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Stability Conditions for Linear Difference System With Two Delays

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Abstract. In this paper, we give new necessary and sufficient conditions for the asymptotic stability of a linear delay difference system with two delays

$$x_{n+1} - ax_n + A\left(x_{n-k} + x_{n-l}\right) = 0, \quad n \in \{0, 1, 2, ...\},$$

where A is a 2×2 constant matrix, $a \in [-1, 1] - \{0\}$ is a real number and l, k are positive integers such that $1 \leq l < k$.

Key Words and Phrases: difference equations, difference equations systems, stability, asymptotic stability.

2010 Mathematics Subject Classifications: 39A06, 39A30

1. Introduction

We consider a linear delay difference system with two delays

$$x_{n+1} - ax_n + A\left(x_{n-k} + x_{n-l}\right) = 0, \quad n \in \{0, 1, 2, ...\}, \tag{1}$$

where A is a 2×2 constant matrix, $a \in [-1, 1] - \{0\}$ is a real number and l, k are positive integers such that $1 \leq l < k$. Also, suppose that the solutions of the system (1) are uniquely determined by initial values $x_{-k}, x_{-k+1}, ..., x_0 \in \mathbb{R}^2$. In this paper, we give new necessary and sufficient conditions for the asymptotic stability of the system (1).

Dynamic models are a useful tool to study in all areas of life and get a better understanding of relevant phenomena such as populations and economics. These models constitute mathematical representations for discrete processes by difference equations. Especially, stability of these difference equations has been

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extensively studied in the literature. Recently, they have been investigated by many researchers. For example, Levin and May [1] , Clark [2] , Kuruklis [3] , Matsunaga and Hara [4], Matsunaga [5] , Nagabuchi [6] , Deger and Bolat [8], Kipnis and Malygina [9], Ivanov, Kipnis and Malygina [10] and Cermák and Jánsky [11, 12] have investigated stability of linear delay difference equations. Especially, the studies of Matsunaga [5] and Nagabuchi [6] have been a source of inspiration for this study. Matsunaga [5] generalized the study of Matsunaga and Hara [4] with one delay. Nagabuchi [6] investigated the study of Matsunaga and Hara [4] for two delay.The aim of this paper is to give new results that generalize the study of Nagabuchi [6] for asymptotically stable linear delay difference system. System (1) can be written with $x_n = Py_n$ for a nonsingular matrix P as

$$y_{n+1} - ay_n + P^{-1}AP\left(y_{n-k} + y_{n-l}\right) = 0 \quad n \in \{0, 1, 2, ...\}.$$

Thus we have merely to take into account system (1), where the matrix A can be one of the following two matrices in Jordan form [7]:

$$(i) \quad A = \begin{pmatrix} q_1 & p \\ 0 & q_2 \end{pmatrix} \qquad\qquad (ii) \quad A = q\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where q, q_1, q_2, p and θ are real constants with $0 < |\theta| \leq \frac{\pi}{2}$.

2. Main Results

Theorem 1. *Assume that* $0 < |\theta| \leq \frac{\pi}{2}$, $a \in [-1, 1] - \{0\}$ *and the matrix* A *in the system* (1) *has the form* (ii). *Then system* (1) *is asymptotically stable iff*

$$0 < q < \frac{\sin\left(\dfrac{\pi - 2|\theta|}{k+l}\right)}{2\cos\left(\dfrac{(k-l)(\pi - 2|\theta|)}{2(k+l)}\right)}.$$

Theorem 2. *Assume that* $a \in [-1, 1] - \{0\}$ *and the matrix* A *in the system* (1) *has the form* (i). *Then system* (1) *is asymptotically stable iff*

$$0 < q < \frac{\sin\left(\dfrac{\pi}{k+l}\right)}{2\cos\left(\dfrac{(k-l)\pi}{2(k+l)}\right)}.$$

The proof of the theorems. The purpose of this paper is to obtain necessary and sufficient conditions for asymptotic stability of the system (1). It is required to show that all the roots of characteristic equation of system (1) lie inside unit disk D^2 [7]. For the proof, root analysis is applied to the system (1). The characteristic equation of the system (1) is

$$F(\lambda) \equiv \det\left(\left(\lambda^{k+1} - a\lambda^k\right) I + A\left(\lambda^{k-l} + 1\right)\right) = 0, \tag{2}$$

where I is a 2×2 identity matrix. We will prove our theorems for the case (ii) of the matrix A. The proof for the case (i) of the matrix A is similar. We consider the Eq. (2). Then

$$F(\lambda) =$$

$$\begin{aligned}
&= \det\begin{pmatrix} \lambda^{k+1} - a\lambda^k + q\cos\theta\left(\lambda^{k-l}+1\right) & -q\sin\theta\left(\lambda^{k-l}+1\right) \\ q\sin\theta\left(\lambda^{k-l}+1\right) & \lambda^{k+1} - a\lambda^k + q\cos\theta\left(\lambda^{k-l}+1\right) \end{pmatrix} \\
&= \left(\lambda^{k+1} - a\lambda^k + q\cos\theta\left(\lambda^{k-l}+1\right)\right)^2 + \left(q\sin\theta\left(\lambda^{k-l}+1\right)\right)^2 \\
&= \left(\lambda^{k+1} - a\lambda^k + qe^{i\theta}\left(\lambda^{k-l}+1\right)\right)\left(\lambda^{k+1} - a\lambda^k + qe^{-i\theta}\left(\lambda^{k-l}+1\right)\right)
\end{aligned}$$

If we get

$$f(\lambda) = \lambda^{k+1} - a\lambda^k + qe^{i\theta}\left(\lambda^{k-l}+1\right) = 0, \tag{3}$$

then

$$F(\lambda) = f(\lambda)\,\overline{f\left(\overline{\lambda}\right)} = 0,$$

where $\overline{\lambda}$ is the complex conjugate of λ. Consider the equation

$$\overline{f\left(\overline{\lambda}\right)} = \lambda^{k+1} - a\lambda^k + qe^{i\widetilde{\theta}}\left(\lambda^{k-l}+1\right) = 0. \tag{4}$$

If $\widetilde{\theta} = -\theta$ in (4) with $-\frac{\pi}{2} \le \widetilde{\theta} < 0$, then (3) is obtained with $0 < \theta \le \frac{\pi}{2}$. Thus, we have merely to consider the case $f(\lambda) = 0$ with $0 < \theta \le \frac{\pi}{2}$. Furthermore, system (1) is asymptotically stable iff all the roots of characteristic equation $f(\lambda) = 0$ with $0 < \theta \le \frac{\pi}{2}$ lie inside unit disk D^2. It should be noted that, for $q = 0$, the roots of (3) are 0 (multiplicity k) and a (simple), furthermore, (3) has no real root when $q \ne 0$. Now we are ready to give lemmas.

2.1. Some auxiliary lemmas

Firstly, we will calculate arguments of complex roots of the (3) on D^2 for $q \ne 0$.

Lemma 1. *Suppose that $0<\theta\leq\frac{\pi}{2}$ and $a\in[-1,1]-\{0\}$. Then the arguments of complex roots of (3) on ∂D^2 are as follows:*

$$\omega_\nu=\frac{(2\nu+1)\,\pi+2\theta}{k+l},$$

where $\nu=\left\{-\left\|\frac{k+l+1}{2}+\frac{\theta}{\pi}\right\|+1,...-1,0,1,...,\left\|\frac{k+l-1}{2}-\frac{\theta}{\pi}\right\|\right\}-\left\{-\left\|\frac{k+l+2}{4}+\frac{\theta}{\pi}\right\|,\left\|\frac{k+l-2}{4}-\frac{\theta}{\pi}\right\|\right\}$, $\|.\|$ denotes greater integer function and l,k are positive integers such that $1\leq l<k$.

Proof. Let $\lambda\in\partial D^2$. Then $\lambda=e^{i\omega}$ with $\omega\in(-\pi,\pi]$. Thus λ satisfy $f(\lambda)=\lambda^{k+1}-a\lambda^k+qe^{i\theta}\left(\lambda^{k-l}+1\right)=0$. Then

$$q=\frac{\lambda^{k+l}\,(a-\lambda)}{e^{i\theta}\,(\lambda^k+\lambda^l)}. \tag{5}$$

Since $\overline{\lambda}=\dfrac{1}{\lambda}$, we obtain conjugate of (5) as

$$q=\frac{e^{i\theta}\,(a\lambda-1)}{\lambda\,(\lambda^k+\lambda^l)}. \tag{6}$$

From (5) and (6) , we have

$$e^{2i\theta}=\lambda^{k+l+1}\frac{(\lambda-a)}{(1-a\lambda)}. \tag{7}$$

Since $a\in[-1,1]-\{0\}$, we can choose $a=\cos\omega$ for $\omega\in(-\pi,\pi]-\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$. If we write $\lambda=e^{i\omega}$ in (7), we have

$$-e^{2i\theta}=e^{i\omega(k+l+1)}e^{-i\omega}. \tag{8}$$

So we get

$$\omega\equiv\omega_\nu=\frac{(2\nu+1)\,\pi+2\theta}{k+l}. \tag{9}$$

The proof is completed. ◄

Now we will determine the value of $q\neq0$ in terms of a and the argument ω_ν when a root of (3) lies on ∂D^2.

Lemma 2. *Suppose that $0<\theta\leq\frac{\pi}{2}$ and $a\in[-1,1]-\{0\}$. If $\lambda=e^{i\omega}$ is a root of (3) on ∂D^2, then*

$$q_\nu=\frac{\sin\omega_\nu}{2\sin\left(k\omega_\nu-\theta\right)},$$

where $q=\{q_\nu\}$ with $\omega_\nu=\dfrac{(2\nu+1)\,\pi+2\theta}{k+l}$. Also, for each q_ν, the simple root $\lambda=e^{i\omega_\nu}$ is the only root on ∂D^2.

Proof. We consider (6) and (8):

$$\begin{aligned}
-q &= \frac{e^{i\theta}\left(1-a\lambda\right)}{\lambda\left(\lambda^k+\lambda^l\right)} \\
&= \frac{\left(1-a\lambda\right)}{e^{-i\theta}\lambda\left(\lambda^k+\lambda^l\right)} \\
&= \frac{\left(1-a\lambda\right)}{e^{-i\theta}\lambda^{-k+1}\left(\lambda^{2k}+\lambda^{k+l}\right)} \\
&= \frac{\left(1-a\lambda\right)}{e^{-i\theta}\lambda^{-k+1}\left(\lambda^{2k}-e^{2i\theta}\right)} \\
&= \frac{\left(1-a\lambda\right)}{\lambda\left(\lambda^k e^{-i\theta}-\lambda^{-k}e^{i\theta}\right)}.
\end{aligned}$$

Since $a=\cos\omega$ for $\omega\in(-\pi,\pi]-\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$ and $\sin\omega=\dfrac{e^{i\omega}-e^{-i\omega}}{2i}$, we have

$$-q=\frac{\dfrac{e^{i\omega}-e^{-i\omega}}{2i}}{\dfrac{2\left(e^{i(\omega k-\theta)}-e^{-i(\omega k-\theta)}\right)}{2i}}$$

or

$$q=\frac{\sin\omega}{2\sin\left(\omega k-\theta\right)}.$$

Now let us show that (3) has a simple root. It is sufficient to show that $\dfrac{\partial f}{\partial\lambda}\Big|_{\lambda=\lambda_\nu}\neq 0$. If $\dfrac{\partial f}{\partial\lambda}\Big|_{\lambda=\lambda_\nu}\neq 0$, it can be said that $f\left(\lambda\right)$ is locally homeomorphic in a neighborhood of $\lambda=\lambda_\nu$. This expression indicates that λ_ν is a simple root. From (3) we obtain

$$\begin{aligned}
\frac{\partial f}{\partial\lambda}\Big|_{\lambda=\lambda_\nu} &= \left(k+1\right)\lambda_\nu^k-ka\lambda_\nu^{k-1}+qe^{i\theta}\left(k-l\right)\lambda_\nu^{k-l-1} \\
&= \left(k+1\right)\lambda_\nu^k-ka\lambda_\nu^{k-1}-\frac{\lambda_\nu^{k+1}-a\lambda_\nu^k}{\lambda_\nu^{k-l}+1}\left(k-l\right)\lambda_\nu^{k-l-1} \\
&= \frac{\lambda_\nu^{k-1}\left[\left(l+1\right)\lambda_\nu^{k-l+1}-la\lambda_\nu^{k-l}+\left(k+1\right)\lambda_\nu-ka\right]}{\lambda_\nu^{k-l}+1}. \qquad (10)
\end{aligned}$$

Suppose that $\dfrac{\partial f}{\partial\lambda}\Big|_{\lambda=\lambda_\nu}=0$. Then from (10), we get

$$\left(l+1\right)\lambda_\nu^{k-l+1}-la\lambda_\nu^{k-l}+\left(k+1\right)\lambda_\nu-ka=0. \qquad (11)$$

Since $\overline{\lambda_\nu} = \dfrac{1}{\lambda_\nu}$, we obtain conjugate of (11) as

$$\frac{(l+1)}{\lambda_\nu^{k-l+1}} - \frac{la}{\lambda_\nu^{k-l}} + \frac{(k+1)}{\lambda_\nu} - ka = 0$$

or

$$-ka\lambda_\nu^{k-l+1} + (k+1)\,\lambda_\nu^{k-l} - la\lambda_\nu + l + 1 = 0. \tag{12}$$

Using (11) and (12) , the following equality can be derived:

$$(k+l+2)\,e^{(2\theta+\omega)i} - (k+l+2)\,e^{(2\theta-\omega)i} - (k+l+2)\,e^{i\omega} + (k+l+2)\,e^{-i\omega} = 0$$

$$\begin{aligned}(k+l+2)\left(e^{i\omega(k-l)} + 1\right)\left(e^{i\omega} - e^{-i\omega}\right) &= 0\\ (k+l+2)\left(\lambda_\nu^{k-l} + 1\right)\sin\omega &= 0.\end{aligned}$$

Since $q \neq 0$, $k+l+2 \neq 0$ and $\lambda_\nu^{k-l} + 1 \neq 0$, $\dfrac{\partial f}{\partial \lambda}\underset{\lambda=\lambda_\nu}{|} \neq 0$. Hence the proof is completed. ◀

Remark 1. *From Lemma* 1 *and Lemma* 2*, it can be seen that the value of* q_ν *is symmetric with respect to* k *and* l*. Thus we have* $q_\nu = \dfrac{\sin\omega_\nu}{2\sin(l\omega_\nu - \theta)}$.

In the next lemma, the movement of the roots of (3) will be investigated on ∂D^2 as q varies. Here, note that $\lambda\underset{q=q_\nu}{|} = \lambda_\nu(q_\nu) = \lambda_\nu$ and $\lambda\underset{q=0}{|} = \lambda_\nu(0) = a$ are the roots of $f(\lambda) = 0$ around $q = q_\nu$ and $q = 0$.

Lemma 3. *Suppose that* $0 < \theta \leq \frac{\pi}{2}$ *and* $a \in [-1,1] - \{0\}$*.* $|\lambda|$ *increases as* $|q|$ *increases in the neighborhood of* $q = q_\nu$*.*

Proof. Consider $f \in \mathbb{C}^2$ and define

$$f(q,\lambda) = \lambda^{k+1} - a\lambda^k + qe^{i\theta}\left(\lambda^{k-l} + 1\right).$$

Since λ_ν is the root of $f(\lambda) = 0$, $f(q_\nu, \lambda_\upsilon) = 0$. From the proof of Lemma 2 we know that $\dfrac{\partial f(q_\nu, \lambda_\upsilon)}{\partial \lambda}\underset{\lambda=\lambda_\nu}{|} \neq 0$. Also, by implicit function theorem, λ_ν is holomorphic in the neighborhood of $q = q_\nu$. Thus, from $f(q, \lambda_\nu) = 0$, we have

$$\frac{d\lambda_\upsilon}{dq} = -\frac{\dfrac{\partial f}{\partial q}}{\dfrac{\partial f}{\partial \lambda}}. \tag{13}$$

Also, since $\lambda_\upsilon = re^{i\omega}$, we have

$$\begin{aligned}
\frac{d\lambda_\upsilon}{dq} &= \frac{\partial\lambda_\upsilon}{\partial\operatorname{Re}(q)} = \frac{\partial\lambda_\upsilon}{\partial r}\frac{\partial r}{\partial\operatorname{Re}(q)} + \frac{\partial\lambda_\upsilon}{\partial\omega}\frac{\partial\omega}{\partial\operatorname{Re}(q)} \\
&= e^{i\omega}\frac{\partial r}{\partial\operatorname{Re}(q)} + ire^{i\omega}\frac{\partial\omega}{\partial\operatorname{Re}(q)} \\
&= \frac{re^{i\omega}}{r}\left(\frac{\partial r}{\partial\operatorname{Re}(q)} + ir\frac{\partial\omega}{\partial\operatorname{Re}(q)}\right) \\
&= \frac{\lambda_\upsilon}{r}\left(\frac{\partial r}{\partial\operatorname{Re}(q)} + ir\frac{\partial\omega}{\partial\operatorname{Re}(q)}\right).
\end{aligned}$$

Since q is limited within real values and from (13) we obtain

$$\frac{dr}{dq} = \operatorname{Re}\left(\frac{r}{\lambda_\upsilon}\frac{d\lambda_\upsilon}{dq}\right) = \operatorname{Re}\left(\frac{-\dfrac{r}{\lambda_\upsilon}\dfrac{\partial f}{\partial q}}{\dfrac{\partial f}{\partial\lambda}}\right) = \frac{\operatorname{Re}\left(-\dfrac{r}{\lambda_\upsilon}\dfrac{\partial f}{\partial q}\overline{\dfrac{\partial f}{\partial\lambda}}\right)}{\left|\dfrac{\partial f}{\partial\lambda}\right|^2}. \tag{14}$$

Since (10) and $\dfrac{\partial f}{\partial q} = -\dfrac{1}{q}\left(\lambda_\upsilon^{k+1} - a\lambda_\upsilon^k\right)$ for $q \neq 0$, we have

$$-\frac{r}{\lambda_\upsilon}\frac{\partial f}{\partial q}\overline{\frac{\partial f}{\partial\lambda}} =$$

$$\begin{aligned}
&\frac{r}{\lambda_\upsilon}\frac{1}{q}\left(\lambda_\upsilon^{k+1} - a\lambda_\upsilon^k\right)\left[\frac{1}{\lambda_\upsilon^k}\frac{\lambda_\upsilon^{k-l+1}}{\lambda_\upsilon^{k-l}+1}\left(\frac{l+1}{\lambda_\upsilon^{k-l+1}} - \frac{la}{\lambda_\upsilon^{k-l}} + \frac{k+1}{\lambda_\nu} - ka\right)\right] \\
&= \frac{r}{\lambda_\upsilon}\frac{1}{q}\lambda_\upsilon^k(\lambda_\upsilon - a)\left[\frac{1}{\lambda_\upsilon^k}\frac{1}{\lambda_\upsilon^{k-l}+1}\left((l+1) - la\lambda_\nu + \frac{k+1}{\lambda_\nu^{l-k}} - \frac{ka}{\lambda_\nu^{l-k-1}}\right)\right] \\
&= \frac{r}{\lambda_\upsilon}\frac{1}{q}(\lambda_\upsilon - a)\left[\frac{l+1}{\lambda_\upsilon^{k-l}+1} - \frac{la\lambda_\nu}{\lambda_\upsilon^{k-l}+1} + \frac{k+1}{\lambda_\nu^{l-k}+1} - \frac{ka\lambda_\nu}{\lambda_\nu^{l-k}+1}\right] \\
&= \frac{r}{q}(\lambda_\upsilon - a)\left(\frac{1}{\lambda_\upsilon} - a\right)\left[\frac{l}{\lambda_\upsilon^{k-l}+1} + \frac{k}{\lambda_\nu^{l-k}+1} + \frac{1}{1-\lambda_\nu a}\right] \\
&= \frac{r}{q}|\lambda_\upsilon - a|^2\left[\frac{l}{\lambda_\upsilon^{k-l}+1} + \frac{k}{\lambda_\nu^{l-k}+1} + \frac{1}{1-\lambda_\nu a}\right].
\end{aligned}$$

If $a = \cos\omega$ can be chosen with $\omega = \dfrac{(2\nu+1)\pi + 2\theta}{k+l}$, then

$$-\frac{r}{\lambda_\upsilon}\frac{\partial f}{\partial q}\overline{\frac{\partial f}{\partial\lambda}} = \frac{r}{q}|\lambda_\upsilon - a|^2\left[\frac{l}{e^{i\omega(k-l)}+1} + \frac{k}{e^{i\omega(l-k)}+1} + \frac{1}{1-e^{i\omega}\cos\omega}\right].$$

Taking the real part of last equality, we get

$$-\frac{r}{\lambda_\upsilon}\frac{\partial f}{\partial q}\overline{\frac{\partial f}{\partial \lambda}} = \frac{r}{q}\left|\lambda_\upsilon - a\right|^2\left(\frac{l+k}{2}+1\right). \tag{15}$$

If we substitute (15) into (14), we get

$$\frac{dr}{dq} = \frac{\left|\lambda_\upsilon - a\right|^2\left(\frac{l+k}{2}+1\right)r}{q\left|\frac{\partial f}{\partial \lambda}\right|^2} \quad \text{for } q \neq 0. \tag{16}$$

We notice that $\lambda_\upsilon \neq a$. From (16), the proof is completed. ◀

Lemma 4. *In the neighborhood of $q = 0$, the following inequalities are true:*
(I) If $0 < \theta < \frac{\pi}{2}$ and $a \in (0,1]$, then $sgn\left(|\lambda| - a\right).sgn\left(q\right) < 0$ for $q \neq 0$.
(II) If $\theta = \frac{\pi}{2}$ and $a \in (-1,1] - \{0\}$, then $|\lambda| > a$ for $q \neq 0$.

Proof. (I) Using (13) we get $\frac{d\lambda_\upsilon}{dq}\Big|_{q=0} = -\dfrac{\frac{\partial f}{\partial q}}{\frac{\partial f}{\partial \lambda}}\Big|_{(q,\lambda)=(0,a)} = -\dfrac{e^{i\theta}\left(a^{k-l}+1\right)}{a^k}$.

Since

$$\begin{aligned}\frac{dr}{dq} &= \operatorname{Re}\left(\frac{r}{\lambda_\upsilon}\frac{d\lambda_\upsilon}{dq}\right)\\ &= \operatorname{Re}\left(-\frac{e^{i\theta}\left(a^{k-l}+1\right)}{a^{k+1}}\right)\\ &= -\frac{\cos\theta\left(a^{k-l}+1\right)}{a^{k+1}},\end{aligned}$$

we have

$$\frac{dr}{dq} < 0,$$

for $0 < \theta < \frac{\pi}{2}$ and $a \in (0,1]$. On the other hand, since $\lambda_\nu\left(q_\nu\right) = \lambda_\nu$ and $\lambda_\nu\left(0\right) = a$,

$$\begin{aligned}\frac{dr}{dq} &= \lim_{q\to 0}\frac{\left|\lambda_\nu\left(q\right)\right| - \left|\lambda_\nu\left(0\right)\right|}{q-0}\\ &= \lim_{q\to 0}\frac{\left|\lambda_\nu\left(q\right)\right| - a}{q} < 0.\end{aligned}$$

Thus (I) holds for $q \neq 0$.

(II) In the case of $\theta = \frac{\pi}{2}$, since $\frac{dr}{dq}\underset{q=0}{|} = 0$, the sign of $\frac{d^2r}{dq^2}\underset{q=0}{|}$ should be considered. Since $f(q, \lambda_\nu(q)) = 0$, we obtain

$$\frac{\partial^2 f}{\partial q^2} + 2\frac{\partial^2 f}{\partial q \partial \lambda_\nu}\frac{d\lambda_\nu}{dq} + \frac{\partial^2 f}{\partial \lambda_\nu^2}\left(\frac{d\lambda_\nu}{dq}\right)^2 + \frac{\partial f}{\partial \lambda_\nu}\frac{d^2\lambda_\nu}{dq^2} = 0.$$

From here, we have

$$\frac{d^2\lambda_\nu}{dq^2} = \frac{2\left(a^{k-l}+1\right)\left(la^{k-l}+k\right)}{a^{2k+1}},$$

thus

$$\frac{1}{a}\frac{d^2\lambda_\nu}{dq^2} > 0.$$

Since $\lambda_\nu = re^{i\omega}$,

$$\frac{\partial^2\lambda_\nu}{\partial q^2} = \frac{\lambda_\nu}{r}\left(2i\frac{dr}{dq}\frac{d\omega}{dq} - r\left(\frac{d\omega}{dq}\right)^2 + \frac{d^2r}{dq^2} + ir\frac{d^2\omega}{dq^2}\right).$$

By restrictions on the real values of q, we conclude

$$\frac{d^2\lambda_\nu}{dq^2} = \frac{\lambda_\nu}{r}\left(-r\left(\frac{d\omega}{dq}\right)^2 + \frac{d^2r}{d\,\mathrm{Re}\,(q)^2}\right).$$

Via polar representation, we have

$$\frac{d^2r}{dq^2} = \mathrm{Re}\left(\frac{r}{\lambda_\nu}\frac{d^2\lambda_\nu}{dq^2}\right) + r\left(\frac{d\omega}{dq}\right)^2 \geq \mathrm{Re}\left(\frac{r}{\lambda_\nu}\frac{d^2\lambda_\nu}{dq^2}\right).$$

Especially,

$$\frac{d^2r}{dq^2}\,|_{q=0} \geq \mathrm{Re}\left(\frac{r}{\lambda_\nu}\frac{d^2\lambda_\nu}{dq^2}\right) \geq \frac{2\left(a^{k-l}+1\right)\left(la^{k-l}+k\right)}{a^{2k+2}} > 0.$$

So, the assertion of this lemma holds for $q \neq 0$. ◄

Now, using Lemma 3 and Lemma 4, we can debate the asymptotic stability of the system (1).

It follows immediately from Lemma 4 (I);

If q increases away from 0, the root $\lambda(0) = a$ on ∂D^2 lies inside D^2. Thus $q > 0$ is a necessary condition for the asymptotic stability of the system (1).

Consider $q > 0$;

From Lemma 4 (I) it follows that when $0 < \theta < \frac{\pi}{2}$, all the roots of $f(\lambda) = 0$ lie inside D^2 in the neighborhood of $q = 0$. From Lemma 3 and Lemma 4 (II) it follows that when $\theta = \frac{\pi}{2}$ and the roots of $f(\lambda) = 0$ depend continuously on q and if q increases away from 0, then the root $\lambda(0) = a$ may overflow D^2, thus $f(\lambda) = 0$ might have at least one root in $\mathbb{C} - D^2$ as long as $q > 0$.

So all the roots of $f(\lambda) = 0$ are inside D^2 iff $q^* > q > 0$ and $0 < \theta < \frac{\pi}{2}$ hold, where $q^* = \min\{q_\nu : q_\nu > 0\}$. Hence $q^* > q > 0$ is also a sufficient condition.

Lemma 5. *The system* (1) *is asymptotically stable iff* $0 < q < q^*$, *where*

$$q^* = \begin{cases} q_{-1}, & 0 < \theta \leq \frac{\pi}{2}, \\ q_0, & -\frac{\pi}{2} \leq \theta < 0, \end{cases}$$

$$q_{-1} = \frac{\sin\left(\dfrac{\pi - 2\theta}{k+l}\right)}{2\cos\left(\dfrac{(k-l)(\pi - 2\theta)}{2(k+l)}\right)} \text{ and } q_0 = \frac{\sin\left(\dfrac{\pi + 2\theta}{k+l}\right)}{2\cos\left(\dfrac{(k-l)(\pi + 2\theta)}{2(k+l)}\right)}.$$

Hence we can write

$$q^* = \frac{\sin\left(\dfrac{\pi - 2|\theta|}{k+l}\right)}{2\cos\left(\dfrac{(k-l)(\pi - 2|\theta|)}{2(k+l)}\right)}. \tag{17}$$

We notice that $0 < |\theta| < \frac{\pi}{2}$ *is true for* $q^* > 0$, *so necessary and sufficient condition is only reduced to* $0 < q < q^*$.

Proof. Both q_{-1} and q_0 have similar proofs, thus the proof will be provided for q_{-1}. It is enough to show that the inequality $q_\nu \geq q_{-1}$ is satisfied for every $q_\nu > 0$, namely:

$$\frac{\sin\omega_\nu}{2\sin(k\omega_\nu - \theta)} \geq \frac{\sin\omega_{-1}}{2\sin(k\omega_{-1} - \theta)} \quad \text{for } 0 < \theta \leq \frac{\pi}{2}. \tag{18}$$

Since $\omega_{-1} = 0$, (17) is obvious for $\theta = \dfrac{\pi}{2}$. Thus, we will provide the proof for $0 < \theta < \dfrac{\pi}{2}$. Furthermore, the following cases are considered for the proof:

(i) $0 < \omega_\nu \leq \dfrac{\pi}{2}$ for $\nu \geq 0$. Then, the following equalities hold:

$$\begin{aligned} k\omega_\nu - \theta &= (2\nu + 1)\pi - \left(\frac{\xi\eta}{\varphi} - \theta\right) \\ k\omega_{-1} - \theta &= -\pi + \eta + \theta, \end{aligned}$$

where $\varphi_\theta \equiv \varphi = -\omega_{-1} = \dfrac{\pi - 2\theta}{M}$,$\varphi_0 = \dfrac{\pi}{M}$, $\eta = (M-k)\varphi$ and $\xi = \omega_\nu = \dfrac{(2\nu+1)\pi + 2\theta}{M}$ as $k+l = M$. So, (17) can be rewritten as

$$\frac{\sin\xi}{2\sin\left(\dfrac{\xi\eta}{\varphi} - \theta\right)} \geq \frac{\sin\varphi}{2\sin(\eta+\theta)}.$$

Then we have

$$\sin\xi\sin(\eta+\theta) \geq \sin\varphi\sin\left(\frac{\xi\eta}{\varphi} - \theta\right). \tag{19}$$

Since $1 \leq l < k$,

$$\begin{aligned} M &\leq 2k \leq 2M-2 \\ \varphi &\leq (M-k)\varphi \leq \frac{M\varphi}{2} \end{aligned}$$

$$\varphi \leq \eta \leq \frac{\pi}{2} - \theta \tag{20}$$

and

$$\frac{\pi+2\theta}{M} \leq \omega_\nu \leq \frac{\pi}{2}$$

$$\frac{2\theta}{M} + \varphi_0 \leq \xi \leq \frac{\pi}{2}. \tag{21}$$

We take $H(\xi,\eta) = \sin\xi\sin(\eta+\theta) - \sin\varphi\sin\left(\dfrac{\xi\eta}{\varphi} - \theta\right)$. Thus the proof of (18), on the rectangular region

$$D := \left\{(\xi,\eta) \in \mathbb{R}^2 : \frac{2\theta}{M} + \varphi_0 \leq \xi \leq \frac{\pi}{2},\ \varphi \leq \eta \leq \frac{\pi}{2} - \theta\right\},$$

is reduced to inequality

$$H(\xi,\eta) \geq 0. \tag{22}$$

For the proof of (21), investigations on D and ∂D will be conducted, respectively.

Step 1 : $H(\xi,\eta) \geq 0$ on $Cr(H) := \{(\xi,\eta) \in D : dH(\xi,\eta) = 0\}$. From $dH(\xi,\eta) = 0$, we get

$$\begin{aligned} \frac{\partial H}{\partial \xi} &= \cos\xi\sin(\eta+\theta) - \sin\varphi\cos\left(\frac{\xi\eta}{\varphi} - \theta\right)\frac{\eta}{\varphi} = 0 \\ \frac{\partial H}{\partial \eta} &= \sin\xi\cos(\eta+\theta) - \sin\varphi\cos\left(\frac{\xi\eta}{\varphi} - \theta\right)\frac{\xi}{\varphi} = 0. \end{aligned}$$

From the above two equalities, we have $\dfrac{\xi}{\tan\xi} = \dfrac{\eta}{\tan(\eta+\theta)}$ on $Cr(H)$. Hence ξ can be expressed as a function Φ of η. Then $\xi = \Phi(\eta)$ can be written for $\varphi \leq \eta \leq \frac{\pi}{2} - \theta$. Since the function $\dfrac{u}{\tan u}$ is monotonically decreasing for $0 \leq u \leq \dfrac{\pi}{2}$ and $\dfrac{\xi}{\tan\xi} < \dfrac{\eta+\theta}{\tan(\eta+\theta)}$, $\Phi(\eta) \geq \eta+\theta$ is obtained for $\varphi \leq \eta \leq \frac{\pi}{2} - \theta$. Consider (21) again. Then

$$\begin{aligned} H(\Phi(\eta),\eta) &= \sin\Phi(\eta)\sin(\eta+\theta) - \sin\varphi\sin\left(\frac{\Phi(\eta)\eta}{\varphi} - \theta\right), \\ \frac{H(\Phi(\eta),\eta)}{\Phi(\eta)(\eta+\theta)} &= \frac{\sin\Phi(\eta)\sin(\eta+\theta)}{\Phi(\eta)(\eta+\theta)} - \sin\varphi\frac{\sin\left(\frac{\Phi(\eta)\eta}{\varphi} - \theta\right)}{\Phi(\eta)(\eta+\theta)}. \end{aligned}$$

By $\dfrac{\Phi(\eta)}{\varphi} > 1$, we have

$$\frac{H(\Phi(\eta),\eta)}{\Phi(\eta)(\eta+\theta)} \geq \frac{\sin\Phi(\eta)}{\Phi(\eta)}\frac{\sin(\eta+\theta)}{(\eta+\theta)} - \frac{\sin\varphi}{\varphi}\frac{\sin\left(\frac{\Phi(\eta)\eta}{\varphi} - \theta\right)}{\frac{\Phi(\eta)\eta}{\varphi} - \theta}.$$

Since $\dfrac{\sin u}{u}$ is monotonically decreasing for $0 \leq u \leq \pi$,

$$\frac{\sin(\eta+\theta)}{(\eta+\theta)} \geq \frac{\sin\Phi(\eta)}{\Phi(\eta)} \quad \text{for } \Phi(\eta) \geq \eta+\theta. \tag{23}$$

From (22), we obtain

$$\left(\frac{\sin\Phi(\eta)}{\Phi(\eta)}\right)^2 \geq \frac{\sin\varphi}{\varphi}\frac{\sin\left(\frac{\Phi(\eta)\eta}{\varphi} - \theta\right)}{\frac{\Phi(\eta)\eta}{\varphi} - \theta} \quad \text{for } \varphi \leq \eta \leq \frac{\pi}{2} - \theta\ . \tag{24}$$

Thus (21) is reduced to (23). The proof of (23) will be made as follows. Since $\dfrac{\Phi(\eta)\eta}{\varphi} - \theta \geq \Phi(\eta) - \theta > \varphi > 0$, the following three cases are available:

case (1) $\quad 0 < \dfrac{\Phi(\eta)\eta}{\varphi} - \theta \leq \pi$: Since the function $\dfrac{\sin u}{u}$ is monotonically decreasing for $0 \leq u \leq \pi$, we can write

$$\left(\frac{\sin\Phi(\eta)}{\Phi(\eta)}\right)^2 \geq$$

$$\geq \frac{\sin\varphi}{\varphi}\frac{\sin\left(\dfrac{\Phi(\eta)\eta}{\varphi}-\theta\right)}{\dfrac{\Phi(\eta)\eta}{\varphi}-\theta} \geq \left(\frac{\sin\left(\dfrac{\Phi(\eta)\eta}{\varphi}-\theta\right)}{\dfrac{\Phi(\eta)\eta}{\varphi}-\theta}\right)^2 \quad \text{for } \varphi \leq \eta \leq \frac{\pi}{2}-\theta\ . \tag{25}$$

Consider $\dfrac{\Phi(\eta)\eta}{\varphi}-\theta-\Phi(\eta)$. Then we obtain $\Phi(\eta)\left(\dfrac{\eta}{\varphi}-1\right)-\theta>0$. Thus we get

$$\Phi(\eta) < \frac{\Phi(\eta)\eta}{\varphi}-\theta. \tag{26}$$

Since the function $\dfrac{\sin u}{u}$ is monotonically decreasing for $0 \leq u \leq \pi$, by (25) , the proof is completed.

case (2) $\pi < \dfrac{\Phi(\eta)\eta}{\varphi}-\theta \leq 2\pi$: It is obvious that $\sin\left(\dfrac{\Phi(\eta)\eta}{\varphi}-\theta\right) \leq 0$.

case (3) $\dfrac{\Phi(\eta)\eta}{\varphi}-\theta > 2\pi$:

$$\left(\frac{\sin\Phi(\eta)}{\Phi(\eta)}\right)^2 - \frac{\sin\varphi}{\varphi}\frac{\sin\left(\dfrac{\Phi(\eta)\eta}{\varphi}-\theta\right)}{\dfrac{\Phi(\eta)\eta}{\varphi}-\theta} \geq \frac{4}{\pi^2}-\frac{1}{\pi^2}>0 \quad \text{for } \varphi \leq \eta \leq \frac{\pi}{2}-\theta\ .$$

Thus, the proof of Step 1 is completed.

Step 2 : $H(\xi,\eta) \geq 0$ on ∂D; For this, investigations at $\eta=\dfrac{\pi}{2}-\theta$, $\xi=\dfrac{\pi}{2}$, $\eta=\varphi$ and $\xi=\dfrac{2\theta}{M}+\varphi_0$ on ∂D are made one by one.

As $\eta=\dfrac{\pi}{2}-\theta$ on ∂D,

$$\begin{aligned} H\left(\xi,\frac{\pi}{2}-\theta\right) &= \sin\xi\sin\left(\frac{\pi}{2}-\theta+\theta\right)-\sin\varphi\sin\left(\frac{\xi\left(\frac{\pi}{2}-\theta\right)}{\varphi}-\theta\right) \\ &\geq \sin\xi-\sin\varphi>0. \end{aligned}$$

As $\xi=\dfrac{\pi}{2}$ on ∂D,

$$\begin{aligned} H\left(\frac{\pi}{2},\eta\right) &= \sin\frac{\pi}{2}\sin(\eta+\theta)-\sin\varphi\sin\left(\frac{\pi\eta}{2\varphi}-\theta\right) \\ &\geq \sin(\eta+\theta)+\sin\varphi>0. \end{aligned}$$

If we investigate the case $\eta = \varphi$ on ∂D, then

$$\begin{aligned} H(\xi,\varphi) &= \sin\xi\sin(\varphi+\theta) - \sin\varphi\sin\left(\frac{\xi\varphi}{\varphi} - \theta\right) \\ &= \sin\xi\sin(\varphi+\theta) - \sin\varphi\sin(\xi-\theta) \\ &= \sin\xi\cos\varphi\sin\theta + \sin\varphi\sin\theta\cos\xi > 0. \end{aligned}$$

As $\xi = \dfrac{2\theta}{M} + \varphi_0$ on ∂D,

$$H\left(\frac{2\theta}{M}+\varphi_0,\eta\right) = \sin\left(\frac{2\theta}{M}+\varphi_0\right)\sin(\eta+\theta) - \sin\varphi\sin\left(\frac{\left(\frac{2\theta}{M}+\varphi_0\right)\eta}{\varphi} - \theta\right).$$

We need to examine

$$\begin{aligned} \frac{\left(\frac{2\theta}{M}+\varphi_0\right)\eta}{\varphi} - \theta - (\eta+\theta) &= 2\theta\left(\frac{2\eta}{M\varphi} - 1\right) \\ &= \frac{2\theta}{M\varphi}(2\eta - M\varphi) \\ &\leq \frac{2\theta}{M\varphi}\left(2\left(\frac{\pi}{2}-\theta\right) - M\varphi\right) = 0. \end{aligned}$$

Thus, we have

$$\frac{\left(\frac{2\theta}{M}+\varphi_0\right)\eta}{\varphi} - \theta \leq \eta + \theta \leq \frac{\pi}{2}.$$

Hence, we can write

$$\sin\left(\frac{\left(\frac{2\theta}{M}+\varphi_0\right)\eta}{\varphi} - \theta\right) \leq \sin(\eta+\theta).$$

Since $\varphi = \dfrac{\pi - 2\theta}{M}$ and $\varphi_0 = \dfrac{\pi}{M}$,

$$\varphi < \frac{2\theta}{M} + \varphi_0 \leq \frac{\pi}{2}.$$

Hence, $H(\xi,\eta) \geq 0$ on ∂D as $\xi = \dfrac{2\theta}{M} + \varphi_0$.

(ii) $-\dfrac{\pi}{2} \leq \omega_\nu < 0$ for $\nu < 0$. We know that $q_\nu = \dfrac{\sin\omega_\nu}{2\sin(\omega_\nu l - \theta)}$ with $\omega_\nu = \dfrac{(2\nu+1)\pi + 2\theta}{k+l}$. Also we get

$$\frac{2\theta-\pi}{2l} \leq \frac{2\theta-\pi}{M} = \omega_{-1}.$$

We define $h(u) = \dfrac{\sin u}{2\sin(lu-\theta)}$ which is monotonically decreasing for $u \in \left(\dfrac{2\theta-\pi}{2l}, 0\right)$. We obtain

$$h\left(\frac{2\theta-\pi}{2l}\right) \geq h(\omega_{-1}). \tag{27}$$

Firstly, we consider $\omega_\nu \in \left(\dfrac{2\theta-\pi}{2l}, 0\right)$. Then

$$\omega_\nu < \omega_{-1},$$

$$q_\nu = h(\omega_\nu) \geq h(\omega_{-1}) = q_{-1}.$$

Secondly, we consider $\omega_\nu \in \left[-\dfrac{\pi}{2}, \dfrac{2\theta-\pi}{2l}\right]$. Besides, for

$$\omega_\nu \in \left(\left(\frac{2\theta-(2m+2)\pi}{2l}\right), \left(\frac{2\theta-(2m+1)\pi}{2l}\right)\right)$$

with $m = 1, 2, \ldots$, using $\omega_\nu \in \left[-\dfrac{\pi}{2}, \dfrac{2\theta-\pi}{2l}\right]$ and (26), we have

$$\begin{aligned} q_\nu &= \frac{|\sin\omega_\nu|}{2|\sin(l\omega_\nu-\theta)|} \\ &\geq \frac{\left|\sin\left(\frac{2\theta-\pi}{2l}\right)\right|}{2} \\ &= h\left(\frac{2\theta-\pi}{2l}\right) \\ &\geq h(\omega_{-1}) = q_{-1} \end{aligned}$$

Thus, the proof is completed. ◄

Hence, the proof of Theorem 1 is completed with the proof of Lemma 5.

3. A higher dimensional linear delay difference system with two delays

Finally, a higher dimensional linear delay difference system with two delays is considered:

$$x_{n+1} - ax_n + A\left(x_{n-k} + x_{n-l}\right) = 0, \tag{28}$$

where A is a $d \times d$ constant matrix, $a \in [-1,1] - \{0\}$ and l, k are positive integers such that $1 \leq l < k$.

Theorem 3. *Let $q_j e^{i\theta_j}$ $(j = 1, 2, ..., d)$ be the eigenvalues of A. Then the system* (28) *is asymptotically stable iff*

$$0 < q < \frac{\sin\left(\dfrac{\pi - 2\left|\theta_j\right|}{k+l}\right)}{2\cos\left(\dfrac{(k-l)\left(\pi - 2\left|\theta_j\right|\right)}{2(k+l)}\right)} \quad j = 1, 2, ..., d,$$

where q_j, θ_j are real numbers and $\left|\theta_j\right| \leq \dfrac{\pi}{2}$.

Proof. Since $q_j e^{i\theta_j}$ $(j = 1, 2, ..., d)$ are the eigenvalues of A, the characteristic equation of the system (28) is given by

$$f(\lambda) = \prod_{j=1}^{d}\left(\lambda^{k+1} - a\lambda^{k} + q_j e^{i\theta_j}\left(\lambda^{k-l} + 1\right)\right) = 0.$$

Thus, Theorem 3 can be seen as a result of Theorem 1 and Theorem 2. ◀

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