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Regularized Trace Formula for One Partial Differential Operator

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Abstract. We propose a new method for obtaining regularized trace formulae. The matter of the method is discussed in details for one partial differential operator with a non-smooth perturbation in a strip.

Key Words and Phrases: regularized trace, eigenvalues, harmonic oscillator, perturbed operator, reduced resolvent.

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1. Introduction

Beginning from 1950, for simplest operators like Sturm-Liouville operators, many authors proposed numerous ideas that formed main directions in studying regularized trace formulae. In the mid of 1960s, there appeared a paper by V.B. Lidskii and V.A. Sadovnichii [1], which presented a method for finding regularized trace formulae for a wide class of differential operators and this stimulated a rapid development of the theory of regularized traces. In 1980s, the main attention was paid to partial differential operators. A significant progress in this direction was made by V.A. Sadovnichii and V.V. Dubrovskii [2], V.A. Sadovnichii and V.E. Podol'skii [3], V.A. Sadovnichii and Z.Yu. Fazullin [4], Z.Yu. Fazullin and Kh.Kh. Murtazin [5], [6]. The most complete survey of the works of other authors can be found in [7]. In recent years, there has been significant progress in weakening the conditions on potential. The most significant works in this direction are [8], [9]. It should be noted that in almost all spectral problems where the spectrum of the perturbed operator is studied, the authors inevitably encounter an evaluation of the solutions of the undisturbed operator. The need to study the core of the resolvent of an undisturbed operator arises already for one-dimensional operators. For example, in [10] the authors had to construct a solution to the inverse problem.

In the present work we propose a new method for obtaining regularized trace formulae, first of all, for self-adjoint partial differential operators. The novelty of the method is that the problem is reduced to representation for the resolvents of one-dimensional operators without using contour integration.

As an example, we consider the two-dimensional operator

$$-\frac{\partial^2 u(x,y)}{\partial x^2}-\frac{\partial^2 u(x,y)}{\partial y^2}+x^2+V(x,y)$$

in the strip $\Pi=\{(x,y):\ -\infty<x<\infty,\ 0\le y\le\pi\}$ with the boundary conditions $u(x,0)=u(x,\pi)=0$, $u(x,y)\in\mathbb{L}^2(\Pi)$ and a compactly supported perturbation $V(x,y)\in\mathbb{L}^2(\Pi)$.

2. Description of method

Let H^0 be a self-adjoint operator in the Hilbert space $\mathbb{H}$ with a discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}$ $(\lambda_1<\lambda_2<\ldots)$ and associated spectral projectors P_k and $\inf\limits_{k\ge1}(\lambda_{k+1}-\lambda_k)>0$. By $R^0(\lambda)$ we denote the resolvent of this operator.

We consider a perturbed operator $H=H^0+V$. If V is a symmetric operator and H^0-bounded with a H^0-bound less than 1, by the known Kato-Rellich theorem (see, for instance, [11], p. 185), the operator H is closed on the domain of H^0 and has a discrete spectrum. Let $d_n=\min(\lambda_{n+1}-\lambda_n,\,\lambda_n-\lambda_{n-1})/2$ and let there exist a sequence ρ_n such that

$$0<\rho_n\le d_n,\quad \inf_{n\ge2}\rho_n>0,\quad \lim_{n\to\infty}\sup_{|\lambda-\lambda_n|\le\rho_n}\|R_n^0(\lambda)V\|=0,$$

where $R_n^0(\lambda)=R^0(\lambda)-P_n/(\lambda_n-\lambda)$. According to [12], the spectrum of the operator $H=H^0+V$ is determined by the equation

$$\lambda=\lambda_n+P_nVP_n-P_nVR_n(\lambda)VP_n, \tag{1}$$

where $R_n(\lambda)=\sum_{k=0}^{\infty}(-1)^k\left[R_n^0(\lambda)V\right]^kR_n^0(\lambda)$.

Expression (1) is a formula for the spectrum of a finite-dimensional operator A_n in the vicinity of an eigenvalue λ_n, $|\lambda-\lambda_n|<\rho_n$ for fixed n; namely, we have $\det(A_n-\lambda)=0$. Indeed, let $\varphi_k^{(n)}$, $1\le k\le\nu_n$, be eigenfunctions associated with an eigenvalue λ_n of a multiplicity ν_n. Then $P_nh=\sum_{k=1}^{\nu_n}(h,\varphi_k^{(n)})\varphi_k^{(n)}$ and formula (1) becomes

$$\begin{vmatrix} c_{11}^n + \lambda_n - \lambda & c_{12}^n & \dots & c_{1\nu_n}^n \\ c_{21}^n & c_{22}^n + \lambda_n - \lambda & \dots & c_{2\nu_n}^n \\ \vdots & \vdots & \ddots & \vdots \\ c_{\nu_n 1}^n & c_{\nu_n 2}^n & \dots & c_{\nu_n \nu_n}^n + \lambda_n - \lambda \end{vmatrix} = 0, \tag{2}$$

where $c_{ij}^n = (V\varphi_i^{(n)}, \varphi_j^{(n)}) - (VR_n(\lambda)V\varphi_i^{(n)}, \varphi_j^{(n)})$. Formula (2) can be written as $\det(A_n - \lambda) = 0$, where $A_n = C_n + \lambda_n I$, $C_n = \{c_{ij}^n\}$, $1 \le i,\, j \le \nu_n$. Employing the fact that for finite-dimensional matrices the spectral trace is equal to the matrix one, equation (2) implies immediately the representation

$$\nu_n\lambda_n + \operatorname{sp} P_n V P_n - \gamma_n = \sum_{k=1}^{v_n} \mu_k^{(n)}, \tag{3}$$

where $\mu_k^{(n)}$ are the eigenvalues of the operator H, $|\lambda_n - \mu_k^{(n)}| < \rho_n$,

$$\operatorname{sp} P_n V P_n = \sum_{k=1}^{\nu_n} \left(V\varphi_k^{(n)}, \varphi_k^{(n)} \right), \quad \gamma_n = \sum_{k=1}^{\nu_n} \left(VR_n(\mu_k^{(n)}) V\varphi_k^{(n)}, \varphi_k^{(n)} \right).$$

If the perturbation V is such that the sequence

$$\sum_{n=1}^{N} \gamma_n = \sum_{n=1}^{N} \sum_{k=1}^{\nu_n} \left(VR_n(\mu_k^{(n)}) V\varphi_k^{(n)}, \varphi_k^{(n)} \right) \to 0 \quad \text{as} \quad N \to \infty, \tag{4}$$

by formula (3) we get immediately a regularized trace formula for the operator H:

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\nu_n} \mu_k^{(n)} - \nu_n\lambda_n - \sum_{k=1}^{\nu_n} \left(V\varphi_k^{(n)}, \varphi_k^{(n)} \right) \right) = 0. \tag{5}$$

3. Regularized trace formula for one partial differential operator

We consider the operator $H^0 = T \otimes I_2 + I_1 \otimes L$ in the Hilbert space $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$, where $\mathbb{H}_1 = L^2(\mathbb{R})$, $\mathbb{H}_2 = L^2[0, \pi]$, I_1 and I_2 are identity mappings in the corresponding spaces,

$$Tf = -f'' + x^2 f, \quad f(x) \in L^2(\mathbb{R}); \tag{6}$$

$$Lg = -g'', \quad g(0) = g(\pi) = 0. \tag{7}$$

Lemma 1. *The spectrum of the operator H^0 consists of the number $\lambda_0 = 2$ with the multiplicity $\nu_0 = 1$, and the numbers $\lambda_n = n+3$, $n \geq 1$, with the multiplicities*

$$\nu_n = \begin{cases} \left[\dfrac{\sqrt{n+2}}{2}\right] & \textit{if} \quad n \textit{ is even};\\ \left[\dfrac{1+\sqrt{n+2}}{2}\right] & \textit{if} \quad n \textit{ is odd}. \end{cases} \tag{8}$$

Proof. The eigenvalues of the operators T and L are well-known and these are the numbers $2k+1$, $k \geq 0$, and m^2, $m \geq 1$, respectively. Let us show that the numbers of the operator H^0 determined by the identities $\lambda_{(m,k)} = m^2 + 2k + 1$, $m \geq 1$, $k \geq 0$, can be represented as $\lambda_n = n + 3$, $n \geq 1$, $\lambda_0 = 2$.

Indeed, $\lambda_{(m,k+1)} - \lambda_{m,k} = 2$ for each fixed m. Let $m = 2N - 1$, $N \geq 1$, be fixed. Then

$$\lambda_{(2N-1,k)} = 4N^2 - 4N + 2k + 2 = 2(2N^2 - 2N + k + 1) = 2s, \text{where} \quad s \geq 1. \tag{9}$$

In the same way, if $m = 2N$, $N \geq 1$, then

$$\lambda_{(2N,k)} = 4N^2 + 2k + 1 = 2(2N^2 + k) + 1 = 2l + 1, \text{where} \quad l \geq 2. \tag{10}$$

Combining formulae (9) and (10), we can represent the eigenvalues $\lambda_{(m,k)} = m^2 + 2k + 1$ as $\lambda_n = n + 3$, $n \geq 1$, $\lambda_0 = 2$; these eigenvalues are provided in the following infinite table:

$k\backslash m$	1	2	3	4	5	6	7	$\cdots$
0	2	5	10	17	26	37	50	$\cdots$
1	4	7	12	19	28	39	52	$\cdots$
2	6	9	14	21	30	41	54	$\cdots$
3	8	11	16	23	32	43	56	$\cdots$
4	10	13	18	25	34	45	58	$\cdots$
5	12	15	20	27	36	47	60	$\cdots$
6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

In particular, this table shows that the eigenvalues $\lambda_0 = \lambda_{(1,0)} = 2$, $\lambda_1 = \lambda_{(1,1)} = 4$, $\lambda_2 = \lambda_{(2,0)} = 5$, $\lambda_3 = \lambda_{(1,2)} = 6$, $\lambda_4 = \lambda_{(2,1)} = 7$, $\lambda_5 = \lambda_{(1,3)} = 8$, $\lambda_6 = \lambda_{(2,2)} = 9$, $\lambda_8 = \lambda_{(2,3)} = 11$, $\lambda_{10} = \lambda_{(2,4)} = 13$, $\lambda_{12} = \lambda_{(2,5)} = 15$ appear only once, that is, they are simple.

We are going to determine the multiplicity of the eigenvalues λ_n. Let $m = 2N - 1$ for some fixed $N \geq 1$, that is, we consider all odd columns. It is clear that all eigenvalues $\lambda_{2s-1} = 2s + 2$, $s \geq 1$, $\lambda_0 = 2$ are in first column. At that, it is easy to confirm that the eigenvalues $\lambda_{2s-1} = 2s + 2$ of multiplicity $\nu_n = n$ are determined by the inequalities

$$(2n-1)^2 + 1 \le 2s + 2 \le (2n+1)^2 - 1. \tag{11}$$

Indeed, $\nu_n = 1$ as $1 \le s \le 3$; $\nu_n = 2$ as $4 \le s \le 11$; $\nu_n = 3$ as $12 \le s \le 23$ and so forth. Resolving $\nu_n = n$ with respect to s, by inequality (11) we get

$$\nu_n = \left[\frac{\sqrt{2s+1}+1}{2}\right], \quad n = 2s-1, \quad s \ge 1. \tag{12}$$

In the same way, considering even columns, we see that the eigenvalues $\lambda_{2s} = 2s + 3$, $s \ge 1$, of multiplicity ν_n obey the inequalities

$$4n^2 + 1 \le 2s + 3 \le 4(n+1)^2 - 1.$$

Thus,

$$\nu_n = \left[\sqrt{\frac{s+1}{2}}\right], \quad n = 2s, \quad s \ge 1. \tag{13}$$

Formulae (12), (13) imply (8). The proof is complete. ◄

Corollary 1. *If m takes odd values, $m = 2l - 1$, $l \ge 1$, then n takes only odd values, $n = 2s - 1$. At that, $\lambda_{(2l-1,k)} = \lambda_{2s-1} = 2s + 2$ as $1 \le l \le \nu_{2s-1}$, $k = s - 2l^2 + 2l$, $s \ge 1$. Similary, for even m, $m = 2l$, $l \ge 1$, the number n is also even: $n = 2s$. At that, $\lambda_{(2l,k)} = \lambda_{2s} = 2s + 3$ as $1 \le l \le \nu_{2s}$, $k = s + 1 - 2l^2$, $s \ge 1$.*

Theorem 1. *The kernel of the resolvent $R^0(\lambda)$ of the operator H^0 is represented as*

$$R^0((x,t);(y,\tau);\lambda) = \frac{2}{\pi}\sum_{m=1}^{\infty} r(x,t,\lambda - m^2)\sin my \sin m\tau,$$

where $r(x,t,\lambda - m^2)$ is the kernel of the resolvent $(T + m^2 - \lambda)^{-1}$.

Proof. We consider the inhomogeneous equation

$$-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} + (x^2 - \lambda)u(x,y) = h(x,y), \tag{14}$$

subject to the boundary conditions

$$u(x,0) = u(x,\pi) = 0, \quad u(x,y) \in \mathbb{H}. \tag{15}$$

Then for all $h(x,y) \in \mathbb{H}$, the solution of the problem (14), (15) can be represented as

$$u(x,y) = \int\limits_0^{\pi} \int\limits_{-\infty}^{+\infty} R^0((x,t);(y,\tau);\lambda) h(t,\tau)\, dt\, d\tau, \tag{16}$$

where $R^0((x,t);(y,\tau);\lambda)$ is the kernel of the resolvent $R^0 = (H^0 - \lambda)^{-1}$.

We seek the solution of the equation (14) as

$$u(x,y) = \sum_{m=1}^{\infty} u_m(x) g_m(y), \quad h(x,y) = \sum_{m=1}^{\infty} h_m(x) g_m(y), \tag{17}$$

where $g_m(y)$ are orthonormalized eigenfunctions of the operator L, $g_m(y) = \sqrt{\dfrac{2}{\pi}} \sin my$, $m \geq 1$.

We rewrite equation (14) as

$$-\frac{\partial^2 u(x,y)}{\partial x^2} + (x^2 + m^2 - \lambda) u(x,y) - \frac{\partial^2 u(x,y)}{\partial y^2} - m^2 u(x,y) = h(x,y).$$

Then, substituting expressions (17) for $u(x,y)$ and $h(x,y)$, we obtain

$$\sum_{m=1}^{\infty} [-u_m''(x) + (x^2 + m^2 - \lambda) u_m(x)] g_m(y) = \sum_{m=1}^{\infty} h_m(x) g_m(y),$$

since $-g_m''(y) - m^2 g_m(y) = 0$, $g_m(0) = g_m(\pi) = 0$. This gives the inhomogeneous equation

$$-u_m''(x) + (x^2 - \lambda_1) u_m(x) = h_m(x), \quad \text{where} \quad \lambda_1 = \lambda - m^2. \tag{18}$$

The solution of equation (18) satisfying the condition $u_m(x) \in L^2(\mathbb{R})$ can be represented as

$$u_m(x) = \int\limits_{-\infty}^{+\infty} r(x,t,\lambda_1) h_m(t)\, dt, \tag{19}$$

where $r(x,t,\lambda_1)$ is the kernel of the resolvent $(T - \lambda_1)^{-1}$, $h_m(t)$ are the coefficients in the Fourier series with respect to the eigenfunctions $g_m(t)$:

$$h_m(t) = \int\limits_0^{\pi} h(t,\tau) g_m(\tau) d\tau = \sqrt{\frac{2}{\pi}} \int\limits_0^{\pi} h(t,\tau) \sin m\tau\, d\tau. \tag{20}$$

Substituting (19), (20) into equation (17), we get:

$$u(x,y)=\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{\pi}\sum_{m=1}^{\infty} r(x,t,\lambda_1)g_m(y)g_m(\tau)h(t,\tau)\,dt\,d\tau.$$

Then it follows from (16) that

$$R^0((x,t);(y,\tau);\lambda)=\frac{2}{\pi}\sum_{m=1}^{\infty} r(x,t,\lambda-m^2)\sin my\sin m\tau.$$

The proof is complete. ◄

Let us find out the conditions on the perturbation V that ensure the formula (1).

Theorem 2. *Let V be the operator of the multiplication by a real compactly supported function $V(x,y)\in\mathbb{H}$, $V(x,y)=0$ as $|x|\geq A$. Then as $|\lambda_n-\lambda|<1/2$, the identity*

$$\lim_{n\to\infty}\sup_{|\lambda_n-\lambda|<1/2}\left\|R_n^0(\lambda)V\right\|=0$$

holds true.

Proof. Since the kernel of the resolvent $R^0(\lambda)$ is represented as

$$R^0((x,t);(y,\tau);\lambda)=\frac{2}{\pi}\sum_{m=1}^{\infty} r(x,t,\lambda_1)\sin my\sin m\tau,$$

where $\lambda_1=\lambda-m^2$, the singularity of $R^0((x,t);(y,\tau);\lambda)$ coincides with the singularity of the kernel $r(x,t,\lambda_1)$. Hence, the kernel of the reduced resolvent $R_n^0(\lambda)$ reads as

$$R_n^0((x,t);(y,\tau);\lambda)=\frac{2}{\pi}\sum_{m=1}^{\infty} \widehat{r}(x,t,\lambda_1)\sin my\sin m\tau, \tag{21}$$

where $\widehat{r}(x,t,\lambda_1)$ is the kernel of the reduced resolvent

$$\widehat{r}(\lambda_1)=r(\lambda_1)-\frac{P_k}{2k+1-\lambda_1},$$

P_k are the projectors associated with the eigenvalues $2k+1$.

According to [13], as $|x|\leq A$ and $|2k+1-\lambda_1|<1/2$ (that is, as $|2k+1+m^2-\lambda|=|\lambda_n-\lambda|<1/2$, where the relation between n, m and k was described in Corollary 1), the kernel $\widehat{r}(x,t,\lambda_1)$ satisfies the estimate

$$|\widehat{r}(x,t,\lambda_1)|\leq\frac{C}{\sqrt[4]{\lambda_1-x^2}\sqrt[4]{\lambda_1-t^2}}\leq\frac{C}{\sqrt{\lambda_1-A^2}}. \tag{22}$$

Let $f(x,y) \in \mathbb{H}$, $\|f\| = 1$. Then

$$\left|[R_n^0(\lambda)Vf](x,y)\right|^2 \le \int_{-A}^{A}\int_{0}^{\pi} \left|R_n^0((x,t);(y,\tau);\lambda)\right|^2 |V(t,\tau)|^2 \, dt\, d\tau.$$

Hence,

$$\left\|R_n^0(\lambda)V\right\|^2 \le \int_{-A}^{A}\int_{-A}^{A}\int_{0}^{\pi}\int_{0}^{\pi} \left|R_n^0((x,t);(y,\tau);\lambda)\right|^2 |V(t,\tau)|^2 \, dt\, d\tau\, dx\, dy. \tag{23}$$

Expression (21) is the expansion of the function $R_n^0((x,t);(y,\tau);\lambda)$ in $\sin my$. Then, by the Parseval identity, we obtain

$$\int_{0}^{\pi} \left|R_n^0((x,t);(y,\tau);\lambda)\right|^2 dy \le C \sum_{m=1}^{\infty} |\widehat{r}(x,t,\lambda_1)|^2 . \tag{24}$$

Then in view of (22), (23) and (24), the identity $\lambda_1 = \lambda - m^2$ and the assumptions of the theorem, we have the estimate

$$\left\|R_n^0(\lambda)V\right\|^2 \le C \sum_{m=1}^{\infty} \frac{1}{\lambda - m^2 - A^2}. \tag{25}$$

The series in the right hand side of the above formula converges absolutely and uniformly on each bounded closed set containing no points $m^2 + A^2$. The sum of this series is calculated as follows:

$$\sum_{m=1}^{\infty} \frac{1}{\lambda - m^2 - A^2} = \frac{\pi}{2\sqrt{\lambda - A^2}} \left(\cot \pi\sqrt{\lambda - A^2} - \frac{1}{\pi\sqrt{\lambda - A^2}} \right).$$

This implies immediately that

$$\lim_{n\to\infty} \sup_{|\lambda_n - \lambda| < 1/2} \left\|R_n^0(\lambda)V\right\| = 0.$$

The proof is complete. ◄

Remark 1. *While calculating the sum of series (25), we have used the partial fraction decomposition of the function* $\cot z$*:*

$$\cot z = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z - m\pi} + \frac{1}{z + m\pi} \right)$$

as $z = \pi\sqrt{\lambda - A^2}$.

Thus, under the assumptions of Theorem 2, the operator $H = H^0 + V$ possesses a discrete spectrum and formula (1) holds. This implies formula (3). It remains to find the conditions ensuring regularized trace formula (5).

Lemma 2. *Under assumptions of Theorem 2 on perturbation, the representation*

$$\sum_{n=1}^{N} \gamma_n = \sum_{n=1}^{N} \sum_{k=1}^{\nu_n} \left(V R_n(\mu_k^{(n)}) V \varphi_k^{(n)}, \varphi_k^{(n)} \right) = \sum_{n=1}^{N} \alpha_n + \sum_{n=1}^{N} \beta_n$$

holds true, where $\alpha_n = \sum\limits_{i=1}^{\nu_n} (V R_n^0(\lambda_n) V \varphi_i^{(n)}, \varphi_n^{(n)})$, $\beta_n = o(\alpha_n)$.

Proof. Under the assumptions of Theorem 2, the following representation holds:

$$R_n(\lambda_n) = \sum_{k=0}^{\infty} (-1)^k [R_n^0(\lambda_n) V]^k R_n^0(\lambda_n) \tag{26}$$

or, equivalently,

$$R_n(\lambda_n) = R_n^0(\lambda_n) - R_n^0(\lambda_n) V R_n(\lambda_n). \tag{27}$$

Substituting the latter into the Hilbert-Schmidt identity

$$R_n(\mu_i^{(n)}) = R_n(\lambda_n) + \left(\mu_i^{(n)} - \lambda_n \right) R_n(\lambda_n) R_n(\mu_i^{(n)}),$$

where $\left| \mu_i^{(n)} - \lambda_n \right| < 1/2, \quad 1 \le i \le \nu_n$, we get

$$\begin{aligned} R_n(\mu_i^{(n)}) =& R_n^0(\lambda_n) - R_n^0(\lambda_n) V R_n(\lambda_n) + \left(\mu_i^{(n)} - \lambda_n \right) R_n(\mu_i^{(n)}) R_n^0(\lambda_n) \\ & - \left(\mu_i^{(n)} - \lambda_n \right) R_n(\mu_i^{(n)}) R_n^0(\lambda_n) V R_n(\lambda_n). \end{aligned}$$

Then

$$\begin{aligned} \left(V R_n(\mu_i^{(n)}) V \varphi_i^{(n)}, \varphi_i^{(n)} \right) = & \left(V R_n^0(\lambda_n) V \varphi_i^{(n)}, \varphi_i^{(n)} \right) - \\ & - \left(V R_n^0(\lambda_n) V R_n(\lambda_n) V \varphi_i^{(n)}, \varphi_i^{(n)} \right) \\ & + \left(\mu_i^{(n)} - \lambda_n \right) \left(V R_n(\mu_i^{(n)}) R_n^0(\lambda_n) V \varphi_i^{(n)}, \varphi_i^{(n)} \right) \\ & - \left(\mu_i^{(n)} - \lambda_n \right) \left(V R_n(\mu_i^{(n)}) R_n^0(\lambda_n) V R_n(\lambda_n) V \varphi_i^{(n)}, \varphi_i^{(n)} \right). \end{aligned} \tag{28}$$

The second and other terms in this sum are infinitesimals of an order higher than the first term as $n \to \infty$. The Hilbert-Schmidt identity and (27) imply the estimate

$$\left\| R_n(\mu_i^{(n)}) V \right\| \le \| R_n(\lambda_n) V \| \le \left\| R_n^0(\lambda_n) V \right\| = \delta_n < 1. \tag{29}$$

Then the first term in the right hand side of (28) satisfies the following estimate:

$$\left|\left(VR_n^0(\lambda_n)V\varphi_i^{(n)},\varphi_i^{(n)}\right)\right| \le \left\|R_n^0(\lambda_n)V\varphi_i^{(n)}\right\| \left\|V\varphi_i^{(n)}\right\| \le C\left\|R_n^0(\lambda_n)V\right\|.$$

According (29), the other terms in (28) obey the estimates:

$$\begin{aligned}
&\left|\left(VR_n^0(\lambda_n)VR_n(\lambda_n)V\varphi_i^{(n)},\varphi_i^{(n)}\right)\right| \le \left\|R_n^0(\lambda_n)V\right\| \left\|R_n(\lambda_n)V\varphi_i^{(n)}\right\| \left\|V\varphi_i^{(n)}\right\| \\
&\qquad\qquad\qquad\qquad\qquad \le C\left\|R_n^0(\lambda_n)V\right\|^2; \\
&\left|\left(\mu_i^{(n)}-\lambda_n\right)\left(VR_n(\mu_i^{(n)})R_n^0(\lambda_n)V\varphi_i^{(n)},\varphi_i^{(n)}\right)\right| \\
&\qquad \le C\left\|R_n^0(\lambda_n)V\varphi_i^{(n)}\right\| \left\|R_n(\mu_i^{(n)})V\varphi_i^{(n)}\right\| \le C\left\|R_n^0(\lambda_n)V\right\|^2; \\
&\left|\left(\mu_i^{(n)}-\lambda_n\right)\left(VR_n(\mu_i^{(n)})R_n^0(\lambda_n)VR_n(\lambda_n)V\varphi_i^{(n)},\varphi_i^{(n)}\right)\right| \\
&\qquad \le C\left\|R_n^0(\lambda_n)V\right\| \left\|R_n(\lambda_n)V\varphi_i^{(n)}\right\| \left\|R_n(\mu_i^{(n)})V\varphi_i^{(n)}\right\| \le C\left\|R_n^0(\lambda_n)V\right\|^3.
\end{aligned}$$

This is why the expression (28) can be represented as

$$\gamma_n = \sum_{i=1}^{\nu_n}\left(VR_n(\mu_i^{(n)})V\varphi_i^{(n)},\varphi_i^{(n)}\right) = \alpha_n + \beta_n,$$

$$\text{where} \quad \alpha_n = \sum_{i=1}^{\nu_n}\left(VR_n^0(\lambda_n)V\varphi_i^{(n)},\varphi_i^{(n)}\right), \quad \beta_n = o(\alpha_n).$$

The proof is complete. ◄

Thus, to prove (4), it is sufficient to show that

$$\sum_{n=1}^{N}\alpha_n = \sum_{n=1}^{N}\sum_{i=1}^{\nu_n}\left(VR_n^0(\lambda_n)V\varphi_i^{(n)},\varphi_i^{(n)}\right) \to 0 \quad \text{as} \quad N\to\infty.$$

Theorem 3. *Let $V(x,y)\in\mathbb{H}$, $V(x,y)=0$ as $|x|\ge A$. Then*

$$\sum_{n=1}^{N}\alpha_n = \sum_{n=1}^{N}\sum_{l=1}^{\nu_n}(VR_n^0(\lambda_n)V\varphi_i^{(n)},\varphi_i^{(n)}) \to 0 \quad \textit{as} \quad N\to\infty.$$

Proof. It was shown in [13] that the study of the kernel of the reduced resolvent $\widehat{r}(x,t,\lambda)$ for one-dimensional harmonic oscillator is reduced to the study of the kernels $B_D^+(x,t,\lambda)$, $B_N^+(x,t,\lambda)$ of the resolvents $B_D^+(\lambda)$ and $B_N^+(\lambda)$ corresponding to the Dirichlet and Neumann problems for one-dimensional harmonic oscillator in $L^2(0,\infty)$:

$$L_D^+u = -u'' + x^2u, \quad u(0) = 0 \quad \text{and} \quad L_N^+u = -u'' + x^2u, \quad u'(0) = 0.$$

As $x \geq 0$, $t \geq 0$, $k \geq 0$, the following representation holds:

$$\widehat{r}(x,t,2k+1) = \frac{1}{2}\begin{cases} B_N^+(x,t,2k+1) + B_{Dk}^+(x,t,2k+1) & \text{for odd } k; \\ B_D^+(x,t,2k+1) + B_{Nk}^+(x,t,2k+1) & \text{for even } k, \end{cases}$$

where $B_{Dk}^+(x,t,\lambda)$, $B_{Nk}^+(x,t,\lambda)$ are the corresponding kernels of the reduced resolvents. At that, the kernel $\widehat{r}(-x,t,2k+1)$ is represented as

$$\widehat{r}(-x,t,2k+1) = \frac{1}{2}\begin{cases} B_N^+(x,t,2k+1) - B_{Dk}^+(x,t,2k+1) & \text{for odd } k; \\ -B_D^+(x,t,2k+1) + B_{Nk}^+(x,t,2k+1) & \text{for even } k. \end{cases}$$

The asymptotics of the kernels $B_D^+(x,t,\lambda)$, $B_N^+(x,t,\lambda)$, $B_{Dk}^+(x,t,\lambda)$, $B_{Nk}^+(x,t,\lambda)$ in the vicinity of the eigenvalue $2k+1$ allow us to write out the asymptotics for the kernel of the reduced resolvent $\widehat{r}(x,t,2k+1)$ of one-dimensional harmonic oscillator as $0 \leq x \leq A,\ 0 \leq t \leq A,\ k \gg 1$:

$$\begin{aligned}
\widehat{r}(x,t,2k+1) = &-\frac{1}{\sqrt[4]{2k+1-x^2}\sqrt[4]{2k+1-t^2}} \\
&\cdot\begin{cases} \cos\left[Q(x,2k+1) - \dfrac{\pi}{4}\right]\sin\left[Q(t,2k+1) - \dfrac{\pi}{4}\right], & t \leq x; \\ \cos\left[Q(t,2k+1) - \dfrac{\pi}{4}\right]\sin\left[Q(x,2k+1) - \dfrac{\pi}{4}\right], & x \leq t. \end{cases} \\
&- \frac{\cos\left[Q(x,2k+1) - \dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-x^2}}\frac{\sin\left[Q(t,2k+1) - \dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-t^2}}\arccos\frac{t}{\sqrt{2k+1}} \\
&- \frac{\cos\left[Q(t,2k+1) - \dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-x^2}}\frac{\sin\left[Q(x,2k+1) - \dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-t^2}} \\
&\cdot\arccos\frac{x}{\sqrt{2k+1}} + \rho_1(x,t,2k+1),
\end{aligned}$$

where $\rho_1(x,t,2k+1) = O\left(k^{-5/4}\right)$. As $0 \leq x \leq A,\ 0 \leq t \leq A,\ k \gg 1$, for the kernel $\widehat{r}(-x,t,2k+1)$ we also have the following asymptotic formulae:

$$\widehat{r}(-x,t,2k+1) = -\frac{\cos\left[Q(x,2k+1)-\dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-x^2}}\frac{\sin\left[Q(t,2k+1)-\dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-t^2}} \cdot \arccos\frac{t}{\sqrt{2k+1}} - \frac{\cos\left[Q(t,2k+1)-\dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-x^2}}\frac{\sin\left[Q(x,2k+1)-\dfrac{\pi}{4}\right]}{\pi\sqrt[4]{2k+1-t^2}} \cdot \arccos\frac{x}{\sqrt{2k+1}} + \rho_2(x,t,2k+1),$$

where $\rho_2(x,t,2k+1) = O\left(k^{-5/4}\right)$.

The asymptotics of the eigenvalues of one-dimensional harmonic oscillator as $0 \le x \le A$, $k \gg 1$ is represented as

$$f_k(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{2k+1-x^2}}\cos\left[Q(x,2k+1)-\frac{\pi}{4}\right] + \widehat{f}_k(x), \tag{30}$$

where $\widehat{f}_k(x) = O\left(k^{-5/4}\right)$, $\qquad Q(x,2k+1) = \int\limits_x^{\sqrt{2k+1}} \sqrt{2k+1-t^2}\,dt.$

In view of the definition of the eigenfunctions and Corollary 1 we obtain:

$$\varphi_i^{(n)}(x,y) = \sqrt{\frac{2}{\pi}}\sin(2p-1)y f_{s-2l^2+2l}(x) \tag{31}$$

as $n = 2s-1$, $m = 2l-1$, $1 \le l \le \nu_{2s-1}$, $i = 2p-1$, $1 \le i \le \nu_{2s-1}$;

$$\varphi_i^{(n)}(x,y) = \sqrt{\frac{2}{\pi}}\sin 2py f_{s-2l^2+1}(x) \tag{32}$$

as $n = 2s$, $m = 2l$, $1 \le l \le \nu_{2s}$, $i = 2p$, $1 \le i \le \nu_{2s}$, where $f_k(x)$ are orthonormalized eigenfunctions of one-dimensional harmonic oscillator. We also observe that $f_k(-x) = (-1)^k f_k(x)$.

Let us find the kernel of the resolvent $R_n^0(\lambda_n)$. By formula (21) we have

$$R_n^0\left((x,t);(y,\tau);\lambda_n\right) = \frac{2}{\pi}\lim_{\lambda\to\lambda_n}\sum_{m=1}^{\infty}\hat{r}\left(x,t,\lambda-m^2\right)\sin my\sin m\tau.$$

The series in the right hand side converges uniformly as $|\lambda - \lambda_n| \le 1/2$ and $x, t \in [-A, A]$, $y, \tau \in [0, \pi]$. Passing to the limit, we obtain the expression for $R_n^0\left((x,t);(y,\tau);\lambda_n\right)$. At that, the eigenvalues $\lambda_n - m^2$ cannot be negative, that is, m cannot be arbitrarily large for a fixed n. Then taking into consideration the dependence of m on n (Corollary 1), we have

$$\begin{aligned} R_{2s-1}^0\left((x,t);(y,\tau);\lambda_{2s-1}\right) =& \frac{2}{\pi}\sum_{l=1}^{\nu_{2s-1}} \hat{r}\left(x,t,2s+2-(2l-1)^2\right) \\ &\cdot \sin(2l-1)y\sin(2l-1)\tau \end{aligned} \tag{33}$$

for odd $n = 2s-1$, $s \ge 1$; and

$$R_{2s}^0\left((x,t);(y,\tau);\lambda_{2s}\right) = \frac{2}{\pi}\sum_{l=1}^{\nu_{2s}} \hat{r}\left(x,t,2s+3-4l^2\right)\sin 2ly\sin 2l\tau, \tag{34}$$

for even $n = 2s$, $s \ge 1$.

Finally, we transform $\sum\limits_{n=1}^{N}\alpha_n$:

$$\begin{aligned} \sum_{n=1}^{N}\alpha_n =& \sum_{n=1}^{N}\sum_{i=1}^{\nu_n}(VR_n^0(\lambda_n)V\varphi_i^{(n)}, \varphi_i^{(n)}) \\ =& \sum_{n=1}^{N}\sum_{i=1}^{\nu_n}\Bigg(\int_{-A}^{A}\int_{0}^{\pi} V(x,y)\varphi_i^{(n)}(x,y) \\ & \cdot \int_{-A}^{A}\int_{0}^{\pi} V(t,\tau)R_n^0\left((x,t);(y,\tau);\lambda_n\right)\varphi_i^{(n)}(t,\tau)\,dx\,dy\,dt\,d\tau\Bigg) \\ =& \sum_{n=1}^{N}\sum_{i=1}^{\nu_n}\Bigg\{\int_{0}^{A}\int_{0}^{A}\int_{0}^{\pi}\int_{0}^{\pi} R_n^0\left((x,t);(y,\tau);\lambda_n\right) \\ & \Big[V(t,\tau)V(x,y)\varphi_i^{(n)}(t,\tau)\varphi_i^{(n)}(x,y) \\ & + V(-x,y)V(-t,\tau)\varphi_i^{(n)}(-x,y)\varphi_i^{(n)}(-t,\tau)\Big]\,dx\,dy\,dt\,d\tau \\ & + \int_{0}^{A}\int_{0}^{A}\int_{0}^{\pi}\int_{0}^{\pi} R_n^0\left((-x,t);(y,\tau);\lambda_n\right)\Big[V(t,\tau)V(-x,y)\varphi_i^{(n)}(t,\tau)\varphi_i^{(n)}(-x,y) \\ & + V(x,y)V(-t,\tau)\varphi_i^{(n)}(x,y)\varphi_i^{(n)}(-t,\tau)\Big]\,dx\,dy\,dt\,d\tau\Bigg\}. \end{aligned}$$

Here we have used the symmetricity of the kernel $\hat{r}\,(x,t,\lambda)$, namely, $\hat{r}\,(x,t,\lambda) = \hat{r}\,(-x,-t,\lambda)$, $\hat{r}\,(-x,t,\lambda) = \hat{r}\,(x,-t,\lambda)$. It is obvious that the kernel $R_n^0\,((x,t);(y,\tau);\lambda)$ possesses similar properties.

We substitute (31)–(34) into the last relation and obtain:

$$\sum_{n=1}^{N}\alpha_n = \frac{4}{\pi^2}\sum_{s=1}^{\left[\frac{N+1}{2}\right]}\sum_{p=1}^{\nu_{2s-1}}\sum_{l=1}^{\nu_{2s-1}}\int\limits_0^A\int\limits_0^A\int\limits_0^\pi\int\limits_0^\pi \widehat{r}(x,t,2s+2-(2l-1)^2)\widehat{V_1}(x,t,y,\tau)$$
$$\cdot f_{s-2p^2+2p}(x)f_{s-2p^2+2p}(t)\sin(2l-1)y\sin(2l-1)\tau$$
$$\cdot\sin(2p-1)y\sin(2p-1)\tau\,dx\,dy\,dt\,d\tau$$
$$+\frac{4}{\pi^2}\sum_{s=1}^{\left[\frac{N}{2}\right]}\sum_{p=1}^{\nu_{2s}}\sum_{l=1}^{\nu_{2s}}\int\limits_0^A\int\limits_0^A\int\limits_0^\pi\int\limits_0^\pi \widehat{r}(x,t,2s+3-4l^2)\widehat{V_1}(x,t,y,\tau)$$
$$\cdot f_{s-2p^2+1}(x)f_{s-2p^2+1}(t)\sin 2ly\sin 2l\tau\sin 2py\sin 2p\tau\,dx\,dy\,dt\,d\tau$$
$$+\frac{4}{\pi^2}\sum_{s=1}^{\left[\frac{N+1}{2}\right]}\sum_{p=1}^{\nu_{2s-1}}\sum_{l=1}^{\nu_{2s-1}}(-1)^s\int\limits_0^A\int\limits_0^A\int\limits_0^\pi\int\limits_0^\pi \widehat{r}(-x,t,2s+2-(2l-1)^2)$$
$$\cdot\widehat{V_2}(x,t,y,\tau)f_{s-2p^2+2p}(x)f_{s-2p^2+2p}(t)\sin(2l-1)y$$
$$\cdot\sin(2l-1)\tau\sin(2p-1)y\sin(2p-1)\tau\,dx\,dy\,dt\,d\tau$$
$$+\frac{4}{\pi^2}\sum_{s=1}^{\left[\frac{N}{2}\right]}\sum_{p=1}^{\nu_{2s}}\sum_{l=1}^{\nu_{2s}}(-1)^{s+1}\int\limits_0^A\int\limits_0^A\int\limits_0^\pi\int\limits_0^\pi \widehat{r}(-x,t,2s+3-4l^2)\widehat{V_2}(x,t,y,\tau)$$
$$\cdot f_{s-2p^2+1}(x)f_{s-2p^2+1}(t)\sin 2ly\sin 2l\tau\sin 2py\sin 2p\tau\,dx\,dy\,dt\,d\tau,$$

where

$$\widehat{V_1}(x,t,y,\tau) = V(t,\tau)V(x,y) + V(-x,y)V(-t,\tau),$$
$$\widehat{V_2}(x,t,y,\tau) = V(t,\tau)V(-x,y) + V(x,y)V(-t,\tau).$$

Let $\nu_{2s-1} = n$. Then $1\le p\le n,\quad 1\le l\le n,\quad 2n^2-2n\le s\le 2n^2+2n-1.$
If $\nu_{2s} = n$, then $1\le p\le n,\quad 1\le l\le n,\quad 2n^2-1\le s\le 2n^2+4n.$

Thus, the sum $\sum\limits_{n=1}^{N}\alpha_n$ becomes

$$\sum_{n=1}^{N}\alpha_n = \frac{4}{\pi^2}\sum_{n=1}^{N_1}\sum_{s=2n^2-2n}^{2n^2+2n-1}\sum_{p=1}^{n}\sum_{l=1}^{n}\int\limits_0^A\int\limits_0^A\int\limits_0^\pi\int\limits_0^\pi \widehat{V_1}(x,t,y,\tau)$$

$$
\begin{aligned}
&\cdot \sin(2l-1)y \sin(2l-1)\tau \sin(2p-1)y \sin(2p-1)\tau \\
&\cdot \widehat{r}(x,t,2s+2-(2l-1)^2) f_{s-2p^2+2p}(x) f_{s-2p^2+2p}(t)\,dx\,dy\,dt\,d\tau \\
&+ \frac{4}{\pi^2}\sum_{n=1}^{N_2}\sum_{s=2n^2-1}^{2n^2+4n}\sum_{p=1}^{n}\sum_{l=1}^{n}\int_0^A\int_0^A\int_0^\pi\int_0^\pi \widehat{V_1}(x,t,y,\tau)\widehat{r}(x,t,2s+3-4l^2) \\
&\cdot \sin 2ly \sin 2l\tau \sin 2py \sin 2p\tau f_{s+1-2p^2}(x) f_{s+1-2p^2}(t)\,dx\,dy\,dt\,d\tau \\
&+ \frac{4}{\pi^2}\sum_{n=1}^{N_1}\sum_{s=2n^2-2n}^{2n^2+2n-1}\sum_{p=1}^{n}\sum_{l=1}^{n}(-1)^s\int_0^A\int_0^A\int_0^\pi\int_0^\pi \widehat{V_2}(x,t,y,\tau) \\
&\cdot \widehat{r}(-x,t,2s+2-(2l-1)^2) \sin(2l-1)y \sin(2l-1)\tau \sin(2p-1)y \\
&\cdot \sin(2p-1)\tau f_{s-2p^2+2p}(x) f_{s-2p^2+2p}(t)\,dx\,dy\,dt\,d\tau \\
&+ \frac{4}{\pi^2}\sum_{n=1}^{N_2}\sum_{s=2n^2-1}^{2n^2+4n}\sum_{p=1}^{n}\sum_{l=1}^{n}(-1)^{s+1}\int_0^A\int_0^A\int_0^\pi\int_0^\pi \widehat{V_2}(x,t,y,\tau) \\
&\cdot \widehat{r}(-x,t,2s+3-4l^2) \sin 2ly \sin 2l\tau \sin 2py \sin 2p\tau \\
&f_{s+1-2p^2}(x) f_{s+1-2p^2}(t)\,dx\,dy\,dt\,d\tau,
\end{aligned}
$$

where $\quad N_1 = 2\left[\dfrac{N+1}{2}\right]^2 + 2\left[\dfrac{N+1}{2}\right] - 1, \quad N_2 = 2\left[\dfrac{N}{2}\right]^2 + 4\left[\dfrac{N}{2}\right].$

Now, using the asymptotics for the kernel of the reduced resolvent and for the eigenfunctions (30) of one-dimensional harmonic oscillator, we can split the sum $\sum_{n=1}^{N}\alpha_n$ into several terms:

$$
\sum_{n=1}^{N}\alpha_n = -\frac{16}{\pi^3}\sum_{n=1}^{N_1}\alpha_n^1 - \frac{16}{\pi^4}\sum_{n=1}^{N_1}\alpha_n^2 - \frac{16}{\pi^3}\sum_{n=1}^{N_2}\alpha_n^3 - \frac{16}{\pi^4}\sum_{n=1}^{N_2}\alpha_n^4 + \sum_{n=1}^{\widehat{N}}\gamma_n, \tag{35}
$$

where $\gamma_n = o(\alpha_n^i)$, $i=1,2,3,4$, $\widehat{N} = \max\{N_1, N_2\}$,

$$
\alpha_n^{2i-1} = \int_0^A\int_0^x\int_0^\pi\int_0^\pi q_n^{2i-1}(x,t,y,\tau)\widehat{V_1}(x,t,y,\tau)\,dx\,dy\,dt\,d\tau, \quad i=1,2; \tag{36}
$$

$$
\begin{aligned}
\alpha_n^{2i} = \int_0^A\int_0^A\int_0^\pi\int_0^\pi & q_n^{2i}(x,t,y,\tau)\Big(\widehat{V_1}(x,t,y,\tau) \\
&- (-1)^{s+i}\widehat{V_2}(x,t,y,\tau)\Big)\,dx\,dy\,dt\,d\tau, \quad i=1,2;
\end{aligned} \tag{37}
$$

$$q_n^i = q_n^i(x,t,y,\tau) = \sum_{s=2n^2-2n}^{2n^2+2n-1} \sum_{p=1}^{n} \sum_{l=1}^{n} \sin(2l-1)y \sin(2l-1)\tau \sin(2p-1)y$$

$$\cdot \sin(2p-1)\tau \frac{\cos\left[Q(x, 2s+2-(2p-1)^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+2-(2p-1)^2-x^2}}$$

$$\cdot \frac{\cos\left[Q(t, 2s+2-(2p-1)^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+2-(2p-1)^2-t^2}} \frac{\cos\left[Q(x, 2s+2-(2l-1)^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+2-(2l-1)^2-x^2}}$$

$$\cdot \frac{\sin\left[Q(t, 2s+2-(2l-1)^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+2-(2l-1)^2-t^2}} \widehat{q}_i(t) \quad \text{as} \quad i = 1, 2,$$

$$\widehat{q}_i(t) = \begin{cases} 1 & i = 1; \\ \arccos \dfrac{t}{\sqrt{2s+2-(2l-1)^2}} & i = 2, \end{cases}$$

$$q_n^i = q_n^i(x,t,y,\tau) = \sum_{s=2n^2-1}^{2n^2+4n} \sum_{p=1}^{n} \sum_{l=1}^{n} \sin 2ly \sin 2l\tau \sin 2py \sin 2p\tau$$

$$\cdot \frac{\cos\left[Q(x, 2s+3-4p^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+3-4p^2-x^2}} \frac{\cos\left[Q(t, 2s+3-4p^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+3-4p^2-t^2}}$$

$$\cdot \frac{\cos\left[Q(x, 2s+3-4l^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+3-4l^2-x^2}} \frac{\sin\left[Q(t, 2s+3-4l^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2s+3-4l^2-t^2}} \widehat{q}_i(t) \quad \text{as} \quad i = 3, 4,$$

$$\widehat{q}_i(t) = \begin{cases} 1 & i = 3; \\ \arccos \dfrac{t}{\sqrt{2s+3-4l^2}} & i = 4. \end{cases}$$

Let us prove that

$$\sum_{n=1}^{N_1} \alpha_n^1 \to 0 \quad \text{as} \quad N_1 \to \infty, \quad \text{where} \quad N_1 = 2\left[\frac{N+1}{2}\right]^2 + 2\left[\frac{N+1}{2}\right] - 1.$$

Indeed, since the finite sums $\sum_{p=1}^{n}\sum_{l=1}^{n} a_p b_l$ are represented as $\sum_{k=1}^{n} c_k$, we can write q_n^1 as

$$q_n^1 = \sum_{k=2n^2-2n}^{2n^2+2n-1} \left(C_1^k + C_2^k + \ldots + C_n^k \right),$$

where

$$\begin{aligned} C_n^k = & C_n^k(x,t,y,\tau) = -\frac{1}{2} \frac{\sin^2(2n-1)y \sin^2(2n-1)\tau}{\sqrt{2k+2-(2n-1)^2-x^2}} \\ & \cdot \frac{\cos^2\left[Q(x,2k+2-(2n-1)^2) - \dfrac{\pi}{4}\right] \cos 2Q(t,2k+2-(2n-1)^2)}{\sqrt{2k+2-(2n-1)^2-t^2}} \\ & - \frac{\sin(2n-1)y \sin(2n-1)\tau \cos\left[Q(x,2k+2-(2n-1)^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2k+2-(2n-1)^2-x^2}\sqrt[4]{2k+2-(2n-1)^2-t^2}} \\ & \cdot \sum_{i=1}^{n-1} \frac{\sin(2i-1)y \sin(2i-1)\tau \cos\left[Q(x,2k+2-(2i-1)^2) - \dfrac{\pi}{4}\right]}{\sqrt[4]{2k+2-(2i-1)^2-x^2}\sqrt[4]{2k+2-(2i-1)^2-t^2}} \\ & \cdot \cos\left[Q(t,2k+2-(2n-1)^2) + Q(t,2k+2-(2i-1)^2)\right]. \end{aligned} \tag{38}$$

Then

$$\begin{aligned} \sum_{n=1}^{N_1} q_n^1 = & \sum_{n=1}^{N_1} \sum_{k=2n^2-2n}^{2n^2+2n-1} (C_1^k + C_2^k + \ldots + C_n^k) = \sum_{k=0}^{3} C_1^k + \sum_{k=4}^{11} (C_1^k + C_2^k) + \ldots \\ & + \sum_{k=2N_1^2-2N_1}^{2N_1^2+2N_1-1} (C_1^k + C_2^k + \ldots + C_n^k) = \sum_{n=1}^{N_1} \sum_{k=2n^2-2n}^{2N_1^2+2N_1-1} C_n^k, \end{aligned} \tag{39}$$

where C_n^k are defined by the formula (38). Hence, in view of the relations (36), (39), we can conclude that

$$\sum_{n=1}^{N_1} \alpha_n^1 = \sum_{n=1}^{N_1} \sum_{k=2n^2-2n}^{2N_1^2+2N_1-1} \int_0^A \int_0^x \int_0^\pi \int_0^\pi C_n^k(x,t,y,\tau) \widehat{V_1}(x,t,y,\tau)\, dx\, dt\, dy\, d\tau, \tag{40}$$

where $C_n^k(x,t,y,\tau) = C_n^k$ are defined by the formula (38).

Let us estimate the terms in the right hand side of (40) as $N_1 \to \infty$. We use the Abel transform

$$\sum_{k=1}^{n} u_k v_k = v_n U_n + \sum_{k=1}^{n-1} U_k (v_k - v_{k+1}), \quad \text{where} \quad U_n = \sum_{k=1}^{n} u_k,$$

to obtain for the second term in (38)

$$\begin{aligned}
&\sum_{i=1}^{n-1}\frac{\sin(2i-1)y\sin(2i-1)\tau\cos\left[Q(x,2k+2-(2i-1)^2)-\frac{\pi}{4}\right]}{\sqrt[4]{2k+2-(2i-1)^2-x^2}\sqrt[4]{2k+2-(2i-1)^2-t^2}}\\
&\quad\cdot\cos\left[Q(t,2k+2-(2n-1)^2)+Q(t,2k+2-(2i-1)^2)\right]\\
&=\frac{U_{n-1}}{\sqrt[4]{2k+2-(2n-3)^2-x^2}\sqrt[4]{2k+2-(2n-3)^2-t^2}}\\
&\quad+\sum_{i=1}^{n-2}U_i\left(\frac{1}{\sqrt[4]{2k+2-(2i-1)^2-x^2}\sqrt[4]{2k+2-(2i-1)^2-t^2}}\right.\\
&\quad\left.-\frac{1}{\sqrt[4]{2k+2-(2i+1)^2-x^2}\sqrt[4]{2k+2-(2i+1)^2-t^2}}\right),
\end{aligned}\tag{41}$$

$$\begin{aligned}
\text{where}\quad U_{n-1}=&\sum_{i=1}^{n-1}\sin(2i-1)y\sin(2i-1)\tau\cos\left[Q(x,2k+2-(2i-1)^2)-\frac{\pi}{4}\right]\\
&\cos\left[Q(t,2k+2-(2n-1)^2)+Q(t,2k+2-(2i-1)^2)\right].
\end{aligned}$$

It is straightforward to show that the second term in (41) is of order $o\left(1/\sqrt{2k+2-(2n-3)^2}\right)$. Then the second term in (38) is of order $O\left(1/\left(2k+2-(2n-1)^2\right)\right)$, that is, it is of the same order as the first term. This is why it is sufficient to consider the first term $C_k^n(x,t,y,\tau)$.

We substitute the first term $C_k^n(x,t,y,\tau)$ into (40) and we get:

$$\begin{aligned}
\sum_{n=1}^{N_1}\alpha_n^1\approx&-\frac{1}{2}\sum_{n=1}^{N_1}\sum_{k=2n^2-2n}^{2N_1^2+2N_1-1}\int_0^A\int_0^x\int_0^\pi\int_0^\pi\widehat{V}_1(x,t,y,\tau)\cos 2Q(t,2k+2-(2n-1)^2)\\
&\cdot\frac{\sin^2(2n-1)y\sin^2(2n-1)\tau}{\sqrt{2k+2-(2n-1)^2-x^2}\sqrt{2k+2-(2n-1)^2-t^2}}\\
&\cdot\cos^2\left[Q(x,2k+2-(2n-1)^2)-\frac{\pi}{4}\right]\,dx\,dy\,dt\,d\tau\\
=&-\frac{1}{2}\sum_{n=1}^{N_1}\sum_{m=1}^{2(N_1+n)(N_1-n+1)}\int_0^{\sqrt{2m-1}/2}\int_0^x\int_0^\pi\int_0^\pi\widehat{V}_1(x,t,y,\tau)\cos 2Q(t,2m-1)\\
&\cdot\frac{\sin^2(2n-1)y\sin^2(2n-1)\tau\cos^2\left[Q(x,2m-1)-\frac{\pi}{4}\right]}{\sqrt{2m-1-x^2}\sqrt{2m-1-t^2}}\,dx\,dy\,dt\,d\tau.
\end{aligned}$$

In the integral with x, we have taken into consideration that the function $\widehat{V_1}(x,t,y,\tau)$ is compactly supported, that is, $\widehat{V_1}(x,t,y,\tau) = 0$ as $A \le x \le \sqrt{2m-1}/2$.

We change the variable $x = \sqrt{2m-1}\xi$, $t = \sqrt{2m-1}\eta$. Then, taking into consideration that

$$Q(t, 2m-1) = (2m-1)Q(\eta, 1), \quad \text{where} \quad Q(\eta, 1) = \int\limits_{\eta}^{1} \sqrt{1-z^2}\, dz,$$

we obtain:

$$\begin{aligned}\sum_{n=1}^{N_1} \alpha_n^1 \approx &-\frac{1}{2}\sum_{n=1}^{N_1}\sum_{m=1}^{K_n} \int\limits_0^{1/2}\int\limits_0^{\xi}\int\limits_0^{\pi}\int\limits_0^{\pi} \frac{\widehat{V}_1(\sqrt{2m-1}\xi, \sqrt{2m-1}\eta, y, \tau)\cos(4m-2)Q(\eta,1)}{\sqrt{1-\xi^2}\sqrt{1-\eta^2}} \\ &\cdot \sin^2(2n-1)y \sin^2(2n-1)\tau \cos^2\left[(2m-1)Q(\xi,1) - \frac{\pi}{4}\right] d\xi\, d\eta\, dy\, d\tau \\ \approx &\frac{1}{16}\sum_{n=1}^{N_1}\sum_{m=1}^{K_n}(-1)^m \int\limits_0^{1/2}\int\limits_0^{\xi}\int\limits_0^{\pi}\int\limits_0^{\pi} \widehat{V}_1(\sqrt{2m-1}\xi, \sqrt{2m-1}\eta, y, \tau) \\ &\cdot \sin(4m-2)\left(\eta + O(\eta^3)\right) d\xi\, d\eta\, dy\, d\tau,\end{aligned}$$

where $K_n = 2(N_1+n)(N_1-n+1)$. Passing back to the variables x and t, we have

$$\begin{aligned}\sum_{n=1}^{N_1} \alpha_n^1 \approx -\frac{1}{16}\sum_{n=1}^{N_1}\sum_{m=1}^{K_n} \frac{(-1)^m}{2m-1} \int\limits_0^{A}\int\limits_0^{x}\int\limits_0^{\pi}\int\limits_0^{\pi} \widehat{V}_1(x,t,y,\tau)\sin 2\sqrt{2m-1} \\ \cdot \left(t + O\left(\frac{t^3}{2m-1}\right)\right) dx\, dt\, dy\, d\tau.\end{aligned}$$

Now even a rough estimate of the external sum leads us to a desired result:

$$\begin{aligned}\left|\sum_{n=1}^{N_1} \alpha_n^1\right| \le \frac{N_1}{16}\Bigg| \sum_{m=1}^{2(N_1+1)N_1} \frac{(-1)^m}{2m-1} \int\limits_0^{A}\int\limits_0^{x}\int\limits_0^{\pi}\int\limits_0^{\pi} \widehat{V}_1(x,t,y,\tau)\sin 2\sqrt{2m-1} \\ \left(t + O\left(\frac{t^3}{2m-1}\right)\right) dx\, dt\, dy\, d\tau\Bigg|.\end{aligned} \tag{42}$$

We apply the Abel transform to the sum in the right hand side of (42):

$$\sum_{m=1}^{2(N_1+1)N_1} \frac{(-1)^m}{2m-1}\sin 2\sqrt{2m-1}\left(t+O\left(\frac{t^3}{2m-1}\right)\right) \\ = \frac{U_{2(N_1+1)N_1}}{4N_1(N_1+1)-1}+2\sum_{m=1}^{2(N_1+1)N_1-1}\frac{U_m}{4m^2-1}, \tag{43}$$

where

$$U_m=\sum_{k=1}^{m}(-1)^k \sin 2\sqrt{2k-1}\left(t+O\left(\frac{t^3}{2k-1}\right)\right). \tag{44}$$

Thus, by formulae (42)-(44) we conclude that

$$\left|\sum_{n=1}^{N_1}\alpha_n^1\right| \le \frac{C}{N_1}\left|\sum_{m=1}^{2(N_1+1)N_1}(-1)^m\int_0^A\int_0^x\int_0^\pi\int_0^\pi \widehat{V}_1(x,t,y,\tau)\sin 2\sqrt{2m-1}\right. \\ \left.\cdot\left(t+O\left(\frac{t^3}{2m-1}\right)\right)dx\,dt\,dy\,d\tau\right| \to 0 \quad \text{as} \quad N_1\to\infty.$$

In the same way we can show that

$$\sum_{n=1}^{N_1}\alpha_n^2 \to 0 \quad \text{as} \quad N_1\to\infty \quad \text{and} \quad \sum_{n=1}^{N_2}\alpha_n^3\to 0,\quad \sum_{n=1}^{N_2}\alpha_n^4\to 0 \quad \text{as} \quad N_2\to\infty,$$

$$\text{where} \quad N_1=2\left[\frac{N+1}{2}\right]^2+2\left[\frac{N+1}{2}\right]-1, \qquad N_2=2\left[\frac{N}{2}\right]^2+4\left[\frac{N}{2}\right].$$

Then formula (35) implies immediately that $\sum\limits_{n=1}^{N}\alpha_n\to 0$ as $N\to\infty$. The proof is complete. ◄

Thus, Lemma 2 and Theorem 3 imply expression (4). Therefore, we have proved the following theorem.

Theorem 4. *Let $V(x,y)\in L^2(\Pi)$, where $\Pi=\{(x,y):-\infty<x<\infty,\ 0\le y\le\pi\}$ and $V(x,y)=0$ as $|x|\ge A$. Then the regularized trace formula of the two-dimensional operator*

$$H^0+V=-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+x^2+V(x,y)$$

in the strip Π *subject to the boundary conditions* $u(x,0)=u(x,\pi)=0$, $u(x,y)\in L^2(\Pi)$ *reads as*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\nu_n}\mu_k^{(n)}-\nu_n\lambda_n-\sum_{k=1}^{\nu_n}(V\varphi_k^{(n)},\varphi_k^{(n)})\right)=0,$$

where $\varphi_k^{(n)}(x,y)$ *are orthonormalized eigenfunctions of the operator* H^0 *associated with the eigenvalues* $\lambda_n=n+3$ *of multiplicity* ν_n, $n\geq 1$.

Corollary 2. *It should be noted that this result is obtained for the first time for partial differential operators with non-smooth potential. Due to the complexity of evaluating the second and subsequent corrections of perturbation theory, where the resolvent of the undisturbed operator is involved, the other authors previously had to impose very strict restrictions on the perturbation. In this paper, we obtain an explicit representation for the resolvent of the undisturbed operator and use the asymptotic kernel of the resolvent of one-dimensional harmonic oscillator. Therefore, the second amendment was presented explicitly, practically calculated.*

Fazullin and Nugaeva [9] obtained, a different result for the same problem under strict conditions for the finite perturbation V(x,y)$\in C_0^2(\Pi)$. *They obtained a constant in the right hand side of the formula (5). Unfortunately, the second amendment is not explicitly presented in the paper. The authors only refer to previous works. They believe that the resolvent of the operator in question has the same appearance as a two-dimensional harmonic oscillator and satisfies the same estimates. Of course this is not the case. This is shown by the fact that Theorem 2 does not hold for a two-dimensional harmonic oscillator.*

The authors also argue that the right-hand side of formula (5) for partial differential operators cannot be zero. However, we can give another example of a partial differential operator that is equal to zero.

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