A}/\mathsf{A}'$ *holds.*

## 4. Harmonic Ky Fan inequalities

In [12] the authors derive the inequalities of type

$$\frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}'} < \frac{1}{\mathsf{P}} - \frac{1}{\mathsf{P}'}$$

from the inequalities $\mathsf{P}/\mathsf{P}' < \mathsf{A}/\mathsf{A}'$ using some elementary algebraic transformation. Here we show an approach that bases on the concept of Seiffert functions.

Suppose $M$ and $N$ are two symmetric and homogeneous means. Using Theorem 1, the inequality

$$\frac{1}{M} - \frac{1}{M'} \le \frac{1}{N} - \frac{1}{N'}$$

can be written as

$$n\left(\frac{|x'-y'|}{x'+y'}\right) - m\left(\frac{|x'-y'|}{x'+y'}\right) \le n\left(\frac{|x-y|}{x+y}\right) - m\left(\frac{|x-y|}{x+y}\right).$$

Since $2-x-y > x+y$ for $0 < x, y < 1/2$, using Theorem 2 we can formulate the following result.

**Theorem 4.** *Suppose the means $M$ and $N$ are symmetric and homogeneous and $m$ and $n$ are their Seiffert means. Then the Ky Fan inequality*

$$\frac{1}{M} - \frac{1}{M'} \le \frac{1}{N} - \frac{1}{N'}$$

*holds if and only if $n - m$ is a Ky Fan function.*

Taking into account the formula (6) we set $s = (1+z)/(1-z)$ and using similar reasoning as in Corollary 1 we obtain

**Corollary 2.** *Under the assumptions of Theorem 4, if the function*

$$g(s) = (s-1)\left(\frac{1}{M(s,1)} - \frac{1}{N(s,1)}\right)$$

*decreases for $s > 1$, then the inequality*

$$\frac{1}{M} - \frac{1}{M'} \le \frac{1}{N} - \frac{1}{N'}$$

*holds.*

**Example 9.** *Consider the chain of inequalities between Seiffert functions (see [13, Lemma 3.1]) valid for* $0 < z < 1$

$$z > \operatorname{arsinh} z > \sin z > \arctan z > \tanh z.$$

*The mean corresponding to the second function is called Neuman-Sándor mean — $\mathsf{NS}$ and the fourth one generates the second Seiffert mean — $\mathsf{T}$.*

*Since* $\cosh^2 z - 1 = (\cosh z + 1)(\cosh z - 1) > 2 \cdot \frac{z^2}{2}$,

$$\frac{d}{dz}(\arctan z - \tanh z) = \frac{\cosh^2 z - 1 - z^2}{(1+z^2)\cosh^2 z} > 0.$$

*Inequality* $\cos z > 1 - z^2/2$ *leads to*

$$\frac{d}{dz}(\sin z - \arctan z) = \frac{(1+z^2)\cos z - 1}{1+z^2} > \frac{(1+z^2)(1-z^2/2) - 1}{1+z^2} = \frac{z^2(1-z^2)}{2(1+z^2)} > 0.$$

*On the other hand,* $\cos z < 1 - z^2/2 + z^4/6$ *, so*

$$\frac{d}{dz}(\operatorname{arsinh} z - \sin z) = \frac{1 - \sqrt{1+z^2}\cos z}{\sqrt{1+z^2}} > \frac{1 - \sqrt{(1+z^2)\left(1 - \frac{z^2}{2} + \frac{z^4}{6}\right)^2}}{\sqrt{1+z^2}}$$
$$= \frac{1 - \sqrt{1 - \frac{5}{12}z^4(1-z^2) - \frac{1}{36}z^8(5-z^2)}}{\sqrt{1+z^2}} > 0.$$

*And finally*

$$\frac{d}{dz}(z - \operatorname{arsinh} z) = 1 - \frac{1}{\sqrt{1+z^2}} > 0.$$

*Therefore by Theorem 4*

$$\frac{1}{\mathsf{S}_{\tanh}} - \frac{1}{\mathsf{S}'_{\tanh}} \le \frac{1}{\mathsf{T}} - \frac{1}{\mathsf{T}'} \le \frac{1}{\mathsf{S}_{\sin}} - \frac{1}{\mathsf{S}'_{\sin}} \le \frac{1}{\mathsf{NS}} - \frac{1}{\mathsf{NS}'} \le \frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}'}.$$

**Example 10.** *On the other side of the arithmetic mean there are two chains of inequalities for Seiffert means involving sine and tangent (see [13, Lemma 3.2]):*

$$z < \sinh z < \left\{ \begin{matrix} \tan z \\ \arcsin z \end{matrix} \right\} < \operatorname{artanh} z. \tag{8}$$

The two Seiffert functions in curly brackets are not comparable, arcsine defines the first Seiffert mean and inverse hyperbolic tangent corresponds to the logarithmic mean.

The difference $\sinh z - z$ increases, because this is the gap between a convex function and its supporting line.

To show that $\frac{d}{dz}(\tan z - \sinh z) = \frac{1-\cos^2 z \cosh z}{\cos^2 z}$ is positive note that

$$1 - \cos^2 z \cosh z > 1 - \cos z \cosh z =: f(z).$$

The function $f$ vanishes at $z = 0$ and

$$f'(z) = \cos z \cosh z(\tan z - \tanh z) > 0,$$

because $\tan z > z > \tanh z$. Thus $f'$ is positive, and so is $f$.

The difference between inverse hyperbolic tangent and tangent also increases, since

$$\frac{d}{dz}(\operatorname{artanh} z - \tan z)' = \frac{1}{1-z^2} - \frac{1}{\cos^2 z} = \frac{z^2 - \sin^2 z}{(1-z^2)\cos^2 z} > 0.$$

To deal with the lower chain of inequalities (8) note that

$$\begin{aligned} \cosh z =& 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots \\ <& 1 + \frac{z^2}{2} + \frac{z^4}{12}\left(\frac{1}{2} + \frac{1}{4} + \dots\right) = 1 + \frac{z^2}{2} + \frac{z^4}{12}. \end{aligned} \tag{9}$$

Now

$$\frac{d}{dz}(\arcsin z - \sinh z) = \frac{1 - \sqrt{1-z^2}\cosh z}{\sqrt{1-z^2}} > 0,$$

because by (9)

$$\begin{aligned} 1 - \sqrt{1-z^2}\cosh z &> 1 - \sqrt{(1-z^2)\left(1 + \frac{z^2}{2} + \frac{z^4}{12}\right)^2} \\ &= 1 - \sqrt{1 - \frac{7z^4}{12} - \frac{z^6}{3} - \frac{11z^8}{144} - \frac{z^{10}}{144}} > 0. \end{aligned}$$

Comparison of the last pair is simple:

$$\frac{d}{dz}(\operatorname{artanh} z - \arcsin z) = \frac{1}{1-z^2} - \frac{1}{\sqrt{1-z^2}} > 0.$$

Now we can use Theorem 4 to write the chain of inequalities

$$\frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}'} < \frac{1}{\mathsf{S}_{\sinh}} - \frac{1}{\mathsf{S}'_{\sinh}} < \left\{ \begin{matrix} \frac{1}{\mathsf{S}_{\tan}} - \frac{1}{\mathsf{S}'_{\tan}} \\ \frac{1}{\mathsf{P}} - \frac{1}{\mathsf{P}'} \end{matrix} \right\} < \frac{1}{\mathsf{L}} - \frac{1}{\mathsf{L}'}.$$

**Example 11.** *Consider the means in Example 8. The function*

$$v(z) - z = \begin{cases} 0 & \textit{if} \quad z \le 1/2, \\ 1 - z & \textit{otherwise} \end{cases}$$

*is a non-monotonic Ky Fan function, so the inequality*

$$\frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}'} < \frac{1}{V} - \frac{1}{V'}$$

*holds.*

NOTE: An example of a function that satisfies the inequality

$$f\left(\frac{|x-y|}{x+y}\right) > f\left(\frac{|x'-y'|}{x'+y'}\right)$$

for all $0 < x, y < 1/2$, but is not monotone has been provided by *tometomek91* - user of the mathematical portal `matematyka.pl`.

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