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Bitsadze–Samarski Problem For Elliptic Systems of Second Order

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Abstract. Under general assumptions with respect to the shift, Bitsadze-Samarski problem for elliptic systems of second order on the plane with constant and only leading coefficients is considered. The Fredholm theorem for this problem is proved and the index formula is obtained.

Key Words and Phrases: elliptic systems, boundary value problems, singular integral equations, functions analytic in the sense of Douglis.

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1. Introduction

Let D be a bounded domain with a piecewise smooth boundary Γ in the complex plane $z = x + iy$. We consider the elliptic system of second order

$$a_{11}\frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21})\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

with constant coefficients $a_{ij} \in R^{l\times l}$ for an unknown vector-valued function $u = (u_1, \cdots u_l) \in C^2(D)$. Suppose the set $F = \{\tau_1, \ldots, \tau_m\} \subseteq \Gamma$ contains all the angular points of the curve. Let us consider a continuous differential map $\alpha : \Gamma \setminus F \to D$ such that there is one-sided limits $\alpha(\tau_j \pm 0) \in F$ and $\alpha'(\tau_j \pm 0) \neq 0$, $1 \leq j \leq m$.

Bitsadse-Samarski problem is formulated as follows: find a solution $u(z) = u(x, y) \in C(\overline{D})$ satisfying the boundary condition

$$(u + bu \circ \alpha)\big|_{\Gamma} = f, \tag{2}$$

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where $b(t)$ is an $l \times l$-matrix-valued function which is piecewise continuous on Γ. We can also consider the case where the coefficient $b(t)$ is only defined on a part Γ' of Γ. In this case (2) is transformed to the following form:

$$u\big|_{\Gamma\setminus\Gamma'} = f_0, \quad (u + bu \circ \alpha)\big|_{\Gamma'} = f_1. \tag{3}$$

We can always reduce this problem to the form (2) by extending α on the whole Γ and setting $b = 0$ on $\Gamma \setminus \Gamma'$.

The problem (1), (3) was first stated by Bitsadze and Samarski [1]. It was also investigated for a general elliptic equation by Skubachevski [2]-[5], Gurevich [6, 7] and others. Attention should be paid to the case where $\alpha(\Gamma')$ divides the domain D into two parts. In this case the problem (1), (3) can be reduced to the generalized Riemann-Hilbert problem studied in [8]. This case of the problem was treated by Sidorova [9] and Zhura [10].

In the present paper a new approach for investigating this problem is developed. It is based on the reduction of Bitsadse-Samarski problem to a system of singular integral equations on Γ of non-classical type; the corresponding theory was developed in [11].

We will consider the problem in Holder spaces. Let $C^\mu(\overline{D})$ and $C^{1,\mu}(\overline{D})$ be the ordinary Holder spaces. Let $C^\mu_\lambda(\overline{D}, F)$, $\lambda \in \mathbb{R}$, be the space of all the functions $\varphi \in C(\overline{D} \setminus F)$ such that $\varphi \in C^\mu(K)$ for every compact subset $K \subseteq \overline{D} \setminus F$ and $\varphi(z) = O(1)|z - \tau|^\lambda$ as $z \to \tau \in F$. To be more precise, in the curvilinear sectors

$$D_j = D \cap \{|z - \tau_j| < \delta\},\ j = 1, \dots, m, \tag{4}$$

where $\delta > 0$ is small enough, we have

$$\varphi_j(z) = \varphi(z)|z - \tau_j|^{\mu-\lambda} \in C^\mu(\overline{D_j}), \quad \varphi_j(\tau_j) = 0. \tag{5}$$

Let $C^\mu_{(\lambda)}(\overline{D}, F)$, $0 < \lambda < 1$, be the space of all the functions $\varphi \in C(\overline{D})$ such that $\varphi \in C^\mu(K)$ for every compact subset $K \subseteq \overline{D} \setminus F$ and $\varphi(z) - \varphi(\tau_j) \in C^\mu_\lambda(\overline{D}_j, \tau_j)$, $1 \le j \le m$.

The spaces $C^\mu_\lambda(\Gamma, F)$ and $C^\mu_{(\lambda)}(\Gamma, F)$ are defined analogously.

We will also use these spaces for piecewise continuous functions φ on Γ which are continuous on $\Gamma\setminus F$. By definition, a function $c \in C(D\setminus F)$ belongs to $C^\nu(\Gamma, F)$ if $c \in C^\nu(\Gamma_0)$ for every smooth arc $\Gamma_0 \subseteq \Gamma$ such that $\tau \in F$ are not inner points of this arc. The space $C^{1,\nu}(\Gamma, F)$ is defined analogously. We also assume that the pair (Γ, F) belongs to the class $C^{1,\nu}$ e.i. $\Gamma_0 \in C^{1,\nu}$ for every smooth arc $\Gamma_0 \subseteq \Gamma$ such that $\tau \in F$ are not inner points of this arc. The smoothness assumptions concerning the problem data are as follows:

$$b \in C^\nu(\Gamma, F), \quad \alpha \in C^{1,\nu}(\Gamma, F), \quad \mu < \nu < 1. \tag{6}$$

The boundary of the sector D_j consists of two smooth arcs $\Gamma_{jk},\ k=1,2$, with a common point τ_j, called lateral sides, and an arc of the circumference $|z-\tau_j|=\delta$. It is convenient to denote the one-sided limits $\varphi(\tau\pm 0)$ at the point τ_j by $\varphi(\tau_{jk})=\lim\varphi(t)$ as $t\to\tau_j, t\in\Gamma_{jk}$. For certainty, we enumerate the lateral sides Γ_{jk} as follows:

$$\varphi(\tau_{j1})=\varphi(\tau-0),\ \varphi(\tau_{j2})=\varphi(\tau+0). \tag{7}$$

Let $q_{jk}\in\partial Q_j$ be the unit tangent vector of Γ_{jk} at the point τ_j. In what follows we suppose that $q_{j1}\neq q_{j2},\quad j=1,\dots,m$. In particular, we can introduce a non-zero angle Q_j between the rays $\{rq_{jk},\ r>0\}$, $k=1,2$, corresponding to D_j. We assume also that the curve $\alpha(\Gamma)$ doesn't touch Γ. In other words, the vector $\alpha'(\tau_{jk})$ belongs to Q_i for $\alpha(\tau_{jk})=\tau_i$. Here we suppose that the derivative α' on Γ_{jk} is meant with respect to the parameter of arc length which counts from the point τ_j. In particular, if $e(t)=t$ is an identical shift on Γ, then $q_{jk}=e'(\tau_{jk})$.

The solution of the problem (1), (2) is sought in the class $C^{\mu}_{(\lambda)}(\overline{D},F)$ or in a wider class $C^{\mu}_{(+0)}=\cup_{\lambda>0}C^{\mu}_{(\lambda)}$. Under our assumptions the operator of the problem (2) is bounded $C^{\mu}_{(\lambda)}(\overline{D},F)\to C^{\mu}_{(\lambda)}(\overline{\Gamma},F)$, $0<\lambda<1$, and we are interested in its Fredholm solvability.

2. Reduction to the problem of function theory

The condition of ellipticity for (1) means that $\det a_{22}\neq 0$ and the characteristic polynomial

$$\chi(w)=\det P(w),\quad P(w)=a_{11}+(a_{12}+a_{21})w+a_{22}w^2, \tag{8}$$

has no real roots. In fact, its roots coincide with eigenvalues of the block 2×2-matrix

$$A_*=\begin{pmatrix} 0 & 1 \\ a_{22}^{-1}a_{11} & -a_{22}^{-1}(a_{12}+a_{21}) \end{pmatrix}\in R^{2l\times 2l},$$

which can be reduced to the Jordan form

$$B_*^{-1}A_*B_*=\begin{pmatrix} J & 0 \\ 0 & \overline{J} \end{pmatrix},\quad B_*=\begin{pmatrix} B & \overline{B} \\ BJ & \overline{BJ} \end{pmatrix}. \tag{9}$$

Here the matrix J is block diagonal; it consists of Jordan blocks corresponding to eigenvalues ν, $\mathrm{Im}\nu>0$.

Due to [12], a general solution $u\in C^2(D)$ of (1) can be represented in the form

$$u=\mathrm{Re}\ B\phi, \tag{10}$$

where l-vector-valued function ϕ satisfies the first order elliptic system

$$\frac{\partial \phi}{\partial x_2} - J\frac{\partial \phi}{\partial x_1} = 0.$$

The solutions of this system are called analytic functions in the sense of Douglis or, shortly, J-analytic functions. The function from (10) is defined uniquely up to a constant vector $\eta \in C^l$, $\mathrm{Re} B\eta = 0$. In general it is a multi-valued function in a multiply connected domain D. To be more precise, its derivative $\phi' = \partial\phi/\partial x$ is an univalent function.

If D is a simply connected domain, then the function ϕ is univalent and (10) can be rewritten in the form

$$u = \mathrm{Re}\ B\phi + \xi,\ \ \xi \in R^l, \phi(0) = 0.$$

For certainty, we assume throughout this paper that $z = 0 \in D$.

Let now D be a multiply connected domain and let s be the number of connected components of ∂D. Then we can make the representation (10) more precise in the following way [13]:

$$u = \mathrm{Re}\ B\phi + \sum\nolimits_1^s u_j\xi_j, \quad \xi_j \in R^l,\ \phi(0) = 0, \tag{11}$$

where $u_j \in C^\infty(\overline{D})$ are some known matrix-valued functions whose columns satisfy (1).

We will consider the solution of (1) in the form (11), where $\phi \in C^\mu_{(\lambda)}(\overline{D}, F)$ and the pair (ϕ, ξ) is defined uniquely by u. The function ϕ can be also represented in the form

$$\phi(z) = \phi_0(z) + \sum\nolimits_1^m (\mathrm{Re}\ z + J\mathrm{Im}\ z)^j \eta_j, \quad \eta_j \in C^l,$$

where $\phi_0 \in C^\mu_\lambda(\overline{D}, F)$, i.e. $\phi_0(\tau_i) = 0,\ \ 1 \le i \le m$, and the pair (ϕ_0, η) is defined uniquely by ϕ. Hence we can write the general representation in the same form (11) changing s into $s + 2m$. So we can reformulate the problem (1), (2) in the following equivalent form:

$$\mathrm{Re}(B\phi + bB\phi \circ \alpha) + \sum_{j=1}^{s+2m} b_j\xi_j = f \quad \text{on} \quad \Gamma \tag{12}$$

for a J-analytic function $\phi \in C^\mu_\lambda$, $\phi(0) = 0$, and vectors $\xi_j \in R^l$ with some given piecewise continuous matrices $b_j \in C^{\mu+0}_{(\lambda+0)}(\Gamma, F)$.

Let us introduce scalar functions $\chi_{jk}(z) \in C^{\mu+0}_{(\lambda+0)}(\Gamma, F)$, $1 \le j \le m$, $k = 1, 2$, satisfying the conditions $\chi_{jk}(\tau_{jk}) = 1$, $\chi_{jk}(\tau_{ir}) = 0$, $(j,k) \ne (i,r)$, and a projector P of the space $C^{\mu}_{(\lambda)}(\Gamma, F)$ into $C^{\mu}_{\lambda}(\Gamma, F)$ by the formula

$$(Pf)(t) = f(t) - \sum\nolimits_{j,k} \chi_{jk}(t) f(\tau_{jk}).$$

Then the problem (12) is equivalent to the system

$$\mathrm{Re}\,(B\phi + bB\phi \circ \alpha) + \sum_{j=1}^{s+2m} (Pb_j)\xi_j = Pf \quad \text{on} \quad \Gamma, \tag{13}$$

$$\sum_{j=1}^{s+2m} b_j(\tau_{ik})\xi_j = f(\tau_{ik}), \quad 1 \le i \le m, \ k = 1, 2, \quad \phi(0) = 0, \tag{14}$$

with respect to $\phi \in C^{\mu}_{\lambda}$ and $\xi_j \in R^l$. Note that the solvability conditions for the equation (14) can be considered as compatibility conditions with respect to the right side f. They are equivalent to the following one: there is a function $\tilde{u} \in C(\overline{D})$ such that the piecewise continuous function

$$\tilde{f} = (\tilde{u} + b\tilde{u} \circ \alpha)\big|_{\Gamma}$$

coincides with f at the points $\tau \in F$, i.e. $\tilde{f}(\tau_{jk}) = f(\tau_{jk})$, $1 \le j \le m$, $k = 1, 2$.

Consider a linear operator $T: \ (R^l)^m \to (R^l)^{2m}$ acting as follows:

$$(T\xi)_{ik} = (u + bu \circ \alpha)(\tau_{ik}), \quad 1 \le i \le m, \quad k = 1, 2, \tag{15}$$

where $u \in C(\overline{D})$ and $u(\tau_i) = \xi_i$. Then the function f satisfies the compatibility conditions if and only if there exists $\xi \in (R^l)^m$ such that $(T\xi)_{ik} = f(\tau_{ik})$. So a number of linear independent compatibility conditions coincides with codim $T =$ dim $((R^l)^{2m}/\mathrm{Im}\ T)$. As ind $T =$ dim $T -$ codim $T =$ dim $(R^l)^m -$ dim $(R^l)^{2m} = -ml$, this number is equal to $ml +$ dim $T \ge ml$.

If $b = 0$, then the operator T is injective and codim $T = ml$. Describe a more general situation of this type.

Lemma 1. *Suppose that there are sets*

$$F = F_0 \supset F_1 \supset \ldots \supset F_{n+1} = \emptyset, \ F_p \setminus F_{p+1} \ne \emptyset, \quad 0 \le p \le n, \tag{16}$$

such that

$$\alpha(\tau \pm 0) \in F_{p+1} \ \text{for} \ \tau \in F_p, \ b(\tau \pm 0) \ne 0, \quad 0 \le p \le n. \tag{17}$$

In particular, $b(\tau + 0) = b(\tau - 0) = 0$ *for* $\tau \in F_n$.

Then the operator T *in* (15) *is injective and therefore a number of linear independent compatibility conditions is equal to* ml.

Proof. By definition, $\alpha(\tau_{ik}) = \tau_{\sigma(ik)}$ for some mapping $\sigma:\ \{1 \le i \le m, k = 1,2\} \to \{1,\dots,m\}$. So we can rewrite (15) in the form $(T\xi)_{ik} = \xi_i + b(\tau_{ik})\xi_{\sigma(ik)}$.

Suppose that $T\xi = 0$ for some $\xi = (\xi_1,\dots,\xi_m) \ne 0$ and let $F_* = \{\tau_i \mid \xi_i \ne 0\}$. As $\xi_i = -b(\tau_{ik})\xi_{\sigma(ik)},\ i = 1,\dots,m$, this set is invariant with respect to mappings $\tau \to \alpha(\tau \pm 0)$. But this fact contradicts the assumption (17). ◄

3. Bitsadze-Samarski problem for J-analytic functions

Let us consider the main part of the problem (13), (14): find a $J-$analytic function $\phi \in C^{\mu}_{\lambda}(\overline{D}, F)$ which satisfies the boundary condition

$$\text{Re }\ (G\phi + G^0\phi \circ \alpha)\ \big|_{\Gamma} = f, \tag{18}$$

where G and G^0 are given piecewise continuous matrix-valued coefficients from the class $C^{\mu+0}_{(+0)}(\Gamma, F)$.

This problem was investigated in [14] where a theorem of Fredholm solvability and an index theorem are proved. In order to formulate the corresponding results, we need in some special matrices. For $w \in C,\ \ \text{Im } w \ne 0$, let us consider the affine transformation on the complex plane acting as follows:

$$q \to q(w) = \text{Re } q + w\text{Im } q. \tag{19}$$

Obviously, the points of real axis remain stationary, so for a given q one can consider the following branch of logarithm:

$$\ln q(w) = \ln|q(w)| + i\arg q(w), \quad \arg q(\overline{w}) = -\arg q(w), \tag{20}$$

which is analytic in the half-planes $\pm\text{Im } w > 0$, and the degree $\exp[\zeta \ln q(w)]$ is analytic on $\zeta \in C$ and $w,\ \ \text{Im } w \ne 0$. Hence for a matrix $W \in C^{l\times l}$ which has no real eigenvalues we can set

$$q^{\zeta}(W) = q^{\zeta}(w)\big|_{w=W}, \quad q^{\zeta}(w) = \exp[\zeta \ln(w)] \tag{21}$$

as an analytic function depending on matrix argument. For example, if W has a unique eigenvalue ν, then the matrix $W - \nu$ is nilpotent and

$$q^{\zeta}(W) = q^{\zeta}(\nu)\sum_{r=0}^{l-1}\frac{\zeta(\zeta-1)\dots(\zeta-r+1)}{r!}\left[\frac{\text{Im } q}{q(\nu)}\right]^r (W-\nu)^r.$$

If the matrix W is triangular, then its diagonal elements coincide with the eigenvalues $\nu \in \sigma(W)$, while the cardinality of the set $\{i \mid W_{ii} = \nu\}$ is equal to the multiplicity of ν. In this case the matrix $q^\zeta(W)$ has the same structure with respect to $q^\zeta(\nu),\ \nu\in\sigma(W)$.

We will use the degree (21) for the Jordan matrices J, $\overline{J}$ from (9) and the vectors changing in the angle $Q=Q_j$ associated with the sector D_j in (4). It is assumed that the branch (20) is continuous on $q\in\overline{Q}$. Let $Q_j(w)$ be the image of the angle Q_j under transformation (19) and let us denote by $\theta_j(w)\in(0,2\pi)$ its span. So $\theta_j(w)=|\arg q_{j2}(w)-\arg q_{j1}(w)|$ and by virtue of (7) for $q\in Q_j$ we have:

$$\begin{gathered}\arg q_{j1}(w)<\arg q(w)<\arg q_{j2}(w),\ \operatorname{Im} w>0,\\ \arg q_{j2}(w)<\arg q(w)<\arg q_{j1}(w),\ \operatorname{Im} w<0.\end{gathered}\tag{22}$$

It implies that

$$q_{j2}^\zeta(w)=e^{\pm i\theta_j(w)\zeta}q_{j1}^\zeta(w),\quad \pm\operatorname{Im} w>0,$$

and hence

$$q_{j2}^\zeta(J)=e^{i\theta_j(J)\zeta}q_{j1}^\zeta(J),\quad q_{j2}^\zeta(\overline{J})=e^{-i\theta_j(J)\zeta}q_{j1}^\zeta(\overline{J}).\tag{23}$$

Here the diagonal matrix $\theta_j(J)$ is defined as a value from J of a function, which is equal to $\theta_j(\nu)$ identically in a neighborhood of $\nu\in\sigma(J)$.

Let us introduce now the families of matrices X_{ijkr},Y_{ijkr} and X^0_{ijkr}, $1\le i,j\le m$, $\ 1\le k,r\le 2$, by the formulas

$$X_{ijkr}=\begin{cases}G(\tau_{ik})q_{ik}^\zeta(J),\ r=1,i=j,\\ \overline{G(\tau_{ik})}q_{ik}^\zeta(\overline{J}),\ r=2,i=j,\\ 0,\ i\neq j,\end{cases}\qquad Y_{ijkr}=\begin{cases}q_{ik}^\zeta(J),\ r=1,i=j,\\ q_{ik}^\zeta(\overline{J}),\ r=2,i=j,\\ 0,\ i\neq j,\end{cases}$$

$$X^0_{ijkr}=\begin{cases}G^0(\tau_{ik})[\alpha'(\tau_{ik})]^\zeta(J),\ r=1,\alpha(\tau_{ik})=\tau_j,\\ \overline{G^0(\tau_{ik})}[\alpha'(\tau_{ik})]^\zeta(\overline{J}),\ r=2,\alpha(\tau_{ik})=\tau_j,\\ 0,\ \alpha(\tau_{ik})\neq\tau_j.\end{cases}$$

These families define block matrices

$$X_{ij}=(X_{ijkr})_1^2,\quad X=(X_{ij})_1^m,$$

and Y_{ij}, X^0_{ij} and Y, X^0 have the same meaning. As matrix valued functions of ζ, they are analytic on the whole complex plane.

It is easy to calculate the determinant $\det Y=\prod\det Y_{jj}$ of the block-diagonal matrix Y explicitly. By virtue of (23) we have:

$$\det Y_{jj}(\zeta)=y_j^0(\zeta)y_j(\zeta),\tag{24}$$

where

$$y_j^0(\zeta)=\det[-q_{j1}^\zeta(J)q_{j1}^\zeta(\overline{J})]=(-1)^l\prod\nolimits_{\nu\in\sigma(J)}|q_{j1}(\nu)|^{2l_\nu\zeta},$$

$$y_j(\zeta)=\det[e^{i\theta_j(J)\zeta}-e^{-i\theta_j(J)\zeta}]=(2i)^l\prod\nolimits_{\nu\in\sigma(J)}\sin^{l_\nu}[\theta_j(\nu)\zeta],$$

and l_ν is a multiplicity of the eigenvalue ν. In particular,

$$\det Y(\zeta)\neq 0,\ \ |\mathrm{Re}\ \zeta|<1/2,\ \ \zeta\neq 0. \tag{25}$$

Let's consider the matrix -valued function $(X+X^0)(\zeta)$ on the line $\mathrm{Re}\ \zeta=\lambda$. By virtue of (7), (22), we obtain

$$(X+X^0)_{ij}(\zeta)\begin{pmatrix} q_{j1}^{-\zeta}(J) & 0\\ 0 & q_{j2}^{-\zeta}(\overline{J})\end{pmatrix}\ \to\ \begin{pmatrix} G(\tau_{j1}) & 0\\ 0 & \overline{G(\tau_{j2})}\end{pmatrix}\ \text{as}\ \mathrm{Im}\ \zeta\ \to+\infty,$$

$$(X+X^0)_{ij}(\zeta)\begin{pmatrix} q_{j2}^{-\zeta}(J) & 0\\ 0 & q_{j1}^{-\zeta}(\overline{J})\end{pmatrix}\ \to\ \begin{pmatrix} 0 & \overline{G(\tau_{j1})}\\ G(\tau_{j2}) & 0\end{pmatrix}\ \text{as}\ \mathrm{Im}\ \zeta\ \to-\infty.$$

It follows from this that

$$\det[(X+X^0)Y^{-1}](\zeta)\ \to\ \begin{cases}\prod_1^m\det[G(\tau_{j1})\overline{G(\tau_{j2})}] & \text{as}\ \ \mathrm{Im}\ \zeta\ \to+\infty,\\ \\ \prod_1^m\det[\overline{G(\tau_{j1})}G(\tau_{j2})] & \text{as}\ \ \mathrm{Im}\ \zeta\ \to-\infty.\end{cases} \tag{26}$$

It is said that the problem (18) is of normal type if $\det G(t)\neq 0,\ t\in\Gamma\setminus F$, and $\det G(\tau_{jk})\neq 0, 1\le j\le m,\ k=1,2$. If this condition holds, then the function $\det(X+X^0)(\zeta)$ has a finite number of zeroes in the each strip $\lambda_1<\mathrm{Re}\,\zeta<\lambda_2$. Let us denote this number by æ(λ_1,λ_2) taking into account its multiplicity and set æ$(\lambda_2,\lambda_1)=-$æ(λ_1,λ_2). It follows from (26) that under the additional condition

$$\det(X+X^0)(\zeta)\neq 0,\ \mathrm{Re}\ \zeta=\lambda, \tag{27}$$

we can define a continuous branch $\ln\det(X+X^0)(\zeta)$ on the line $\mathrm{Re}\ \zeta=\lambda$. With regard to (25), for $0<|\lambda|<1/2$ we can also define the increment

$$\ln\det[(X+X^0)Y^{-1}]\big|_\lambda=$$

$$=\ln\det[(X+X^0)Y^{-1}](\lambda+i\infty)-\ln\det[(X+X^0)Y^{-1}](\lambda-i\infty).$$

For the problem of normal type we can introduce the increment of a piecewise continuous branch $\arg\det G(t)$ on $\Gamma\setminus F$ by the formula

$$\arg\det G\big|_\Gamma=\sum\nolimits_1^m[(\arg\det G)(\tau_j-0)-(\arg\det G)(\tau_j+0)].$$

From (25) and Rouche's theorem it also follows that

$$\ln\det[(X+X^0)Y^{-1}]\big|_{+0}-\ln\det[(X+X^0)Y^{-1}]\big|_{-0}=2\pi i[\text{æ}(-0,+0)-ml]. \tag{28}$$

Here the increments and æ are considered with respect to the lines $\mathrm{Re}\ \zeta=\pm\epsilon$, where $\epsilon>0$ is so small that æ$(0,\epsilon)=$ æ$(0,-\epsilon)=0$.

Theorem 1. *The problem* (18) *is Fredholm in the class* C^μ_λ, $\lambda \in R$, *if and only if it is of normal type and the condition* (27) *holds. In this case its index* æ *is given by the formula*

$$æ = -\frac{1}{\pi} \arg\det G\big|_\Gamma - \frac{1}{2\pi i} \ln\det[(X + X^0)Y^{-1}]\big|_{-0} - æ(-0,\lambda) + l(2 - s). \quad (29)$$

Note that according to (7), (25) the right hand side of this formula is an integer.

Let us consider the case where the problem (18) is of normal type but (27) is not fulfilled. Then the matrix-valued function $(X + X^0)^{-1}$ has a finite number of poles on the line $\mathrm{Re}\,\zeta = \lambda$. It is convenient to denote by $r(\zeta)$ a degree of the pole of the function $(X + X^0)^{-1}(z + \zeta)$ at the point $z = 0$ supposing $r(\zeta) = 0$ at regular points. Then the inequality $r(\zeta) > 0$ is valid on the line $\mathrm{Re}\,\zeta = \lambda$ for points ζ of a finite set only.

Let us introduce classes

$$C^\mu_{\lambda-0} = \bigcap_{\epsilon>0} C^\mu_{\lambda-\epsilon}, \quad C^\mu_{\lambda+0} = \bigcup_{\epsilon>0} C^\mu_{\lambda+\epsilon},$$

and consider the problem (18) in $C^\mu_{\lambda-0}(\overline{D}, F)$ with the right hand side $f \in C^\mu_{\lambda+0}(\Gamma, F)$. In order to formulate the corresponding result [14], let us extend continuously the branch (20) associated with the angle Q_j to all the vectors $z - \tau_j$, $z \in D_j$. Then, analogously to (21), we can define the matrix-valued functions $\ln(z - \tau_j)(J)$ and $(z - \tau_j)^\zeta(J)$. Note that

$$(z - \tau_j)^\zeta(J)[\ln(z - \tau_j)(J)]^k \in C^\mu_{\lambda-0}(\overline{D}_j, \tau_j), \quad \mathrm{Re}\,\zeta = \lambda,\ k = 0, 1, \ldots.$$

Theorem 2. *Suppose the problem* (18) *is of normal type and* $\phi \in C^\mu_{\lambda-0}(\overline{D}, F)$ *is its solution with the right hand side* $f \in C^\mu_{\lambda+0}(\Gamma, F)$. *Then for any sector* D_j *in* (4) *there is* $c_k(\zeta) \in C^l$, $0 \le k \le r(\zeta) - 1$, *such that*

$$\phi(z) - \sum_{\mathrm{Re}\ \zeta=\lambda} \sum_{k=0}^{r(\zeta)-1} (z - \tau_j)^\zeta(J)[\ln(z - \tau_j)(J)]^k c_k(\zeta) \in C^\mu_{\lambda+0}(\overline{D}_j, \tau_j).$$

Certainly the interior sum is equal to zero for $r(\zeta) = 0$.

Corollary 1. *Suppose the problem* (18) *is of normal type and*

$$r(\zeta) \le \delta_{\zeta,0}, \quad \mathrm{Re}\ \zeta = 0. \quad (30)$$

Then every solution $\phi \in C^\mu_{-0}(\overline{D}, F)$ *with the right hand side* $f \in C^\mu_{+0}(\Gamma, F)$ *belongs to the class* $C^\mu_{(+0)}(\overline{D}, F)$.

Note that (30) is equivalent to the conditions $\det(X + X^0)(\zeta) \neq 0$, $\zeta \neq 0, \operatorname{Re}\zeta = 0$, and $(X + X^0)^{-1}(\zeta) = O(|\zeta|^{-1})$ as $\zeta \to 0$.

If $G^0 = 0$, then the problem (18) transforms into the classic Riemann- Hilbert problem

$$\operatorname{Re} G\phi|_\Gamma = f.$$

When the contour Γ is smooth and the function G is continuous this problem for general elliptic systems was investigated by Gilbert and Buchanan [15], Begehr and Wen, Guo Chun [16], Wendland [17] and others. For general piecewise smooth case see, for example, Mushelishvili [18] for $l = 1$ and Soldatov [13] for $l > 1$.

The condition (27) is equivalent to $\det X_{jj}(\zeta) \neq 0,\ \operatorname{Re}\zeta = \lambda,\ 1 \le j \le m$, for the problem (31). We can also consider more general situation when the degree λ in (5) depends on j, e. i. $\lambda = (\lambda_1, \dots, \lambda_m)$. Analogously to (24), we can write

$$\det X_{jj}(\zeta) = x_j^0(\zeta)x_j(\zeta), \tag{31}$$

where

$$x_j^0(\zeta) = \det[-q_{j1}^\zeta(J)q_{j1}^\zeta(\overline{J})G(\tau_{j1})\overline{G}(\tau_{j1})],$$
$$x_j(\zeta) = \det[G(\tau_{j2})e^{i\theta_j(J)\zeta}G^{-1}(\tau_{j1}) - \overline{G}(\tau_{j2})e^{-i\theta_j(J)\zeta}\overline{G}^{-1}(\tau_{j1})].$$

So we can reformulate Theorem 1 for the Riemann-Hilbert problem in the following way. This problem is Fredholm in the class $C_\lambda^\mu,\ \lambda \in R^m$, if and only if it is of normal type and the condition

$$x_j(\zeta) \neq 0,\ \operatorname{Re}\zeta = \lambda_j,\ 1 \le j \le m, \tag{32}$$

holds. In this case its index æ is given by the formula

$$æ = -\frac{1}{\pi}\arg\det G\big|_\Gamma - \sum_{j=1}^m \left[\frac{1}{2\pi i}\ln\frac{x_j}{y_j}\bigg|_{-0} + æ_j(-0,\lambda_j)\right] + l(2-s), \tag{33}$$

where $æ_j(-0,\lambda_j)$ is defined with respect to the function x_j in (31) above.

Let us consider the special case where the function G is continuous and its values $G(\tau_j)$ commute with $\theta_j(J)$:

$$G(\tau_{jk}) = G(\tau_j),\ k = 1,2; \quad G(\tau_j)\theta_j(J) = \theta_j(J)G(\tau_j),\ 1 \le j \le m.$$

The last condition is obviously fulfilled for $l = 1$. This condition also holds for smooth curve Γ because in this case $\theta_j(w) = \pi$ and therefore $\theta_j(J) = \pi$. In combination with (24) these conditions provide the equality $x_j = y_j$. Hence, with regard to (31) the formula (33) can be transformed into

$$æ = -\frac{1}{\pi}\arg\det G\big|_\Gamma - \sum_{j=1}^m \sum_{\nu\in\sigma(J)} l_\nu \left[1 + \frac{\theta_j(\nu)}{\pi}\lambda_j\right] + l(2-s), \tag{34}$$

4. Fredholm solvability of the original problem

Let us turn to the original problem (1), (2), which is equivalent to (13), (14). The main part of the latter is the problem (18) with specific data

$$G(t) = B,\ G^0(t) = b(t)B, \tag{35}$$

where the matrix $B \in C^{l\times l}$ appears in (9). So the problem (1), (2) is of normal type if

$$\det B \neq 0. \tag{36}$$

The matrix B is not defined uniquely by the system (1). But it may be shown [19] that (36) is, in fact, equivalent to the condition

$$\det\left(\int_{-\infty}^{\infty} P^{-1}(t)dt\right) \neq 0,$$

where the matrix polynomial $P(w)$ from (8) is defined by the coefficients of the system (1). According to Bitsadze [20], the elliptic systems (1) possessing this property are called weakly connected.

The matrices X and X^0 corresponding to the problem (18), (35) have a special kind. Let us put

$$\begin{gathered}\hat{b} = \text{diag}\ (\hat{b}_1, \dots, \hat{b}_m),\quad \hat{b}_j = \text{diag}\ (b(\tau_{j1}, b(\tau_{j2}),\\ X(\alpha) = \{X_{ij}(\alpha)\}_1^m,\quad X_{ij}(\alpha) = \{X_{ijkr}(\alpha)\}_1^2,\\ X_{ijkr}(\alpha) = \begin{cases} B[\alpha'(\tau_{ik})]^\zeta(J),\ r = 1, \alpha(\tau_{ik}) = \tau_j,\\ \overline{B}[\alpha'(\tau_{ik})]^\zeta(\overline{J}),\ r = 2, \alpha(\tau_{ik}) = \tau_j,\\ 0,\ \alpha(\tau_{ik}) \neq \tau_j.\end{cases}\end{gathered} \tag{37}$$

Then $X = X(e), X^0 = \hat{b}X(\alpha)$, where $e(t) = t$ is an identical shift.

The problem (13), (14) is a finite-dimensional perturbation of the problem (18), (35). It follows from the well-known theory of Fredholm operators that these problems are Fredholm equivalent; their indices æ and æ$'$, respectively, are connected with the relation æ $=$ æ$' + (s-2)l$. So Theorem 1 produces the following result.

Theorem 3. *The problem* (1), (2) *is Fredholm in the class* $C^\mu_{(\lambda)}$, $0 < \lambda < 1$, *if and only if the system* (1) *is weakly connected and* $\det(X + \hat{b}X(\alpha))(\zeta) \neq 0,\ \text{Re}\,\zeta = \lambda$. *In this case its index* æ *is given by the formula*

$$æ = -\frac{1}{2\pi i}\ln\det[(X + \hat{b}X(\alpha))Y^{-1}]\big|_{-0} - æ(-0, \lambda). \tag{38}$$

In view of (28) this formula can be rewritten in the form

$$æ = -\frac{1}{2\pi i} \ln \det[(X + \hat{b}X(\alpha))Y^{-1}]\big|_{+0} - æ(+0, \lambda) - ml.$$

In particular,

$$æ = -\frac{1}{2\pi i} \ln \det[(X + \hat{b}X(\alpha))Y^{-1}]\big|_{\lambda} - ml, \; 0 < \lambda < 1/2.$$

In a similar way we can apply Theorem 2 and its corollary to the considered problem. For certainty, let us restrict ourselves to the following case.

Theorem 4. *Suppose that the system* (1) *is weakly connected and it has a solution* $u \in C^{\mu}_{-0}(\overline{D}, F)$ *(more exactly, the function* ϕ *in* (10) *belongs to the class* C^{μ}_{-0}*). Suppose that*

$$f = (u + bu \circ \alpha)\big|_{\Gamma} \in C^{\mu}_{+0}(\Gamma, F).$$

Then for every sector D_j *in* (4) *there is* $c_k(\zeta) \in C^l, \; 0 \le k \le r(\zeta) - 1,$ *such that*

$$u(z) - \sum_{\mathrm{Re}\ \zeta=0} \sum_{k=0}^{r(\zeta)-1} \mathrm{Re}\ \Big\{B(z - \tau_j)^{\zeta}(J)[\ln(z - \tau_j)(J)]^k c_k(\zeta)\Big\} \in C^{\mu}_{+0}(\overline{D}_j, \tau_j).$$

In particular, if $r(\zeta) = 0$ *for* $\zeta \ne 0, \mathrm{Re}\,\zeta = 0,$ *and* $r(0) \le 1,$ *then* $u \in C^{\mu}_{(+0)}$.

Let us consider the Dirichlet problem corresponding to the case $b = 0$. The Fredholm property of this problem is given by the condition (32), where x_j is defined in (31) with respect to $G = B$. With regard to (33), the index formula (38) transforms into

$$æ = -\sum_{j=1}^{m} \left[\frac{1}{2\pi i} \ln \frac{x_j}{y_j}\bigg|_{-0} + æ_j(-0, \lambda_j)\right].$$

As at well as in the end of Section 3, we can consider the special case where $B\theta_j(J) = \theta_j(J)B$ for all j. In this case, with regard to (34), the above index formula in the class $C^{\mu}_{(\lambda)}$,$0 < \lambda < 1$, takes the following form:

$$æ = -\sum_{j=1}^{m} \sum_{\nu \in \sigma(J)} l_{\nu} \left[1 + \theta_j(\nu)\lambda_j/\pi\right], \tag{39}$$

where [ ] means the integer part of a number. In particular, $æ = -ml$ in the class $C^{\mu}_{(\lambda)}$, $0 < \lambda < 1/2$.

There are cases where the last term in the boundary condition (2) has no affect on the Fredholm solvability of the problem.

Lemma 2. *Under assumptions of Lemma 1 the equality*

$$\det[X + \hat{b}X(\alpha)](\zeta) = \det X(\zeta). \qquad (40)$$

is valid.

Proof. According to (16), we can enumerate the set F in such a way that $F \setminus F_1 = \{\tau_j,\ 1 \le j \le m_1\}$, $F_1 \setminus F_2 = \{\tau_j,\ m_1 + 1 \le j \le m_2\}$ and so on. Then with regard to (17), (37) it follows that $\hat{b}_i X_{ij}(\alpha) = 0, \quad i \ge j$. Hence

$$(X + \hat{b}X(\alpha))_{ij} = 0, \quad i > j; \quad (X + \hat{b}X(\alpha))_{ii} = X_{ii}.$$

This property provides the relation (40). ◄

Lemma 2 shows that under its conditions the indices of the problem (2) and the the ones of the Dirichlet problem coincide. As a corollary, let us consider the scalar equation (1), i.e. the case $l = 1$.

Theorem 5. *Suppose that* $l = 1$, *the inequality* $|b| \le 1$ *holds and the conditions* (16),(17) *are fulfilled. Then under compatibility conditions the problem* (1), (2) *is uniquely solved in the class* $C^{\mu}_{(\lambda)}$, $0 < \lambda < 1/2$.

Proof. In scalar case the condition $B\theta_j(J) = \theta_j(J)B$ is obvious and therefore we can follow the formula (38) for the Dirichlet problem. By virtue of Lemma 2 it is also valid for the problem (1), (2). So the index of the problem in the class $C^{\mu}_{(\lambda)}$, $0 < \lambda < 1/2$, is equal to $-m$. Taking into account Lemma 1, it remains to prove the uniqueness of the solution in this class.

Let $u \in C(\overline{D})$ be the solution of the homogeneous problem. By virtue of the inequality $|b| \le 1$ and the maximum of principle we conclude that the maximum of $|u(z)|$ can be taken at the point $\tau \in F$. But in view of Lemma 1 all values $u(\tau) = 0$ for $\tau \in F$ and therefore $u = 0$. ◄

5. Conclusion

Let a domain $D \subseteq \mathbb{R}$ be bounded by piece-wise smooth contour. In this domain Bitsadze-Samarski problem (1), (2) is considered under general assumption with respect to the shift. The Fredholm theorem for this problem is proved and the index formula is obtained.

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