Azerbaijan Journal of Mathematics
V. 11, No 1, 2021, January
ISSN 2218-6816

Locally Invo-Regular Rings

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Abstract. We define and study in a comprehensive way the class of so-called *locally invo-regular rings*. These rings form a proper subclass of the class of almost unit-regular rings due to Chen (Commun. Algebra, 2012) and nontrivially enlarge both the classes of weakly tripotent rings due to Breaz-Cîmpean (Bull. Korean Math. Soc., 2018) and quasi invo-regular rings due to the present author (J. Prime Research Math., 2019). We also somewhat refine the classification of those weakly tripotent rings by using our recent results published in Commun. Korean Math. Soc. (2017) and results obtained by Li et al. in Commun. Korean Math. Soc. (2018).

Key Words and Phrases: invo-regular rings, quasi invo-regular rings, locally invo-regular rings.

2010 Mathematics Subject Classifications: 16U99, 16E50, 16W10, 13B99

1. Introduction and Background

Throughout the text of the current article, all rings R are assumed to be associative, possessing the identity element 1 which differs from the zero element 0 of R, and all subrings excluding the proper ideals are unital (i.e., containing the same identity as that of the former ring). Our standard terminology and notation are mainly in agreement with [18]. For instance, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, $J(R)$ the Jacobson radical of R, and $C(R)$ the center of R. The specific notions and notations will be given explicitly in the sequel.

First of all, let us recall by referring to [14] and [15] that an element a of an arbitrary ring R is said to be *unit-regular* if there exists $u \in U(R)$ such that $a = aua$. If every element of R is equipped with that property, R is called *unit-regular*, too. This kind of rings has very interesting and important properties, which affect the general ring's structure.

Generalizing substantially this critical concept, the so-called *almost unit-regular* rings were explored in [2] which, in the context of formal (triangular and full) matrix rings extensions accomplished with the Morita contexts with zero pairings, are better to be termed as *locally unit-regular* rings. Here we shall examine a subclass of the class consisting of these rings as defined bellow – for more information on a related subject see [16] as well.

On the other hand, an element b of a ring R is called *tripotent* if the equality $b^3 = b$ is fulfilled. If each element of R is with this property, R is said to be *tripotent* as well. The complete description of such rings is given in [17]. Specifically, they are a subdirect product (= a subring of a direct product) of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

On the other hand, the so-called *weakly tripotent* rings were explored in [1] that are rings in which at least one of the elements b or $1-b$ is a tripotent. It is immediate that weakly tripotent rings of characteristic 3 are themselves tripotent as $(1-b)^3 = 1-b^3 = 1-b$ yields $b^3 = b$. An interesting example of a weakly tripotent ring of characteristic 8 is the indecomposable ring $\mathbb{Z}_8$ whose elements are solutions of (one of) the equations $x^2 = 1$ or $(1-x)^2 = 1$ (thus x or $1-x$ is obviously a tripotent). In general, mainly in the non-commutative case, these rings have not a complete characterization yet. By the way, it is worthwhile noting that the same terminology of weakly tripotent rings was used in [6] under the meaning that each element in the ring satisfies (one of) the equations $x^3 = \pm x$. Nevertheless, these two notions differ from each other, since the field $\mathbb{Z}_5$ is weakly tripotent in the sense of [6], but it is definitely *not* weakly tripotent in the sense of [1], whereas the aforementioned indecomposable ring $\mathbb{Z}_8$ is *not* of the weakly tripotent sorts of rings described in [6] (notice that this is true also directly taking into account that $2^3 = 0 \neq \pm 2$).

An element c of a ring R is said to be *invo-regular* if, there exists an involution $v \in R$ (i.e., $v^2 = 1$) such that $c = cvc$ (in fact, c is a special unit-regular element with the inner inverse being an involution). It is pretty clear that any tripotent element has to be invo-regular, because it can be checked by simple manipulations that $b = b^3$ assures $b = bvb$ with $v = 1 + b - b^2$ and $v^2 = 1$. The converse is false, however. If every element of R is invo-regular, then R is called *invo-regular* too. These rings were completely described in [4] by showing that they coincide with the above discussed tripotent rings (see, e.g., [5, Theorem 2]). Even something more, the so-termed *quasi invo-regular rings* were handled in [5] in which each element is invo-regular such that the inner inverse v has the property that v or $1-v$ is an involution. Surprisingly, quasi invo-regular rings are themselves invo-regular, that gives an immediate coincidence of these two ring classes.

So, we come to our pivotal tool.

Definition 1. *We shall say that a ring R is locally invo-regular if, for any $r \in R$, either r or $1-r$ is invo-regular in R. This means that either $r = rvr$ or $1-r = (1-r)v(1-r)$ for some $v \in R$ with $v^2 = 1$.*

A few quick commutative examples are the next ones: Boolean rings, the field $\mathbb{Z}_3$ and its arbitrary (finite or infinite) direct products, the indecomposable ring $\mathbb{Z}_4$, etc. A direct inspection shows that the field $\mathbb{F}_4$ consisting of four elements is non-locally invo-regular ring of characteristic 2.

This new point of view, in accordance with the aforementioned ring classes, allows us to derive the next implications:

$$\textbf{quasi invo-regular} \Rightarrow \textbf{weakly tripotent} \Rightarrow$$

$$\Rightarrow \textbf{locally invo-regular} \Rightarrow \textbf{locally unit-regular.}$$

None of these implications is reversible in general (see, for more detailed information, Example 1 below). However, in the case of abelian (in particular, commutative) rings, all locally invo-regular rings are exactly weakly tripotent in the sense of [1].

As it was demonstrated in the paragraph before [2, Example 2.6], although the indecomposable local ring $\mathbb{Z}_4$ is almost unit-regular, the direct product $\mathbb{Z}_4 \times \mathbb{Z}_4$ is *not* almost unit-regular.

Our motivation in writing up this research paper is to encompass the above mentioned classes of weakly tripotent and invo-regular (= quasi invo-regular) rings into the class of locally invo-regular rings which possesses quite more exotic properties. However, although our rings from Definition 1 are closely related to the aforementioned almost unit-regular rings, they will be properly enclosed in them.

In what follows, our basic achievement will be the complete description of the structure of locally invo-regular rings and, in particular, of the structure of weakly tripotent rings by approaching in a different way than that in [1].

2. Main Results

As it was already emphasized, in the non-commutative case there is *no* a satisfactory complete characterization of the weakly tripotent rings from [1], so that by what we have shown above such full characterizing cannot be happen in the general case of locally invo-regular rings too. Nevertheless, the chief result of ours is the following one:

Theorem 1. *A ring R is locally invo-regular if, and only if, all elements of $J(R)$ satisfy the equation $x^2 = 2x$ and $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is a* *locally invo-regular ring with* $8 = 0$ *(thus it is of characteristic either* 2, 4 *or* 8*) for which* $R_1/J(R_1)$ *is Boolean provided* R_1 *is abelian (i.e., all its idempotents are central) and* $x^4 = 0$ *for all* $x \in J(R_1)$*, and either* $R_2 = \{0\}$ *or* R_2 *is a ring which is a subdirect product of a family of copies of the field of three elements* $\mathbb{Z}_3$*.*

Proof. "**Left-to-right**". Given $z \in J(R)$, one has $z = zvz$ or $1 - z = (1 - z)v(1 - z)$. This yields $z(1 - vz) = 0$, i.e., $z = 0$, or $1 - z = v$ as $1 - vz$ and $1 - z$ both invert taking into account that $vz \in J(R)$. Thus $z = 1 - v$, which by squaring gives $z^2 = 2z$, as required.

Furthermore, for $3 \in R$, we write $3 = 3.v.3 = 9v$ for some $v \in R$ with $v^2 = 1$. By squaring, one obtains $72 = 2^3.3^2 = 0$. In the other possibility, we write $-2 = (-2).v.(-2) = 4v$. Again by squaring, $12 = 2^2.3 = 0$. So, finally, $8.9 = 0$ in R will hold. Next, with the Chinese Remainder Theorem at hand, one decomposes $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is a locally invo-regular ring with $8 = 0$, and either $R_2 = \{0\}$ or R_2 is a locally invo-regular ring with $9 = 0$. We even claim that $3 = 0$ in R_2. In fact, the equality $3 = 9v$ holding in R_2 directly implies what we pursue. In the case where $-2 = 4v$ holds in R_2, one deduces that $-4 = 8v = -v$, i.e., $4 = v$. Squaring that equality, we find that $15 = 0$. But we also have that $18 = 0$. This immediately ensures $3 = 0$, as claimed.

We will now describe both direct factor components separately:

Describing R_1**:** Here $2 = 0$ or $4 = 0$ or $8 = 0$. By what we have shown above, $x^2 = 2x$ holds for every $x \in J(R_1)$. Therefore, $x^3 = 2x^2 = 4x$ and thus $x^4 = 4x^2 = 8x = 0$, as expected. That is why $J(R_1)$ is always nil.

Let now R_1 be abelian. As 2 is a central nilpotent, it follows that $2 \in J(R_1)$ and hence the quotient ring $R_1/J(R_1)$ is locally invo-regular of characteristic 2. However, as it is well-known, R_1 being abelian guarantees that $R_1/J(R_1)$ is abelian too, because $J(R_1)$ is nil. According to [13], all nilpotents of $R_1/J(R_1)$ are also central and thus they are zero as this factor-ring is obviously semiprimitive (= semi-simple in the sense of Jacobson). If now $\overline{v}$ is an involution in $R_1/J(R_1)$, i.e., $\overline{v}^2 = \overline{1}$, then $(\overline{v} - \overline{1})^2 = \overline{0}$ since $\overline{2} = \overline{0}$ and thus $2\overline{v} = \overline{0}$, deriving from this that $\overline{v} = \overline{1}$ which gives the assertion.

Describing R_2**:** Here $3 = 0$. We intend to prove now that each element of R_2 satisfies the equation $x^3 = x$ and so [17] will then apply to get the expected claim. We assert that $Nil(R_2) = \{0\}$ whence R_2 will be abelian (that is, every one of its idempotents is central). To show this, given $q \in R_2$ with $q^2 = 0$, we consider the difference $q - 1 \in U(R_2)$. It must be $q - 1 = (q - 1)v(q - 1)$ or $2 - q = (2 - q)v(2 - q)$, for some $v \in R_2$ with $v^2 = 1$ as $1 - (q - 1) = 2 - q = -1 - q \in U(R_2)$. In the first situation, canceling by $q - 1$, we have $q - 1 = v$ and so squaring $q = v + 1$

it follows that $2 + 2v = 0$, i.e., $v = -1$ as $3 = 0$. Therefore, $q = -1 + 1 = 0$, as needed. In the second situation, canceling by $2 - q$, we arrive at $-1 - q = v$ and thus squaring $q = -1 - v = -(1 + v)$ it again follows that $v = -1$ and hence $q = 0$, as required.

Furthermore, the equality $x = xvx$ ensures that xv is an idempotent forcing that $x = x^2v$. So, $x^2 = x^3v$ yields $x = x^3v^2 = x^3$, as promised. Similarly, $1 - x = (1 - x)^3 = 1 - x^3$, because $3 = 0$. Consequently, $x = x^3$ in both aspects, as required.

"**Right-to-left**". If R_1 is a locally invo-regular ring and R_2 is a tripotent ring, then we assert that the direct product $R_1 \times R_2$ is too a locally invo-regular ring. As the verification is only a routine technical exercise, we leave it to the interested reader for a direct check. ◀

It was shown in [1, Lemma 1(1)] that a subring of a weakly tripotent ring is also weakly tripotent. As this cannot happen in the case of locally invo-regular rings, we now have to prove that the centers of locally invo-regular rings are always locally invo-regular (and thus they are commutative weakly tripotent rings).

Proposition 1. *The center of a locally invo-regular ring is also a locally invo-regular ring (in fact, it is a commutative weakly tripotent ring).*

Proof. Given $z \in C(R)$, we write $z = zvz$ or $1 - z = (1 - z)v(1 - z)$ for some $v \in R$ with $v^2 = 1$. Consequently, in the first case, one writes that $z = z^2v$ whence $z^2 = z^3v$, and so $z = z^3v^2 = z^3$. Thus z is a tripotent element and hence it can be written as $z = z(1 + z - z^2)z$, where $1 + z - z^2$ is an involution in $C(R)$ as by a direct inspection $(1 + z - z^2)^2 = 1$. The second possibility follows in the same manner by replacing $z \to 1 - z$, so we omit the details. ◀

Let us recall that an element a of a ring R is *strongly regular* if there is an element $d \in R$ such that $a = a^2d$. Thus, if each element of a ring is strongly regular, the ring is also said to be strongly regular. So, generalizing this, we call a ring R *locally strongly regular* if, for every $r \in R$, at least one of the elements r or $1 - r$ is strongly regular.

It is well known that the center of a unit-regular ring is also a unit-regular ring (indeed, it is a commutative strongly regular ring, and thus a subdirect product of fields) – this follows elementarily observing that for any c in $C(R)$, where R is unit-regular, it will follow that $c = cuc = c^2u = uc^2$ for some $u \in U(R)$. Now, by using the same idea, we are able to extend this to locally unit-regular rings as follows: *The center of a locally unit-regular ring is a locally strongly regular ring.*

It seems that non-commutative weakly tripotent rings have not been comprehensively studied in [1], so a new additional material concerning that theme with a new approach is definitely needed. Under this reason, we shall now considerably extend [1, Corollary 4] by shown that weakly tripotent rings are special strongly clean rings, as more exactly they are surely strongly invo-clean, as defined in [3]: A ring R is called *strongly invo-clean* if, for each $r \in R$, there exist $e \in Id(R)$ and $v \in U(R)$ with $v^2 = 1$ such that $r = e + v$ and $ev = ve$. It was shown in [3, Corollary 2.17] (accomplishing this with achievements in [11]) that if a ring R is strongly invo-clean, then R is decomposable as $R_1 \times R_2$, where $R_1 = \{0\}$ or $R_1/J(R_1)$ is Boolean with nil $J(R_1)$ of index of nilpotence at most 3, and $R_2 = \{0\}$ or R_2 is a subdirect product of a family of copies of the field $\mathbb{Z}_3$. This, however, is a strict containment as the ring $\mathbb{Z}_4 \times \mathbb{Z}_4$ is obviously strongly invo-clean, but as already noticed above it is even *not* locally unit-regular.

The next relationship considerably strengthens [1, Corollary 4].

Proposition 2. *Every weakly tripotent ring is strongly invo-clean. The converse implication is untrue.*

Proof. For such a ring R, we have $r^3 = r$ or $(1-r)^3 = 1-r$ whenever $r \in R$. In the first case, we have $r = (1-r^2)+(r^2+r-1)$. A direct manipulation shows that $1 - r^2 \in Id(R)$ as $r^2 \in Id(R)$ and $(r^2 + r - 1)^2 = 1$, observing elementarily that these two elements commute. We also see that $rR \cap (1 - r^2)R = \{0\}$. To look at this, we write $ra = (1 - r^2)b$ for some $a, b \in R$. Since $r^2 \neq 0$ (as for otherwise, $r^2 = 0$ will imply that $0 = r^3 = r$, and we are done), one obtains by multiplying by r^2 from the left that $r^3a = (r^2 - r^4)b$ which is equivalent to $ra = 0$, as required.

Dealing with the other equality $(1 - r)^3 = 1 - r$, by replacing $r \to 1 - r$, we can write by using the trick above that $1 - r = f + w$ for some idempotent $f = 1 - (1 - r)^2 = 2r - r^2$ and an involution $w = 1 - 3r + r^2$. Therefore, $r = (1 - f) + (-w) \in Id(R) + U(R)$, where $(-w)^2 = w^2 = 1$, as required. However, the equality $rR \cap (1 - r)^2R = \{0\}$ cannot properly happen.

About the second part, we refer to [8] (compare with Example 1 presented below). ◀

By what we have shown so far, it will follow that some valuable characterization of weakly tripotent rings could be extracted from already well-known results established in [3, Corollary 2.17].

A question that immediately arises (mainly due to the last proof) is of whether or not weakly tripotent rings are unit-regular (certainly, they are necessarily locally unit-regular being locally invo-regular).

We will be now able to improve [1, Corollary 9] in the following expected way by giving a more direct and transparent proof:

Proposition 3. *A ring R is locally invo-regular without non-trivial idempotents if, and only if, for each $r \in R$, one of the equalities $r^2 = 1$ or $r^2 = 2r$ holds. In particular, such a ring is necessarily weakly tripotent.*

Proof. "$\Rightarrow$". Given $rvr = r$ or $(1-r)v(1-r) = 1-r$ for some $v \in R$ with $v^2 = 1$, one plainly sees that rv or $(1-r)v$ is an idempotent. Hence, $rv = 0$ or $rv = 1$ ensuring that $r = 0$ or $r = v$ whence $r^2 = 1$ in the second case, or $(1-r)v = 0$ or $(1-r)v = 1$ assuring that $1-r = 0$, i.e., $r = 1$ or $1-r = v$ whence in the second case $(1-r)^2 = 1$, that is, $r^2 = 2r$. Since $r = 0$ yields $r^2 = 2r$ and $r = 1$ yields $r^2 = 1$, we are done.

"$\Leftarrow$". If r is an idempotent, it easily follows that $r = 1$ or $r = 0$, as required. If $r^2 = 1$, we have $r = r.r.r$, while if $r^2 = 2r$ we have $(1-r)^2 = 1$ and so $1 - r = (1-r).(1-r).(1-r)$, as needed. ◀

As an immediate consequence to the last assertion, one can derive the following:

Corollary 1. *Suppose that R is a ring with no non-trivial idempotents in which 2 is nilpotent. Then R is locally invo-regular if, and only if, $R/J(R) \cong \mathbb{Z}_2$ and the equation $x^2 = 1$ holds for any element of $U(R)$ if, and only if, $R/J(R) \cong \mathbb{Z}_2$ and the equation $x^2 = 2x = 0$ holds for any element of $J(R)$.*

In particular, such a ring R is locally invo-regular exactly when it is weakly tripotent.

Proof. "**Necessity.**" As $2 \in J(R)$ is a central nilpotent, setting $\overline{R} := R/J(R)$, one observes that $\overline{R}$ is a ring of characteristic 2 (i.e., $\overline{2} = \overline{0}$) such that, in accordance with Proposition 3, the condition $\overline{r}^2 = \overline{1}$ or $\overline{r}^2 = \overline{0}$ is true. This makes up $\overline{R}$ without non-trivial idempotents as well. Furthermore, in the first case, it must be $(\overline{r} - \overline{1})^2 = \overline{0}$ whence $\overline{r} = \overline{1} + (\overline{r} - \overline{1})$, whereas in the second case $\overline{r} = \overline{0} + \overline{r}$. This means that, in both situations, $\overline{R}$ is a strongly nil-clean ring, so that [11] will apply to get that $\overline{R}$ is Boolean and hence isomorphic to $\mathbb{Z}_2$, as asked for.

Certainly, both equalities $x^2 = 1$ and $x^2 = 2x = 0$ cannot be true simultaneously, so the relation $x^2 = 1$ will be fulfilled for all $x \in U(R)$ as the other one $x^2 = 2x \in J(R)$ is impossible in $U(R)$ and, reciprocally, the relation $x^2 = 2x$ will be valid for all $x \in J(R)$ as the other one $x^2 = 1$ makes no sense in $J(R)$.

"**Sufficiency.**" As $U(R/J(R)) \cong U(R)/(1 + J(R)) \cong U(\mathbb{Z}_2) = \{1\}$, it follows at once that $U(R) = 1 + J(R)$. Hence $Nil(R) \subseteq J(R)$ because $1 + Nil(R) \subseteq U(R)$. However, one sees that $Nil(R) = J(R)$ since $J(R)$ is nil. But R being local implies that any element r of R is either nilpotent or unit. That is, $r^2 = 2r = 0$ or $r^2 = 1$. Therefore, $(1-r)^3 = 1 - 3r + 3r^2 - r^3 = 1 - 3r = 1 - r$ as $r^2 = 2r = 0$ or $r^3 = r$, substantiating our claim.

Concerning now the additional statement, one way being self-evident, we concentrate on the other one. Henceforth, we just apply the aforementioned [1, Corollary 9] to get the desired result. ◄

One more critical remark, which clarifies all the things alluded to above, is that in [1, Corollary 9(2)] the condition imposed on the ring R to be "*a local ring*" is superfluous as the quotient $R/J(R) \cong \mathbb{Z}_2$ is surely a field.

The next characterization criterion might be of some applicable purposes (compare with Proposition 2 quoted above) – see also [9] and [10].

Lemma 1. *Suppose R is a ring of characteristic 2. Then the following three issues are equivalent:*

(1) R is weakly tripotent.

(2) All elements of R satisfy (one of) the equations $x^3 = x$ or $x^3 = x^2$.

(3) Each element r of R is presentable as $r = q + e$, where $q, e \in R$ commute with $q^2 = 0$ and $e^2 = e$ such that either $qe = 0$ or $q(1 - e) = 0$.

Proof. First of all, let $\mathrm{char}(R) = 2$.

The equivalence (1) $\iff$ (2) follows by a direct routine check, because $(1 - x)^3 = 1 - x$ is tantamount to $x^3 = x^2$ as $2 = 0$.

Now, we handle the implication (3) $\Rightarrow$ (2). In fact, one derives that $r^2 = e$ and $r^3 = qe + e$. So, if $qe = 0$, we obtain $r^3 = r^2$. However, if $qe = q$, one gets $r^3 = r$, as formulated.

What remains to show is the validity of the converse implication (2) $\Rightarrow$ (3). Indeed, we claim foremost that the nilpotence index of R is exactly 2. To see that, given $q \in Nil(R)$, we have $q^3 = q$ or $q^3 = q^2$. Thus $q(1 - q^2) = 0$ or $q^2(1 - q) = 0$. As both $1 - q^2$ and $1 - q$ invert in R, it must be $q^2 = 0$ in both, as required. Furthermore, we directly can apply Proposition 2 to write that $r = v + e = (v + 1) + (1 + e)$, where $v + 1$ is a nilpotent of order 2 and $1 + e$ is an idempotent. However, a more precise analysis of this record is needed. Another useful approach might be as follows: If $u \in U(R)$, then either $u^3 = u$ or $u^3 = u^2$. Hence, in both cases $u^2 = 1$, which means that $(u - 1)^2 = 0$. This, in turn, forces that $u \in 1 + Nil(R)$, i.e., $U(R) = 1 + Nil(R)$. As these rings have elements as solutions of (one of) the equations $x^3 = x$ or $x^3 = x^2$ and thus they are thereby clearly exchange, the application of [11] shows that $r = q + e$ with $qe = eq$, $q^2 = 0$ and $e^2 = e$. Nevertheless, we need the more detailed relations between q and e stated above. To draw them, we foremost calculate that $r^2 = e$ and $r^3 = qe + e$ as $2 = 0$. Then, $r^3 = r$ yields $qe + e = q + e$, i.e., $q(1 - e) = 0$; whereas $r^3 = r^2$ implies that $qe + e = e$, i.e., $qe = 0$, as promised. ◄

As we already have seen above in Proposition 2, if R is a weakly tripotent ring, then R is strongly invo-clean, but we need a rather more precise description, however. This will successfully be materialized in the next statement.

Theorem 2. *A ring R is weakly tripotent (of characteristic 2, 4 or 8) if, and only if, every element r of R is presentable as $r = q + e$, where $q, e \in R$ commute, with $q^2 = 2q$ (hence $q^2 = 0$ when $2 = 0$, $q^3 = 0$ when $4 = 0$ and $q^4 = 0$ when $8 = 0$) and $e^2 = e$ such that either $qe = 0$ or $q(1-e) = 0$.*

Proof. "$\Rightarrow$". The case where $2 = 0$ was handled above in Lemma 1.

So, let now $\mathrm{char}(R) = 4$. Then, for any $q \in Nil(R)$, we have $q^3 = q$ or $(1-q)^3 = 1-q$. The first equality immediately ensures that $q = 0$. The second one, however, implies that $(1-q)^2 = 1$ as $1-q$ inverts in R, so that we come to $q^2 = 2q$. Thus, $q^3 = 2q^2 = 4q = 0$ as $4 = 0$. Furthermore, with Proposition 2 at hand, we write $r = q + e$, where $qe = eq$ with $e^2 = e$ and $q^3 = 0$. Consequently, $r = r^3 = q^3 + 3q^2e + 3qe + e = -q^2e - qe + e = q + e$ yields $q^2e + qe = -q$. As $q^2 = 2q$, we derive that $3qe = -q$, i.e., $-qe = -q$, i.e., $q(1-e) = 0$, as desired. If now $(1-r)^3 = 1-r$, then as showed in Proposition 2 we may replace r by $1-r$ (and hence $q \to -q$ and $e \to 1-e$), deducing that $qe = 0$, as wanted.

Assuming now that $\mathrm{char}(R) = 8$, as already observed above, we have $q^2 = 2q$ whence $q^4 = 4q^2 = 8q = 0$ as $8 = 0$. Furthermore, for $r^3 = r$ such that $r = q + e$ with $qe = eq$, one infers that $q^3 + 3q^2e + 3qe = q$. The latter means that $q(1-q)(1+q) = 3qe(1+q)$. Since $1+q$ is invertible in R, this leads to $q(1-q) = 3qe$ and hence to $q(1-q)(1-e) = 0$. But $1-q$ also inverts in R, so we obtain the pursued equality $q(1-e) = 0$. Another way of proving up this equality is to use as given above the relation $q^2 = 2q$. Replacing as above $q \to -q$ and $e \to 1-e$ in the case of $1-r = (1-r)^3$, one concludes that $qe = 0$, as asked for.

"$\Leftarrow$". Writing $r = q + e$ with q and e satisfying the conditions stated above, we can proceed like this: If $qe = q = eq$, then $r^3 = (q+e)^3 = q^3 + 3q^2e + 3qe + e = 4q + 9qe + e = 13q + e = q + e = r$ whenever $4 = 0$. If now $qe = 0 = eq$, then $1 - r = -q + (1-e)$, so $(-q)(1-e) = -q = (1-e)(-q)$ and thereby we may apply the same trick to obtain $(1-r)^3 = 1-r$.

For $8 = 0$, things are little more complicated as follows: In fact, the element $0 \neq 2r$ does not satisfy the equation $x^3 = x$ as for otherwise $8r^3 = 2r = 0$. Hence one has $(1-2r)^3 = 1-2r$ and thus $4r = 4r^2$. For further detailed arguments, we refer to [9]. ◄

Note that even without $8 = 0$ at hand, the equalities $qe = 0 = eq$ (hence $q(1-e) = (1-e)q = q$), $q^2 = 2q$ and $e^2 = e$ force whenever $r = q + e$, i.e. when

$1 - r = -q + (1 - e)$, that $(1 - r)^3 = -q^3 + 3q^2(1 - e) - 3q(1 - e) + (1 - e) = -4q + 3q(1 - e) + (1 - e) = -q + (1 - e) = 1 - r$, as expected.

Likewise, as $4q^2 = 0$ when $8 = 0$, one may derive that $r^3 = q^3 - q^2e + 3qe + e = 4q + qe + e = 5q + e = r + 4q$, provided $qe = eq = q$. Thus $2r^3 = 2r$, and after the squaring of $r^3 = r + 4q$ we deduce that $r^6 = r^2$, that is, r^2 is a tripotent.

Recall that a ring is called *indecomposable* if it does *not* possess non-trivial central idempotents and *strongly indecomposable* if it does *not* possess non-trivial idempotents. An appeal to [13] enables us to state that every ring without central nilpotent elements is indecomposable, but the converse is not always true – however, this is the case for regular rings in the sense of von Neumann (in particular, for unit-regular rings).

We now have accumulated all the ingredients necessary to proceed by proving with the next achievement, which is a significant reminiscent of [1, Corollary 9].

Theorem 3. *Suppose R is a locally invo-regular ring with no central nilpotent elements. Then the following two points are true:*

(i) If char$(R) = 3$, then $R \cong \mathbb{Z}_3$.

(ii) If char(R) is even and not exceeding 8 and all nilpotents of order ≤ 2 are contained in $J(R)$, then char$(R) = 2$ and $R/J(R)$ is Boolean.

Proof. Item (i) follows immediately with the aid of Theorem 1 because as noticed above R has to be indecomposable.

As for item (ii), we first observe that as $2 \in Nil(R)$ is central, it must be $2 = 0$. Hence, if $v \in R$ with $v^2 = 1$, then $(v - 1)^2 = 0$ and so $v = 1 + t$ for some $t \in R$ with $t^2 = 0$. Therefore, in view of our assumption, $v \in 1 + J(R)$. This relation, combined with the local invo-regularity of R, forces that $R/J(R)$ is also locally invo-regular with existing involution exactly 1. That is, both x and $1 - x$ are idempotents for any $x \in R/J(R)$. The latter possibility also implies that x is an idempotent, as needed. ◄

Pertaining these considerations to weakly tripotent rings, we may proceed as in [9]: According to Proposition 2, for each $r \in R$, we have $r = v + e = (1 + v) + (1 + e)$ for some $v \in R$ with $v^2 = 1$ and $e \in Id(R)$. But $(1 + v)^2 = 0$ and $(1 + e)^2 = 1 + e$, so R is nil-clean (see, e.g., [11] or [12]). Moreover, we claim that the index of nilpotence of R is at most 2. In fact, taken an arbitrary $q \in Nil(R)$, we write as above that $q = v + e$. With [3, Corollary 2.6] at hand we arrive at $e = 1$. We, therefore, have $q = v + 1$ and hence $q^2 = 0$ as $2 = 0$. The claim sustained (compare also with [19, Lemma 2.1] where the proof is unnecessarily intricately stated). Another approach to extract this claim could be taken from Lemma 1 above.

Furthermore, since the homomorphic images of a weakly tripotent ring and those of a nil-clean ring retain the same property, one deduces that $R/J(R)$ has to be simultaneously weakly tripotent and nil-clean. However, [19, Theorem 2.2] allows us to conclude that $R/J(R)$ is a subdirect product of a family of copies of $\mathbb{Z}_2$, bearing in mind that the matrix ring $\mathbb{M}_2(\mathbb{Z}_2)$ need not be weakly tripotent (see Example 1(3) below). Consequently, we expect to have only $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, since one expects that the inclusion $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2$ is impossible for $\lambda \geq 3$ because, for example, the triple direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ cannot be isomorphic to the weakly tripotent factor-ring $R/J(R)$. In fact, assuming in a way of contradiction that $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we may refer to, e.g., the ring $\mathbb{T}_3(\mathbb{Z}_2)$ which is surely not weakly tripotent in accordance with Example 1(2) quoted below. Indeed, there is an appropriate proper nil-ideal I of $\mathbb{T}_3(\mathbb{Z}_2)$ such that $\mathbb{T}_3(\mathbb{Z}_2)/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with a consecutive epimorphism $\mathbb{T}_3(\mathbb{Z}_2)/I \to \mathbb{T}_3(\mathbb{Z}_2)/J(\mathbb{T}_3(\mathbb{Z}_2))$ as it is known that $I \subseteq J(\mathbb{T}_3(\mathbb{Z}_2))$. So, resuming, the possibility for $R/J(R)$ to be in general isomorphic to either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is not absolutely realistic.

On the other vein, the elements of a weakly tripotent ring of characteristic 2 satisfy (one of) the equations $x^2 = 0$ or $x^2 = x^3$.

The ring $\mathbb{T}_2(\mathbb{Z}_2)$ possesses the property that the factor-ring modulo of its Jacobson radical is isomorphic to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$. So, about the validity of the reverse implication, a question which immediately arises is whether or *not* an indecomposable ring R of characteristic 2, for which $z^2 = 0$ for any $z \in J(R)$ and for which the factor-ring $R/J(R)$ has the presentation $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, will be isomorphic to $\mathbb{T}_2(\mathbb{Z}_2)$?

The next (possibly non-commutative) examples shed some more light on the currently studied class of rings:

Example 1. *(1) The triangular (upper) matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$ is locally invo-regular.*

In fact, each (nontrivial) element in it is either an idempotent or an involution or a nilpotent of order 2. As the first two types of elements are clearly invo-regular, what remains to verify is that $1 - q$ is invo-regular whenever $q^2 = 0$. But this follows elementarily since $(1 - q)^2 = 1 - q^2 = 1$ with $2 = 0$.

Moreover, it was also shown in [1, Example 11] that $\mathbb{T}_2(\mathbb{Z}_2)$ is even weakly tripotent.

(2) The (upper) triangular matrix ring $\mathbb{T}_3(\mathbb{Z}_2)$ is not locally invo-regular (and thus it is not weakly tripotent).

To see that, consider the invertible matrix $x = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. *Direct calculations show that* $x^3 \neq x$ *and* $x^4 = I_3$, *the identity* 3×3 *matrix. Now, setting,* $y := 1 - x$, *one computes that* $y = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $y^3 = 0 \neq y$. *Further straightforward computations guarantee that any involution matrix* v *in* $\mathbb{T}_3(\mathbb{Z}_2)$, *i.e.,* $v^2 = I_3$, *is of the form* $v = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, *where* $a, b, c \in \mathbb{Z}_2$ *such that* $ac = 0$ *(this certainly excludes the case* $a = c = 1$*).*

Next, it is obvious that x *cannot be written as* $x = xvx$ *for some involution* v *as* $x^2 \neq I_3$. *We will demonstrate now that* y *is also non-presentable in such a way, namely* $y \neq yvy$. *Assuming the contrary, we write*

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, this leads to

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & c+1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is the desired contradiction

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is long known that $\mathbb{T}_3(\mathbb{Z}_2)/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *for a proper nil-ideal* I *of* $\mathbb{T}_3(\mathbb{Z}_2)$. *So, although the quotient* $\mathbb{T}_3(\mathbb{Z}_2)/I$ *is obviously Boolean and so weakly tripotent, what could be extracted from this is that the same cannot be said of the former ring* $\mathbb{T}_3(\mathbb{Z}_2)$.

Finally, it is worth to noticing that $\mathbb{T}_3(\mathbb{Z}_2)$ *is surely locally unit-regular, however (see [2, Corollary 4.5 (2)]).*

(3) The full matrix ring $\mathbb{M}_2(\mathbb{Z}_2)$ *is unit-regular but not locally invo-regular.*

Indeed, by considering the invertible matrix $u = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ *having the property* $u^3 = 1 \neq u$, *one calculates that* $1 - u = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = u^2 \neq (1-u)^3$, *whence* $(1-u)^2 = u$ *and* $(1-u)^3 = 1$. *This shows that* $\mathbb{M}_2(\mathbb{Z}_2)$ *is, definitely, not weakly tripotent. This fact, however, is not directly deducible from the corresponding results presented in [1].*

Further, we shall even show a bit more. The only non-trivial involutions in $\mathbb{M}_2(\mathbb{Z}_2)$ *are the three ones* $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. *What suffices to check is the invalidity of each of the equalities* $u = uvu$, $u = uwu$, $u = uyu$ *as well as of* $1-u = (1-u)v(1-u)$, $1-u = (1-u)w(1-u)$, $1-u = (1-u)y(1-u)$. *In fact, it can be verified by direct computations, which we leave for a check to the interested reader, that the curious relationships* $uvu = v \neq u$, $uwu = w \neq u$, $uyu = y \neq u$ *as well as* $(1-u)v(1-u) = v \neq 1-u$, $(1-u)w(1-u) = w \neq 1-u$, $(1-u)y(1-u) = y \neq 1-u$ *hold. This, manifestly, substantiates our claim after all.*

(4) The (upper) triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_3)$ *is not locally invo-regular (and thus it is not weakly tripotent).*

To substantiate this, we look at the invertible matrix $A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ *which is of order* 6, *that is,* $u^6 = 1$. *For* E *being the identity matrix, namely* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *we consider the difference* $B := E - A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, *which also inverts in the whole ring. As* $B^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *the claim is sustained.*

(5) It could be expected that the almost unit-regular ring R, *having* $U(R)$ *with* $U^2(R) = \{1\}$, *is non-commutative locally invo-regular which is neither weakly tripotent in the sense of [1] nor local. However, the proof still eludes us.*

This example leads us to the following strengthening of [2, Corollary 4.5].

Theorem 4. *Let* R *be a ring and* $n \in \mathbb{N}$. *Then the triangular matrix ring* $\mathbb{T}_n(R)$ *is locally invo-regular if, and only if, either*

(1) $n = 1$ *and* R *is locally invo-regular*

or

(2) $n = 2$ *and* $R \cong \mathbb{Z}_2$.

Proof. "$\Rightarrow$". The first point is straightforward, so we will be concentrated on the second one. In fact, as in the proof of [2, Corollary 4.5 (2)], it will follow that $n = 2$ or $n = 3$ and R is a division ring. Concerning the case where $n = 2$, it easily follows that the factor-ring $\mathbb{T}_2(R)/I \cong R \times R$ is locally invo-regular for some appropriate ideal I of $\mathbb{T}_2(R)$ as being an epimorphic image of the locally invo-regular ring $\mathbb{T}_2(R)$. Hence, we claim that $R \cong \mathbb{Z}_2$. In fact, $R \times R$ being locally invo-regular implies the same property for R. Assuming in a way of contradiction that $R \neq \{0, 1\}$, it follows there is $r \in R$ which is neither 0 nor 1. Assume also that $2 \neq 0$. But as in R each nonzero element is invertible, it follows from the proof of Theorem 1 that $3 = 0$ and hence $R \cong \mathbb{Z}_3$. However, Example 1(4) shows that this is impossible. So, R has characteristic 2. That is why it must be $r^2 = 1$ or $(1-r)^2 = 1$ as both r and $1-r$ invert in R. The first equality gives $(1-r)^2 = 0$, so $r = 1$, contrary to our hypothesis. The second one gives $1-r^2 = 1$, i.e., $r^2 = 0$. Thus $r = 0$, contrary to our assumption. Finally, R really must contain only two elements, namely 0 and 1, as required.

Concerning the case where $n = 3$, we assert that the ring $\mathbb{T}_3(R)$ cannot be locally invo-regular as the ring $\mathbb{T}_3(\mathbb{Z}_2)$ is not so appealing to Example 1 (2) quoted above. Indeed, there is a sequence of epimorphisms $\mathbb{T}_3(R) \to \mathbb{T}_3(R/M) \to \mathbb{T}_3(\mathbb{Z}_2)$ for some maximal ideal M of R.

"$\Leftarrow$". As (1) trivially ensures that $\mathbb{T}_n(R) \cong R$, we are concentrating on (2). What needs to be proved is that $\mathbb{T}_2(\mathbb{Z}_2)$ is locally invo-regular. To this aim, Example 1(1) assures that $\mathbb{T}_2(\mathbb{Z}_2)$ is such a ring. ◄

The next comments are worthwhile.

Remark 1. *As already noted above, in [1], Breaz and Cîmpean investigated those rings R for which either (at least one of) r or $1-r$ is a solution of the equation $x^3 = x$ whenever $r \in R$. In the commutative case, the authors obtain a complete characterization like this: R is a subring of the direct product $K_1 \times K_2 \times K_3$ such that $K_1/J(K_1) \cong \mathbb{Z}_2$, for every $z \in J(K_1) \quad : \quad z^2 = 2z$, K_2 is a Boolean ring (i.e., a subring of a direct product of copies of $\mathbb{Z}_2$), and K_3 is a subring of a direct product of copies of $\mathbb{Z}_3$.*

We shall now illustrate how some of this can be somewhat deduced from already well-known results established in [12]. Indeed, at the beginning, we assert that all indecomposable weakly tripotent rings of characteristic 3 are always isomorphic to the field $\mathbb{Z}_3$ and thus they are commutative. To see that, we state that $x = x^3$ implies that x^2 is an idempotent and so either $x^2 = 0$, which leads to $x = 0$, or $x^2 = 1$. On the other vein, as already seen above, $1 - x = (1-x)^3$ implies that $x = x^3$, which is nothing new. Now, an application of [12] is a guarantor of our initial claim. Furthermore, treating now the case where $8 = 0$ (and hence 2 is *a nilpotent – this case also includes the cases* $2 = 0$ *and* $4 = 0$*), we assert that* $U(R) = 1 + Nil(R)$*. In fact, the containment* $\supseteq$ *being obvious, we need to show the opposite one. So, given* $u \in U(R)$*, it will follow that* $u^2 = 1$ *or* $(1-u)^2 = 1$ *(as* $u^3 = u$ *or* $(1-u)^3 = 1-u$*). Thus, in the first possibility,* $(1-u)^2 = 2(1-u) \in Nil(R)$*, whence* $1-u \in Nil(R)$ *and* $u \in 1 + Nil(R)$*, as required. In the second possibility, one gets* $u^2 = 2u \in Nil(R)$*, which is inadequate. Finally, the assertion is sustained. That is why, as demonstrated above,* $x = x^3$ *implies that* x^2 *is an idempotent and* $(x-x^2)^2 = 2(-x+x^2)$ *is a nilpotent, so that* $x \in Id(R)+Nil(R)$*. In the remaining case,* $1-x = (1-x)^3$ *implies that* $(1-x)^2$ *is an idempotent and using the same trick once again by replacing* x *with* $1-x$*, we arrive at* $1-x \in Id(R)+Nil(R)$ *giving up* $x \in (1-Id(R))+Nil(R) \subseteq Id(R)+Nil(R)$*, as asked for. This means that* R *is nil-clean and, according to [11],* R *must be strongly nil-clean possessing the property* $R/N(R)$ *is Boolean, where the symbol* $N(R)$ *represents* $Nil(R)$*, calling it the nil-radical of* R*.*

We end our work with the following several questions that come to mind immediately and which remain unanswered in the text above.

The first two of them are concerned with corners and say the following:

• If the ring R is locally invo-regular (resp., locally unit-regular), does it follow that the corner subring eRe is also locally invo-regular (resp., locally unit-regular) for any $e \in Id(R)$?

• If R is a locally invo-regular (resp., a locally unit-regular) ring, is either eRe or $(1-e)R(1-e)$ an invo-regular (resp., a unit-regular) ring for every $e \in Id(R)$?

• If R is a locally invo-regular ring of characteristic 2 such that $J(R) = \{0\}$, is it then true that R is a Boolean ring?

Notice that, in view of Theorem 1, this needs to be established when R is non-abelian.

• Describe, up to an isomorphism, the structure of *locally* n*-torsion regular* rings R in the sense that, for any $r \in R$, there exists an n-torsion unit w of R for some natural $n \in \mathbb{N}$ (that is, $w^n = 1$ with $n \in \mathbb{N}$) such that either $r = rwr$ or $1-r = (1-r)w(1-r)$.

For some relevant material on that aspect, the reader can consult with [7].

Acknowledgement

The scientific work in this article is partially supported by the Bulgarian National Science Fund under Grant KP-06 No 32/1 of December 07, 2019.

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Received 20 January 2019
Accepted 21 June 2020