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Absolute and Uniform Convergence of the Spectral Expansion in the Eigenfunctions of an Odd Order Differential Operator

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Abstract. We study the absolute and uniform convergence of spectral expansions of functions of the class $W_p^1(G),\ p>1,\ G=(0,1)$, in the eigenfunctions of an ordinary differential operator of odd order with integrable coefficients. Sufficient conditions for absolute and uniform convergence are obtained and the rate of uniform convergence of these expansions on the interval $\overline{G}=[0,1]$ is found.

Key Words and Phrases: absolute convergence, uniform convergence, eigenfunction, spectral expansion.

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1. Statement of results

On the interval $G=(0,1)$, consider the odd order differential operator

$$Lu=u^{(n)}+P_1(x)u^{(n-1)}+...+P_n(x)u,$$

where $n=2m+1,\ m=1,2,...,P_1(x)\in L_2(G),\ P_l(x)\in L_1(G),\ l=\overline{2,2m+1}$.

By $D_{2m+1}(G)$ we denote the class of functions absolutely continuous together with their derivatives of order $\leq 2m$ on the interval $\overline{G}=[0,1]$.

An eigenfunction of L corresponding to an eigenvalue λ is understood as a function $y(x)\in D_{2m+1}(G)$ that is not identically zero and satisfies the equation $Ly+\lambda y=0$ almost everywhere in G (see [1]).

Let $\{u_k(x)\}_{k=1}$ be a complete orthonormal system in $L_2(G)$ consisting of eigenfunctions of the operator L, and let $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues, $Re\lambda_k=0,\ k=1,2,....$ Parallel with the spectral parameter λ_k, we consider a parameter μ_k:

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$$\mu_k = \begin{cases} (-i\lambda_k)^{1/(2m+1)} & \text{for} \quad Im\lambda_k \geq 0 \\ (i\lambda_k)^{1/(2m+1)} & \text{for} \quad Im\lambda_k < 0. \end{cases}$$

We say that a function $f(x)$ belongs to $W_p^1(G)$, $1 \leq p \leq \infty$, if $f(x)$ is absolutely continuous on $\overline{G}$ and $f'(x)$ belongs to $L_p(G)$. The norm of function $f(x) \in W_p^1(G)$ is given by the equality

$$\|f\|_{W_p^1(G)} = \|f\|_p + \left\|f'\right\|_p ,$$

where $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$.

We now introduce a partial sum of the spectral expansion of the function $f(x) \in W_p^1(G)$, $p > 1$, with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x, f) = \sum_{\mu_k \leq \nu} f_k u_k(x), \quad \nu > 0,$$

where $f_k = (f, u_k) = \int\limits_G f(x)\overline{u_k(x)}dx$, and the difference $R_\nu(x, f) = f(x) - \sigma_\nu(x, f)$.

In this paper, we prove the following results.

Theorem 1. *Assume that* $P_1(x) \equiv 0$, $P_l(x) \in L_1(G)$, $l = \overline{2, 2m+1}$*; a function* $f(x)$ *of the class* $W_p^1(G)$*, where* $p > 1$*, and a system* $\{u_k(x)\}_{k=1}^{\infty}$ *satisfy the condition*

$$\left| f(x)\overline{u^{(2m)}(x)}\,\Big|_0^1 \right| \leq C_1(f)\mu_k^{\alpha} \|u_k\|_{\infty} , \quad 0 \leq \alpha < 2m, \quad \mu_k \geq 1, \tag{1}$$

where $C_1(f)$ *is a constant depending on* $f(x)$.

Then the spectral expansion of the function $f(x)$ *with respect to the system* $\{u_k(x)\}_{k=1}^{\infty}$ *converges absolutely and uniformly on the interval* $\overline{G} = [0, 1]$ *and the following estimate holds:*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const \left\{ C_1(f)\nu^{\alpha-2m} + \nu^{-\beta} \left\|f'\right\|_p + \right.$$

$$\left. +\nu^{-1} \left(\|f\|_{\infty} + \left\|f'\right\|_p \right) \sum_{l=2}^{2m+1} \nu^{2-l} \|P_l\|_1 \right\}, \tag{2}$$

where $\nu \geq 2$, $p^{-1} + q^{-1} = 1$, $\beta = \min\left\{2^{-1}, q^{-1}\right\}$*; and the constant const is independent of function* $f(x)$.

Corollary 1. *If the constant $C_1(f)$ in Theorem 1 is zero or $0 \leq \alpha < 2m-1-\beta$, $\beta = \min\left\{2^{-1}, q^{-1}\right\}$, then the following estimate holds:*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} = o\left(\nu^{-\beta}\right), \ \nu \to +\infty. \tag{3}$$

Corollary 2. *If the function $f(x) \in W_p^1(G)$, $p > 1$, in Theorem 1 satisfies the relations $f(0) = f(1) = 0$, then condition (1) is necessarily satisfied and the following estimate holds:*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const \nu^{-\beta} \|f\|_{W_p^1(G)}, \quad \nu \geq 2, \tag{4}$$

where the constant const is independent of function $f(x)$.

Theorem 2. *Assume that $P_1(x) \in L_2(G)$, $P_l(x) \in L_1(G), l = \overline{2, 2m+1}$; a function $f(x)$ of the class $W_2^1(G)$ and a system $\{u_k(x)\}_{k=1}^\infty$ satisfy the condition (1). Then the spectral expansion of the function $f(x)$ with respect to the system $\{u_k(x)\}_{k=1}^\infty$ converges absolutely and uniformly on the interval $\overline{G} = [0,1]$, and the following estimate holds:*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const \Big\{ C_1(f)\nu^{\alpha-2m} + \nu^{-\frac{1}{2}} \times$$
$$\times \left(\|P_1 f\|_2 + \|f'\|_2\right) + \nu^{-1}\|f\|_\infty \sum_{l=2}^{2m+1} \nu^{2-l}\|P_l\|_1 \Big\}, \ \nu \geq 2, \tag{5}$$

where the constant const is independent of $f(x)$.

Corollary 3. *If the function $f(x)$ in Theorem 2 satisfies condition $f(0) = f(1) = 0$, then condition (1) is necessarily satisfied and the following estimate is true:*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const \nu^{-\frac{1}{2}} \left(\|P_1 f\|_2 + \|f'\|_2\right), \ \nu \geq 2, \tag{6}$$

where the constant const is independent of $f(x)$.

Corollary 4. *If $C_1(f) = 0$ or $0 \leq \alpha < 2m - \frac{1}{2}$, then the following estimate is true:*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} = o\left(\nu^{-\frac{1}{2}}\right), \quad \nu \to +\infty, \tag{7}$$

where the symbol "o" depends on $f(x)$.

Theorem 3. *Suppose that $P_1(x) \in L_2(G)$, $P_l(x) \in L_1(G)$, $l = \overline{2, 2m+1}$, $f(x) \in W_p^1(G)$, $1 < p < 2$, condition (1) is satisfied, and the system $\{u_k(x)\}_{k=1}^\infty$ is uniformly bounded. Then the spectral expansion of the function $f(x)$ with respect* *to the system* $\{u_k(x)\}_{k=1}^{\infty}$ *converges absolutely and uniformly on the interval* $\overline{G} = [0,1]$*, and the following estimate holds:*

$$\|R_\nu(\cdot,f)\|_{C[0,1]} \le const\Big\{C_1(f)\nu^{\alpha-2m} + \nu^{-\frac{1}{2}}\|P_1 f\|_2 +$$

$$+\nu^{-1/q}\left\|f'\right\|_p + \nu^{-1}\|f\|_\infty \sum_{l=2}^{2m+1}\nu^{2-l}\|P_l\|_1\Big\},\ \nu \ge 2, \qquad (8)$$

where $p^{-1}+q^{-1}=1$ *and the constant const is independent of* $f(x)$*.*

Corollary 5. *If the constant* $C_1(f)$ *in Theorem 3 is zero or* $0 \le \alpha < 2m - 1/q$*, then*

$$\|R_\nu(\cdot,f)\|_{C[0,1]} = o\left(\nu^{-1/q}\right),\ \nu \to +\infty, \qquad (9)$$

where the symbol "o" depends on function $f(x)$*.*

Corollary 6. *If the function* $f(x)$ *in Theorem 3 satisfies the relations* $f(0) = f(1) = 0$*, then condition (1) is necessarily satisfied and the following estimate holds:*

$$\|R_\nu(\cdot,f)\|_{C[0,1]} \le const\left\{\nu^{-1/2}\|P_1 f\|_2 + \nu^{-1/q}\left\|f'\right\|_p\right\},\quad \nu \ge 2, \qquad (10)$$

where the constant const is independent of function $f(x)$*.*

Similar results for the even order operators were obtained in [2-8], for the third order operator in [9-10], for the arbitrary odd order operators in the case $f(x) \in W_1^1(G)$ (under some additional conditions) in [11-12]. Uniform convergence rate was studied in [13].

Recall that the uniform convergence of the spectral expansion of a function $f(x)$ from the domain of definition of a self-adjoint differential operator has been considered in the monograph [14] (chap III, sec. 9).

2. Some auxiliary lemmas

To prove our results, we must estimate the Fourier coefficients f_k of the function $f(x) \in W_p^1(G)$. To this end, we use representation of the eigenfunction $u_k(x)$. Let as introduce

$$X_j^{\pm} \equiv X_{jk}^{\pm}(0) = \frac{(i)^{n+1}}{n\mu^{n-1}}\sum_{r=0}^{n-1}(\pm i\mu_k)^r\,\omega_j^{r+1}u_k^{(n-1-r)}(0);$$

$$M\left(\xi,u_k\right)=\frac{(i)^{n-1}}{n\mu_k^{n-1}}\sum_{l=1}^{n}P_l(\xi)u_k^{(n-l)}(\xi),\quad i=\sqrt{-1},\quad n=2m+1,$$

where the numbers $\omega_j,\ j=\overline{1,n}$, are distinct roots of the number $(-1)^n$ of n-th degree.

Lemma 1. *(see [11,12]). If $\lambda_k\neq 0$, then the following representation is valid for the eigenfunction $u_k(x)$:*

$$\mu_k^{-l}u_k^{(l)}(t)=\sum_{Im\omega_j\leq 0}(-i\omega_j)^l X_{jk}^{-}(0)e^{-i\omega_j\mu_k t}+\sum_{Im\omega_j>0}(-i\omega_j)^l B_{jk}^{-}(0)\times$$

$$\times e^{i\omega_j\mu_k(1-t)}+\sum_{Im\omega_j\leq 0}(-i)^l\omega_j^{l+1}\int\limits_0^t M(\xi,u_k)e^{i\omega_j\mu_k(\xi-t)}d\xi-$$

$$-\sum_{Im\omega_j>0}(-i)^l\omega_j^{l+1}\int\limits_t^1 M(\xi,u_k)e^{i\omega_j\mu_k(\xi-t)}d\xi,\quad l=\overline{0,n-1},\tag{11}$$

if $n=4q-1,\ Im\lambda_k>0$ or $n=4q+1,\ Im\lambda_k<0$;

$$\mu_k^{-l}u_k^{(l)}(t)=\sum_{Im\omega_j\geq 0}(i\omega_j)^l X_{jk}^{+}(0)e^{i\omega_j\mu_k t}+\sum_{Im\omega_j<0}(i\omega_j)^l B_{jk}^{+}(0)\times$$

$$\times e^{-i\omega_j\mu_k(1-t)}+\sum_{Im\omega_j\geq 0}(i)^l\omega_j^{l+1}\int\limits_0^t M(\xi,u_k)e^{-i\omega_j\mu_k(\xi-t)}d\xi-$$

$$-\sum_{Im\omega_j<0}(i)^l\omega_j^{l+1}\int\limits_t^1 M(\xi,u_k)e^{-i\omega_j\mu_k(\xi-t)}d\xi,\quad l=\overline{0,n-1},\tag{12}$$

if $n=4q-1,\ Im\lambda_k<0$ or $n=4q+1,\ Im\lambda_k>0$. In these relations

$$B_{jk}^{+}(0)=X_{jk}^{+}(0)e^{i\omega_j\mu_k}+\omega_j\int\limits_0^1 M(\xi,u_k)e^{-i\omega_j\mu_k(\xi-1)}d\xi,$$

$$B_{jk}^{-}(0)=X_{jk}^{-}(0)e^{i\omega_j\mu_k}+\omega_j\int\limits_0^1 M(\xi,u_k)e^{i\omega_j\mu_k(\xi-1)}d\xi.$$

For the coefficients $X^{\pm}_{jk}(0)$ and $B^{\pm}_{jk}(0)$ the following estimates are true (see [15], formulas (42)-(45)):

$$\left|X^{\pm}_{jk}(0)\right| \le const\,\|u_k\|_2 \le const, \text{ if } Im\omega_j = 0; \tag{13}$$

$$\left|X^{\pm}_{jk}(0)\right| \le const\,\|u_k\|_\infty, \text{ if } Im\omega_j \ne 0; \tag{14}$$

$$\left|B^{\pm}_{jk}(0)\right| \le const\,\|u_k\|_\infty. \tag{15}$$

Lemma 2. *Suppose that the function $f(x) \in W^1_p(G)$, $p > 1$, and the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy condition (1). Then the Fourier coefficients f_k satisfy the inequalities* ($\mu_k \ge 1$)

$$|f_k| \le const\left\{C_1(f)\mu_k^{\alpha-2m-1}\|u_k\|_\infty + \mu_k^{-1}\left|\left(\overline{P_1}f, \mu_k^{-2m}u_k^{(2m)}\right)\right| + \right.$$

$$\left. +\mu_k^{-1}\left|\left(f, \mu_k^{-2m}u_k^{(2m)}\right)\right| + \mu_k^{-2}\left(\sum_{l=2}^{2m+1}\mu_k^{2-l}\|P_l\|_1\right)\|u_k\|_\infty\|f\|_\infty\right\}; \tag{16}$$

$$|f_k| \le const\mu_k^{-1}\left\{\left[C_1(f)\mu_k^{\alpha-2m-1} + \sum_{Im\omega_j<0}\left|\int_0^1\overline{f'(t)}e^{-i\omega_j\mu_k t}dt\right| + \right.\right.$$

$$+\sum_{Im\omega_j>0}\left|\int_0^1\overline{f'(1-t)}e^{i\omega_j\mu_k t}dt\right| + \mu_k^{-1}\left(\|f\|_\infty + \|f'\|_1\right)\times$$

$$\left.\left.\times\sum_{l=2}^{2m+1}\mu_k^{2-l}\|P_l\|_1\right]\|u_k\|_\infty + \left|\int_0^1\overline{f'(t)}e^{i\mu_k t}dt\right|\right\}, \tag{17}$$

if $P_1(x) \equiv 0$ and $n = 4q-1$, $Im\lambda_k > 0$ or $n = 4q+1$, $Im\lambda_k < 0$;

$$|f_k| \le const\mu_k^{-1}\left\{\left[C_1(f)\mu_k^{\alpha-2m-1} + \sum_{Im\omega_j>0}\left|\int_0^1\overline{f'(t)}e^{i\omega_j\mu_k t}dt\right| + \right.\right.$$

$$+\sum_{Im\omega_j<0}\left|\int_0^1\overline{f'(1-t)}e^{-i\omega_j\mu_k t}dt\right| + \mu_k^{-1}\left(\|f\|_\infty + \|f'\|_1\right)\times$$

$$\left.\left.\times\sum_{l=2}^{2m+1}\mu_k^{2-l}\|P_l\|_1\right]\|u_k\|_\infty + \left|\int_0^1\overline{f'(t)}e^{-i\mu_k t}dt\right|\right\}, \tag{18}$$

if $P_1(x) \equiv 0$ and $n = 4q-1$, $Im\lambda_k < 0$ or $n = 4q+1$, $Im\lambda_k > 0$.

Proof. Since the eigenfunction is a solution of the equation $Lu_k = -\lambda_k u_k$, we represent the Fourier coefficient f_k of $\mu_k \geq 1$ in the form

$$f_k = (f, u_k) = \left(f, -\lambda_k^{-1} L u_k\right) = -\overline{\lambda}_k^{-1}\left(f, u_k^{(2m+1)}\right) -$$

$$-\overline{\lambda}_k^{-1} \sum_{l=1}^{2m+1} \left(f, P_l u_k^{(2m+1-l)}\right) = -\overline{\lambda}_k^{-1}\left(f, u_k^{(2m+1)}\right) -$$

$$-\overline{\lambda}_k^{-1}\left(f, P_1 u_k^{(2m)}\right) - \overline{\lambda}_k^{-1} \sum_{l=2}^{2m+1} \left(f, P_l u_k^{(2m+1-l)}\right). \tag{19}$$

By virtue of the estimate (see [16], [17])

$$\left\| u_k^{(s)} \right\|_\infty \leq const(1+\mu_k)^{s+1/p} \left\| u_k \right\|_p, \ p \geq 1, \ s = \overline{0, 2m}, \tag{20}$$

we obtain the following estimate for the third term on the right-hand side of (19):

$$\left| \overline{\lambda}_k^{-1} \sum_{l=2}^{2m+1} \left(f, P_l u_k^{(2m+1-l)}\right) \right| \leq \mu_k^{-(2m+1)} \left\| f \right\|_\infty \sum_{l=2}^{2m+1} \left\| P_l \right\|_1 \times$$

$$\times \left\| u_k^{(2m+1-l)} \right\|_\infty \leq const \mu_k^{-(2m+1)} \left\| f \right\|_\infty \left(\sum_{l=2}^{2m+1} \left\| P_l \right\|_1 \mu_k^{2m+1-l} \right) \times$$

$$\times \left\| u_k \right\|_\infty \leq const \mu_k^{-2} \left\| f \right\|_\infty \left\| u_k \right\|_\infty \sum_{l=2}^{2m+1} \mu_k^{2-l} \left\| P_l \right\|_1. \tag{21}$$

Integrating the first term on the right-hand side of equality (19) by parts and using condition (1), we get

$$\left|\lambda_k\right|^{-1} \left| \left(f, u^{(2m+1)}\right) \right| \leq C_1(f) \mu_k^{\alpha-2m-1} \left\| u_k \right\|_\infty + \mu_k^{-2m-1} \left| \left(f', u_k^{(2m)}\right) \right| . \tag{22}$$

Estimate (16) now follows from relations (19), (21), and (22).

We now estimate the expression $\mu_k^{-2m-1} \left| \left(f', u_k^{(2m)}\right) \right|$ in the case where $P_1(x) \equiv 0$. To this end, we use relations (11) and (12) depending on the sign of $Im\lambda_k$. For certainty, we consider the case $n = 2m+1 = 4q-1, \ q \in N, \ Im\lambda_k > 0$ and apply relation (11) with $l = 2m$. Thus, by virtue of estimates (13)-(15), (20), and

$$\left| M(\xi, u_k) \right| \leq \frac{\mu_k^{-2m}}{2m+1} \sum_{l=2}^{2m+1} \left| P_l(\xi) u_k^{(2m+1-l)}(\xi) \right| \leq$$

$$\leq const\mu_k^{-1}\left(\sum_{l=2}^{2m+1}\mu_k^{2-l}\left|P_l(\xi)\right|\right)\|u_k\|_\infty\,,$$

we find

$$\mu_k^{-2m-1}\left|\left(f',u_k^{(2m)}\right)\right|=\mu_k^{-1}\left|\left(f',\mu_k^{-2m}u_k^{(2m)}\right)\right|\leq\mu_k^{-1}\sum_{Im\omega_j\leq0}\left|\left(f',X_{jk}^{-}(0)e^{-i\omega_j\mu_k t}\right)\right|+$$

$$+\mu_k^{-1}\sum_{Im\omega_j>0}\left|\left(f',B_{jk}^{-}(0)e^{i\omega_j\mu_k(1-t)}\right)\right|+\mu_k^{-1}\sum_{Im\omega_j\leq0}\left|\left(f',\int_0^t M(\xi,u_k)e^{i\omega_j\mu_k(\xi-t)}d\xi\right)\right|+$$

$$+\mu_k^{-1}\sum_{Im\omega_j>0}\left|\left(f',\int_t^1 M(\xi,u_k)e^{i\omega_j\mu_k(\xi-t)}d\xi\right)\right|\leq$$

$$\leq\mu_k^{-1}\sum_{Im\omega_j\leq0}\left|X_{jk}^{-}(0)\right|\left|\left(f',e^{-i\omega_j\mu_k t}\right)\right|+\mu_k^{-1}\sum_{Im\omega>0}\left|B_{jk}^{-}(0)\right|\left|\left(f',e^{i\omega_j\mu_k(1-t)}\right)\right|+$$

$$+const\mu_k^{-2}\left(\sum_{l=2}^{2m+1}\mu_k^{2-l}\|P_l\|_1\right)\|u_k\|_\infty\left\|f'\right\|_1\leq$$

$$\leq const\mu_k^{-1}\left(\sum_{Im\omega_j<0}\left|\int_0^1\overline{f'(t)}e^{-i\omega_j\mu_k t}dt\right|\|u_k\|_\infty+\int_0^1\overline{f'(t)}e^{i\mu_k t}dt\right)+$$

$$+const\mu_k^{-1}\sum_{Im\omega_j>0}\int_0^1\overline{f'(1-t)}e^{i\omega_j\mu_k t}dt\,\|u_k\|_\infty+$$

$$+const\mu_k^{-2}\left(\sum_{l=2}^{2m+1}\mu_k^{2-l}\|P_l\|_1\right)\|u_k\|_\infty\|f\|_1\,.$$

Taking into account the last estimate in equality (22) and combining the result with estimate (21), we derive inequality (17) from equality (19). Lemma 2 is proved. ◄

Lemma 3. *(see [17]). Assume that* $P_1(x)\in L_2(G)$, $P_l(x)\in L_1(G)$, $l=\overline{2,2m+1}$. *Then for the orthonormal system of eigenfunctions* $\{u_k(x)\}_{k=1}^{\infty}$ *and the sequence* $\{\mu_k\}_{k=1}^{\infty}$*, the following estimates are true:*

$$\sum_{\tau\leq\mu_k\leq\tau+1}1\leq const\ \ \textit{for any}\ \tau\geq0,\tag{23}$$

$$\sum_{0\leq\mu_k\leq\tau}\|u_k\|_\infty^2\leq const(1+\tau)\ \ \textit{for any}\ \tau>0.\tag{24}$$

Lemma 4. *(see [15], [18]). If the conditions of Lemma 3 are satisfied, then*

$$\left\{\mu_k^{-2m} u_k^{(2m)}(x)\right\}_{k=1}^{\infty}, \ \mu_k \neq 0,$$

is a Bessel system, i.e., for any function $f(x) \in L_2(G)$, the following inequality is true:

$$\left(\sum_{\mu_k>0} \left|\left(f, \mu_k^{-2m} u_k^{(2m)}\right)\right|^2\right)^{1/2} \leq const \left\|f\right\|_2 . \tag{25}$$

Lemma 5. *Under condition (23), the systems $\left\{e^{i\mu_k t}\right\}_{k=1}^{\infty}$ and $\left\{e^{-i\mu_k t}\right\}_{k=1}^{\infty}$ satisfy the Riesz inequality for $1 < p \leq 2$.*

Proof. Note that each of these systems is a Bessel system in $L_2(G)$ (see [19]) under condition (23) and in addition, the following inequality is true:

$$\left|\int_0^1 f(x)\overline{\varphi_k(x)}dx\right| \leq const \left\|f\right\|_1 ,$$

for any $f(x) \in L_2(G)$, where $\{\varphi_k(x)\}$ is any of the above-mentioned systems. Thus, by virtue of the Riesz-Thorin theorem (see [20], chap. XII, sec. 1), the Riesz inequality holds for these systems, i.e., for any function $f(x) \in L_p(G)$, $1 < p \leq 2$, the following inequality is true:

$$\left(\sum_{k=1}^{\infty} \left|\int_0^1 f(x)\overline{\varphi_k(x)}dx\right|^q\right)^{1/q} \leq const \left\|f\right\|_p , \tag{26}$$

where $q = p/(p-1)$. Lemma 5 is proved. ◄

Lemma 6. *Suppose that the conditions of Lemma 3 are satisfied. Then the following estimates hold for the system $\{u_k(x)\}_{k=1}^{\infty}$ for any $\mu \geq 2$:*

$$\sum_{\mu_k \geq \mu} \mu_k^{-(1+\delta)} \left\|u_k\right\|_\infty^2 \leq C_2(\delta)\mu^{-\delta}, \ \delta > 0, \tag{27}$$

$$\sum_{\mu_k \geq \mu} \mu_k^{-p} \left\|u_k\right\|_\infty^p \leq C_3(p)\mu^{1-p}, \ 1 < p \leq 2, \tag{28}$$

where $C_2(\delta)$ and $C_3(p)$ are positive constants.

Proof. Take a positive integer n_0. By the estimates (23), (24), using the Abel transformation, we obtain the inequalities

$$\sum_{\mu\le\mu_k\le[\mu]+n_0}\mu_k^{-(1+\delta)}\left\|u_k\right\|_\infty^2\le\sum_{[\mu]\le\mu_k\le[\mu]+n_0}\mu_k^{-(1+\delta)}\left\|u_k\right\|_\infty^2\le$$

$$\le\sum_{n=[\mu]}^{[\mu]+n_0}n^{-(1+\delta)}\left(\sum_{n\le\mu_k<n+1}\left\|u_k\right\|_\infty^2\right)\le\sum_{n=[\mu]}^{[\mu]+n_0-1}\left(\sum_{1\le\mu_k<n+1}\left\|u_k\right\|_\infty^2\right)\times$$

$$\times\left(n^{-(1+\delta)}-(n+1)^{-(1+\delta)}\right)+\left(\sum_{1\le\mu_k<[\mu]+n_0+1}\left\|u_k\right\|_\infty^2\right)([\mu]+n_0)^{-(1+\delta)}+$$

$$+\left(\sum_{1\le\mu_k<[\mu]}\left\|u_k\right\|_\infty^2\right)[\mu]^{-(1+\delta)}\le const\sum_{n=[\mu]}^{[\mu]+n_0-1}(n+1)\frac{(1+\delta)(1+n)^\delta}{(n(n+1))^{1+\delta}}+$$

$$+const\frac{n_0+[\mu]+1}{(n_0+[\mu])^{1+\delta}}+const\frac{1+[\mu]}{[\mu]^{1+\delta}}\le$$

$$\le const\left\{(1+\delta)\sum_{n=[\mu]}^{\infty}n^{-(1+\delta)}+[\mu]^{-\delta}\right\}\le C_2(\delta)\mu^{-\delta},$$

whence, since the number n_0 is arbitrary, we obtain the estimate (27).

Let us prove the estimate (28). Obviously, for $p=2$ this estimate is a special case of the estimate (27) for $\delta=1$.

Consider the case of $1<p<2$. Then, applying the Holder inequality for $p'=2/p$ and $q'=2/(2-p)$ we obtain

$$\sum_{\mu_k\ge\mu}\mu_k^{-p}\left\|u_k\right\|_\infty^p=\sum_{\mu_k\ge\mu}\mu_k^{-1/2}\left(\mu_k^{-p+1/2}\left\|u_k\right\|_\infty^p\right)\le$$

$$\le\left(\sum_{n=[\mu]}\mu_k^{-1/(2-p)}\right)^{(2-p)/2}\left(\sum_{\mu_k\ge\mu}\mu_k^{-2+1/p}\left\|u_k\right\|_\infty^2\right)^{p/2}\le$$

$$\le\left(\sum_{n=[\mu]}^{\infty}n^{-1/(2-p)}\left(\sum_{n\le\mu_k\le n+1}1\right)\right)^{(2-p)/2}\left(\sum_{\mu_k\ge\mu}\mu_k^{-2+1/p}\left\|u_k\right\|_\infty^2\right)^{p/2}.$$

Now we apply the estimate (27) with $\delta=1-1/p$ and the first estimate (i.e., estimate (23)) in Lemma 3 to obtain

$$\sum_{\mu_k\ge\mu}\mu_k^{-p}\left\|u_k\right\|_\infty^p\le C(p)\mu^{(1/p-1)p/2}[\mu]^{(1-p)/2}\le C_3(p)\mu^{1-p}.$$

The proof of Lemma 6 is complete. ◄

Lemma 7. *(see [15]). Suppose that, for any number $N = 1, 2, ...$, a sequence $\{\alpha_k\}_{k=0}^{\infty}$, $\alpha_k \geq 0$, satisfies the condition*

$$\sum_{k=0}^{N} \alpha_k \leq const \cdot N.$$

Then, for any $f(x) \in L_p(G)$, $1 < p \leq 2$, the following inequality is true:

$$\left(\sum_{k=0}^{\infty} \alpha_k \left| \int_0^1 f(x) e^{-k\beta x} dx \right|^q \right)^{1/q} \leq M_p \|f\|_p \,,$$

where β is a complex number with $Re\beta > 0$, $p^{-1}+q^{-1} = 1$, and M_p is independent of $f(x)$.

Lemma 8. *Under the conditions of Lemma 3, for any system*

$$\left\{ \|u_k\|_{\infty}^{2/q} e^{i\omega j \mu_k t} \right\}_{k=1}^{\infty}, \ Im\omega_j > 0$$

and

$$\left\{ \|u_k\|_{\infty}^{2/q} e^{-i\omega j \mu_k t} \right\}_{k=1}^{\infty}, \ Im\omega_j < 0$$

for $n = 4l - 1$, $Im\lambda_k < 0$ or $n = 4l + 1$, $Im\lambda_k > 0$ and any system

$$\left\{ \|u_k\|_{\infty}^{2/q} e^{-i\omega j \mu_k t} \right\}_{k=1}^{\infty}, \ Im\omega_j < 0$$

and

$$\left\{ \|u_k\|_{\infty}^{2/q} e^{i\omega j \mu_k t} \right\}_{k=1}^{\infty}, \ Im\omega_j > 0$$

for $n = 4l - 1$, $Im\lambda_k > 0$ or $n = 4l + 1$, $Im\lambda_k < 0$ the Riesz inequality is true for $1 < p \leq 2$, where $p^{-1} + q^{-1} = 1$.

Proof. Consider a number j with $Im\omega_j > 0$ and denote $\gamma = Im\omega_j$. Taking into account the relation $\left|e^{i\omega_j \mu_k t}\right| = e^{-\gamma \mu_k t}$, for any function $f(x) \in L_p(G)$ we obtain the chain of inequalities

$$\sum_{k=1}^{\infty} \left(\|u_k\|_{\infty}^{2/q} \left| \int_0^1 \overline{f(t)} e^{i\omega_j \mu_k t} dt \right| \right)^q \leq \sum_{k=1}^{\infty} \|u_k\|_{\infty}^{2} \left(\int_0^1 |f(t)| \, e^{-\gamma \mu_k t} dt \right)^q \leq$$

$$\leq \sum_{n=0}^{\infty} \sum_{n\leq\mu_k<n+1} \|u_k\|_\infty^2 \left(\int_0^1 |f(t)|\, e^{-\gamma\mu_k t} dt \right)^q \leq$$

$$\leq \sum_{n=0}^{\infty} \left(\sum_{n\leq\mu_k<n+1} \|u_k\|_\infty^2 \right) \left(\int_0^1 |f(t)|\, e^{-\gamma n t} dt \right)^q \leq \sum_{n=0}^{\infty} \alpha_n \left(\int_0^1 |f(t)|\, e^{-n\gamma t} dt \right)^q, \tag{29}$$

where $\alpha_n = \sum\limits_{n\leq\mu_k<n+1} \|u_k\|_\infty^2$.

By virtue of inequality (24), for any positive integer N we obtain the estimate

$$\sum_{n=0}^{N} \alpha_n = \sum_{n=0}^{N} \left(\sum_{n\leq\mu_k<n+1} \|u_k\|_\infty^2 \right) = \sum_{0\leq\mu_k<N+1} \|u_k\|_\infty^2 \leq const \cdot N,$$

and hence the assumption of Lemma 7 is satisfied.

Therefore,

$$\left\{ \sum_{n=0}^{\infty} \alpha_n \left(\int_0^1 |f(t)|\, e^{-\gamma n t} dt \right)^q \right\}^{1/q} \leq M_p \|f\|_p .$$

This, together which inequality (29), implies that the Riesz inequality holds for the system

$$\left\{ \|u_k\|_\infty^{2/q} e^{i\omega_j \mu_k t} \right\}_{k=1}^{\infty}, \ Im\omega_j > 0.$$

The proof of the lemma is complete. ◄

3. Proof of the results

Proof of Theorem 1. Consider the case $n = 2m + 1 = 4r - 1, r \in N$, and $1 < p \leq 2$. We prove the uniform convergence of the series

$$\sum_{k=1}^{\infty} |f_k|\, |u_k(x)|$$

on $\overline{G} = [0, 1]$. To this end, we split this series into the sums

$$\sum_{0\leq\mu_k\leq 2} |f_k|\, |u_k(x)| \quad \text{and} \quad \sum_{\mu_k>2} |f_k|\, |u_k(x)| \ .$$

By virtue of estimate (24), the first sum does not exceed the quantity $const\,\|f\|_1$. To study the second series, we apply Lemma 2, i.e., estimates (17) and (18) depending on the sign of $Im\lambda_k$. To this end, we represent this series in the form

$$\sum_{\mu_k>2}|f_k|\,|u_k(x)|=\sum_{k\in I_1}|f_k|\,|u_k(x)|+\sum_{k\in I_2}|f_k|\,|u_k(x)|=J_1+J_2,$$

where $I_1=\{k:\mu_k>2,\ Im\lambda_k<0\}$, $I_2=\{k:\mu_k>2,\ Im\lambda_k>0\}$.

By virtue of estimate (18), we find

$$J_1=\sum_{k\in I_1}|f_k|\,|u_k(x)|\le const\ C_1(f)\sum_{k\in I_1}\mu_k^{\alpha-2m-1}\|u_k\|_\infty^2+$$

$$+const\sum_{k\in I_1}\mu_k^{-1}\sum_{Im\omega_j>0}\left|\int\limits_0^1\overline{f'(t)}e^{i\omega_j\mu_k t}dt\right|\|u_k\|_\infty^2+$$

$$+const\sum_{k\in I_1}\mu_k^{-1}\sum_{Im\omega_j<0}\left|\int\limits_0^1\overline{f'(1-t)}e^{-i\omega_j\mu_k t}dt\right|\|u_k\|_\infty^2+$$

$$+const\left(\|f\|_\infty+\left\|f'\right\|_1\right)\sum_{k\in I_1}\mu_k^{-2}\left(\sum_{l=2}^{2m+1}\mu_k^{2-l}\,\|P_l\|_1\right)\|u_k\|_\infty^2+$$

$$+const\sum_{k\in I_1}\mu_k^{-1}\left|\int\limits_0^1\overline{f'(t)}e^{-i\mu_k t}dt\right|\|u_k\|_\infty^2=const\left(J_1^1+J_1^2+J_1^3+J_1^4+J_1^5\right).$$

Further, we prove convergence of the series $J_1^j,\ j=\overline{1,5}$. By virtue of Lemma 6 and the condition $0\le\alpha<2m,$ we get

$$J_1^1=C_1(f)\sum_{k\in I_1}\mu_k^{\alpha-2m-1}\|u_k\|_\infty^2=C_1(f)\sum_{k\in I_1}\frac{\|u_k\|_\infty^2}{\mu_k^{1+(2m-\alpha)}}\le$$

$$\le CC_1(f)2^{\alpha-2m}<\infty. \tag{30}$$

To estimate the series J_1^2, we first apply the Holder inequality to this sum and then Lemmas 6 and 8. This yields

$$J_1^2=\sum_{k\in I_1}\mu_k^{-1}\sum_{Im\omega_j>0}\left|\int\limits_0^1\overline{f'(t)}e^{i\omega_j\mu_k t}dt\right|\|u_k\|_\infty^2=$$

$$= \sum_{Im\omega_j>0} \sum_{k\in I_1} \mu_k^{-1} \left| \int_0^1 \overline{f'(t)} e^{i\omega_j \mu_k t} dt \right| \|u_k\|_\infty^{2/p+2/q} \le \sum_{Im\omega_j>0} \left(\sum_{k\in I_1} \frac{\|u_k\|_\infty^2}{\mu_k^p} \right)^{1/p} \times$$

$$\times \left(\sum_{k\in I_1} \|u_k\|_\infty^2 \left| \int_0^1 \overline{f'(t)} e^{i\omega_j \mu_k t} dt \right|^q \right)^{1/q} \le const M_p 2^{-1/p} m \left\| f' \right\|_p < \infty.$$

The series J_1^3 is estimated similar to the series J_1^2. To estimate the series J_1^4, we apply Lemma 6 and obtain

$$J_1^4 = const \left(\|f\|_\infty + \left\| f' \right\|_1 \right) \sum_{k\in I_1} \frac{\|u_k\|_\infty^2}{\mu_k^2} \sum_{l=2}^{2m+1} \mu_k^{2-l} \|P_l\|_1 \le$$

$$\le const \left(\|f\|_\infty + \left\| f' \right\|_1 \right) \sum_{l=2}^{2m+1} 2^{1-l} \|P_l\|_1 < \infty. \tag{31}$$

Finally, we estimate the series J_1^5. To this end, we first apply the Holder inequality and then Lemmas 5 and 6. This yields

$$J_1^5 = const \sum_{k\in I_1} \mu_k^{-1} \left| \int_0^1 \overline{f'(t)} e^{-i\mu_k t} dt \right| \|u_k\|_\infty \le const \left(\sum_{k\in I_1} \frac{\|u_k\|_\infty^p}{\mu_k^p} \right)^{1/p} \times$$

$$\times \left\{ \sum_{k\in I_1} \left| \int_0^1 \overline{f'(t)} e^{-i\mu_k t} dt \right|^q \right\}^{1/q} \le const 2^{-1/q} \left\| f' \right\|_p < \infty.$$

Thus, the series J_1 is uniformly convergent on $\overline{G}$. By using estimate (17) for the coefficients f_k, in exactly the same way, we prove the uniform convergence of the series J_2 on $\overline{G}$. Hence, the series $\sum\limits_{k=1}^{\infty} |f_k| \, |u_k(x)|$ uniformly converges on $\overline{G}$. In view of the completeness of the system $\{u_k(x)\}_{k=1}^\infty$ in $L_2(G)$ and continuity of function $f(x)$ on $\overline{G}$, the series $\sum\limits_{k=1}^{\infty} f_k u_k(x)$ uniformly converges to $f(x)$, i.e., the equality

$$f(x) = \sum_{k=1}^{\infty} f_k u_k(x) \tag{32}$$

is true.

We now check the validity of estimate (2). By virtue of equality (32), we find

$$|R_\nu(x,f)| = |f(x) - \sigma_\nu(x,f)| = \left|\sum_{\mu_k>\nu} f_k u_k(x)\right| \le$$

$$\le \sum_{\mu_k\ge\nu} |f_k|\,\|u_k\|_\infty = \sum_{k\in B_1(\nu)} |f_k|\,|u_k(x)| + \sum_{k\in B_2(\nu)} |f_k|\,|u_k(x)| = K_1(\nu) + K_2(\nu),$$

where $B_1(\nu) = \{k : \mu_k \ge \nu,\ Im\lambda_k < 0\}$ and $B_2(\nu) = \{k : \mu_k \ge \nu,\ Im\lambda_k > 0\}$.

Further, the series $K_1(\nu)$ and $K_2(\nu)$ are estimated by using the same procedure as in estimating the series J_1 and J_2. As a result, we get

$$K_j(\nu) \le const\left\{C_1(f)\nu^{\alpha-2m} + \nu^{-\frac{1}{q}}\left\|f'\right\|_p + \nu^{-1}\left(\|f\|_\infty + \left\|f'\right\|_1\right)\times\right.$$

$$\left.\times\sum_{l=2}^{2m+1}\nu^{2-l}\,\|P_l\|_1\right\},\ j=1,2.$$

Hence, estimate (2) is true for $1 < p \le 2$. Thus, Theorem 1 is proved for $1 < p \le 2$. For $p > 2$, the validity of Theorem 1 follows from the embedding $L_p(G) \subset L_2(G)$.

Theorem 1 is proved. ◄

Proof of Theorem 2. Let $P_1(x) \in L_2(G),\ P_l(x) \in L_1(G),\ l = \overline{2, 2m+1}$, $f(x) \in W_2^1(G)$ and let condition (1) be satisfied. We now prove the uniform convergence of the series $\sum\limits_{\mu_k\ge2} |f_k|\,|u_k(x)|$ on $\overline{G}$.

By virtue of estimate (16), we get

$$\sum_{\mu_k\ge2}|f_k|\,|u_k(x)|_\infty \le const\left\{C_1(f)\sum_{\mu_k\ge2}\mu_k^{\alpha-2m-1}\,\|u_k\|_\infty^2 + \right.$$

$$+\sum_{\mu_k\ge2}\mu_k^{-1}\left|\left(\overline{P}_1 f, \mu_k^{-2m}u_k^{(2m)}\right)\right|\,\|u_k\|_\infty + \sum_{\mu_k\ge2}\mu_k^{-1}\,\|u_k\|_\infty\times$$

$$\times\left|\left(f', \mu_k^{-2m}u_k^{(2m)}\right)\right| + \|f\|_\infty\sum_{\mu_k\ge2}\mu_k^{-2}\,\|u_k\|_\infty^2\times$$

$$\left.\times\left(\sum_{l=2}^{2m+1}\mu_k^{2-l}\,\|P_l\|_1\right)\right\} = const\left\{T_1+T_2+T_3+T_4\right\}.$$

We estimate the series T_1 and T_4 similar to the the series J_1^1 and J_1^4. The series T_1 satisfies estimate (30) and the series T_4 satisfies estimate (31) with replacement of the factor $(\|f\|_\infty + \|f'\|_1)$ by the factor $\|f\|_\infty$.

To estimate the series T_2 and T_3 we apply Lemma 4 to the system $\left\{\mu_k^{-2m} u_k^{(2m)}(x)\right\}$, $\mu_k \geq 2$, and also Lemma 6 with $\delta = 1$ and $\mu = 2$. As a result, we obtain

$$T_2 = \sum_{\mu_k \geq 2} \mu_k^{-1} \|u_k\|_\infty \left|\left(\overline{P}_1 f, \mu_k^{-2m} u_k^{(2m)}\right)\right| \leq$$

$$\leq \left(\sum_{\mu_k \geq 2} \mu_k^{-2} \|u_k\|_\infty^2\right)^{1/2} \left(\sum_{\mu_k \geq 2} \left|\left(\overline{P}_1 f, \mu_k^{-2m} u_k^{(2m)}\right)\right|^2\right)^{1/2} \leq const\, 2^{-1/2} \left\|\overline{P}_1 f\right\|_2,$$

$$T_3 = \sum_{\mu_k \geq 2} \mu_k^{-1} \|u_k\|_\infty \left|\left(f', \mu_k^{-2m} u_k^{(2m)}\right)\right| \leq$$

$$\leq \left(\sum_{\mu_k \geq 2} \mu_k^{-2} \|u_k\|_\infty^2\right)^{1/2} \left(\sum_{\mu_k \geq 2} \left|\left(f', \mu_k^{-2m} u_k^{(2m)}\right)\right|^2\right)^{1/2} \leq const\, 2^{-1/2} \left\|f'\right\|_2.$$

Hence, the series $\sum\limits_{k=1}^{\infty} |f_k| \, |u_k(x)|$ is uniformly convergent on $\overline{G}$. This implies the uniform convergence of the series $\sum\limits_{k=1}^{\infty} f_k u_k(x)$. In view of the completeness of the system $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2(G)$ and the continuity of the function $f(x)$, we obtain

$$f(x) = \sum_{k=1}^{\infty} f_k u_k(x), \quad x \in \overline{G}.$$

Note that the remainder $R_\nu(x, f)$ of this series (in the remainder, summation is carried out over the numbers k for which $\mu_k > \nu$) can be estimated as follows:

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const \Big\{ C_1(f) \nu^{\alpha - 2m} + \nu^{-1/2} \left(\|P_1 f\|_2 + \right.$$

$$\left. + \left\|f'\right\|_2 + \nu^{-1} \|f\|_\infty \sum_{l=2}^{2m+1} \nu^{2-l} \|P_l\|_1 \right\}, \ \nu \geq 2.$$

Theorem 2 is proved. ◄

To substantiate estimate (7), it suffices to take into account the fact that, in the proof of Theorem 2 the sequence of remainders of the convergent series tends to zero, i.e.,

$$\sum_{\mu_k \geq \nu} \left|\left(\overline{P}_1 f, \mu_k^{-2m} u_k^{(2m)}\right)\right|^2 = o(1), \quad \nu \to +\infty,$$

$$\sum_{\mu_k \geq \nu} \left| \left(f', \mu_k^{-2m} u_k^{(2m)} \right) \right|^2 = o(1), \quad \nu \to +\infty,$$

where the symbol "o" depends on function $f(x)$.

Proof of Theorem 3. Let $P_1(x) \in L_2(G)$, $P_l(x) \in L_1(G)$, $l = \overline{2, 2m+1}$, $f(x) \in W_p^1(G)$, $1 < p < 2$, condition (1) be satisfied, and the system $\{u_k(x)\}_{k=1}^{\infty}$ be uniformly bounded.

By virtue of the orthonormality of the system $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2(G)$, condition (23) is satisfied. On the other hand,

$$1 = |(u_k, u_k)| \leq \|u_k\|_p \|u_k\|_q \leq \|u_k\|_\infty \|u_k\|_q .$$

This yields $\|u_k\|_q^{-q} \leq \|u_k\|_\infty^q$.

In view of inequality (23) and the uniform boundedness of the system $\{u_k(x)\}_{k=1}^{\infty}$, for any $\tau > 0$, we get

$$\sum_{0 \leq \mu_k \leq \tau} \|u_k\|_\infty^q \|u_k\|_q^{-q} \leq \sum_{0 \leq \mu_k \leq \tau} \|u_k\|_\infty^{2q} \leq const \sum_{0 \leq \mu_k \leq \tau} 1 \leq const\ \tau.$$

Thus, the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfies all conditions of the sufficient part of Theorem 3 in [15,18]. Therefore the system $\left\{ \mu_k^{-2m} u_k^{(2m)}(x) \right\}$, $\mu_k \geq 1$, satisfies the Riesz inequality for $1 < p < 2$.

To prove Theorem 3, it suffices to estimate the series T_3 (the other series T_1, T_2 and T_4 have been estimated in Theorem 2 without the restriction of uniform boundedness of the system $\{u_k(x)\}_{k=1}^{\infty}$). By using the Holder inequality and Riesz inequality for the system $\left\{ \mu_k^{-2m} u_k^{(2m)}(x) \right\}$, $\mu_k \geq 1$, and Lemma 6 we arrive at the following results for the series T_3 and its remainder:

$$T_3 = \sum_{\mu_k \geq 2} \mu_k^{-1} \|u_k\|_\infty \left| \left(f', \mu_k^{-2m} u_k^{(2m)} \right) \right| \leq \left(\sum_{\mu_k \geq 2} \mu_k^{-p} \|u_k\|_\infty^p \right)^{1/p} \times$$

$$\times \left(\sum_{\mu_k \geq 2} \left| \left(f', \mu_k^{-2m} u_k^{(2m)} \right) \right|^q \right)^{1/q} \leq const\ 2^{-1/q} \left\| f' \right\|_p ,$$

$$\sum_{\mu_k \geq \nu} \mu_k^{-1} \|u_k\|_\infty \left| \left(f', \mu_k^{-2m} u_k^{(2m)} \right) \right| \leq \left(\sum_{\mu_k \geq \nu} \mu_k^{-p} \|u_k\|_\infty^p \right)^{1/p} \times$$

$$\times \left(\sum_{\mu_k \geq \nu} \left| \left(f', \mu_k^{-2m} u_k^{(2m)} \right) \right|^q \right)^{1/q} \leq const\ \nu^{-1/q} \left\| f' \right\|_p .$$

Theorem 3 is proved.

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