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Unique Continuation of the Quasilinear Elliptic Equation on Lebesgue Spaces L_p

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Abstract. In this paper we make the convolution between ϕ, the fundamental solution of the Laplace equation, and function V that belongs to the space $L_{\frac{n}{p}}(\mathbb{R}^n)$. Since this convolution solves Poisson's equation $-\Delta z = V$, we use this result to derive Fefferman's inequality, which will be the cornerstone in the proof of our main result, which deals with the unique continuation property of the nonnegative solution of the quasilinear elliptic equation $\operatorname{div}A(x, u, \nabla u) = B(x, u, \nabla u)$, whose coefficients belong to the $L_{\frac{n}{p}}(\mathbb{R}^n)$ space.

Key Words and Phrases: unique continuation, Fefferman's inequality, Hedberg's inequality, doubling condition.

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1. Introduction

Let's define all the function spaces to be used throughout this paper.

Let $\Omega \subset \mathbb{R}^n$. The Sobolev space $W^{1,p}(\Omega)$ consists of all integrable functions $u : \Omega \longrightarrow \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq 1$, $D^\alpha u$ exists in the weak sense and belongs to $L_p(\Omega)$. We write:

$$C_c^2(\Omega) = \{u : \mathbb{R}^n \longrightarrow \mathbb{R} \ : \ u \text{ is two times continuously differentiable with compact support}\},$$

and

$$C_c^\infty(\Omega) = \{u : \mathbb{R}^n \longrightarrow \mathbb{R} \ : \ u \text{ is infinitely continuously differentiable with compact support}\}.$$

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We denote by $W_0^{1,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On the other hand, as usual, we write

$$H_0^1(\Omega) = W_0^{1,p}(\Omega)$$

and

(a) $H^1(\Omega) = W^{1,2}(\Omega)$,

(b) $H^2(\Omega) = W^{2,2}(\Omega)$.

If $u \in W^{1,p}(\Omega)$, we define its norm as

$$\|u\|_{W^{1,p}} = \left(\sum_{|\alpha|\leq 1} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}, \quad \text{if } 1 \leq p < \infty$$

and

$$\|u\|_{W^{1,\infty}} = \sum_{|\alpha|\leq 1} \text{esssup}_\Omega |D^\alpha u|.$$

Let us start with some historical background. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set, let P be a linear operator given by

$$Pu = a^{jk}\partial_{jk}u + b^j\partial_j u + cu,$$

where the coefficients satisfy, for instance, $a^{jk} \in W^{1,\infty}(\Omega)$, $b^j \in L_\infty(\Omega)$, $c \in L_\infty(\Omega)$ and (a^{jk}) is a symmetric matrix satisfying the uniform ellipticity condition for some constant $\lambda > 0$,

$$a^{jk}(x)\xi_j\xi_k \geq \lambda|\xi|^2, \quad \text{for all } x \in \Omega, \text{ and } \xi \in \mathbb{R}^n.$$

A simple example to keep in mind is the elliptic Schrödinger operator $P = -\Delta + V$, where $V \in L_\infty(\Omega)$. The unique continuation principle comes in several different forms:

(c) (Weak) If $u \in H^2(\Omega)$ satisfies $Pu = 0$ in Ω and $u = 0$ in some ball B contained in Ω, then $u = 0$ in Ω.

(d) (Strong) If $u \in H^2(\Omega)$ satisfies $Pu = 0$ in Ω and if u vanishes to infinite order at $x_0 \in \Omega$, in the sense that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x_0,r)} |u|^2 dx = 0, \quad \text{for all } n \geq 0,$$

then $u = 0$ in Ω.

The purpose of this paper is to discuss the unique continuation principle for non-negative solution of the quasilinear elliptic equation

$$\operatorname{div} A(x,u,\nabla u)=B(x,u,\nabla u),$$

where the coefficients belong to the Lebesgue space $L_{\frac{n}{p}}(\mathbb{R}^n)$ $(1\le p<n)$.

In order to do that, we combine the central step in the Moser proof of Harnack's inequality with the Fefferman inequality (Theorem 3 below) and the doubling condition (Corollary 3). To prove our main result (Theorem 5), it is important to point out that in [11, 12, 13] some problems were studied in a different setting and the authors used different techniques than the one we used here to prove it.

2. Basic results

In what follows we gather some known results. However, for completeness and convenience of the reader we include their proofs. We need to state the definition of maximal function. The Hardy-Littlewood maximal function for $x\in\mathbb{R}^n$ is defined as follows:

$$Mf(x)=\sup_{r>0}\frac{1}{m(B(x,r))}\int\limits_{B(x,r)}|f(y)|dy,\quad f\in L^1_{\text{loc}}(\mathbb{R}^n).$$

Here m denotes the Lebesgue measure and $B(x,r)$ is the ball centered at x with radius r.

Lemma 1. *Let μ be a Radon measure in $\mathbb{R}^n$ and $\alpha<n$. Then*

$$\int\limits_{\mathbb{R}^n}\frac{d\mu(y)}{|x-y|^{n-\alpha}}=(n-\alpha)\int\limits_0^\infty r^{\alpha-n-1}\mu(B(x,r))dr.$$

Proof. Using Cavalieri's principle, we can write

$$\begin{aligned}\int\limits_{\mathbb{R}^n}\frac{d\mu(y)}{|x-y|^{n-\alpha}}&=\int\limits_0^\infty\mu\left(\{y:|x-y|^{\alpha-n}>\lambda\}\right)d\lambda\\&=\int\limits_0^\infty\mu\left(\left\{y:|x-y|<(\frac{1}{\lambda})^{\frac{1}{n-\alpha}}\right\}\right)d\lambda\\&=\int\limits_0^\infty\mu\left(B\big(x,(\frac{1}{\lambda})^{\frac{1}{n-\alpha}}\big)\right)d\lambda.\end{aligned}$$

Making the change of variable $\left(\frac{1}{\lambda}\right)^{\frac{1}{n-\alpha}} = r$, we obtain

$$\int_0^\infty \mu\left(B\left(x, \left(\frac{1}{\lambda}\right)^{\frac{1}{n-\alpha}}\right)\right) d\lambda = (n-\alpha)\int_0^\infty r^{\alpha-n-1}\mu\big(B(x,r)\big)dr.$$

Finally

$$\int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}} = (n-\alpha)\int_0^\infty r^{\alpha-n-1}\mu\big(B(x,r)\big)dr. \quad ◀$$

Theorem 1. *If $0<\alpha<n$, $\beta>0$ and $\delta>0$, then for $x\in\mathbb{R}^n$*

$$\int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy \le C_\alpha \delta^\alpha Mf(x),$$

where $C_\alpha = \frac{n}{\alpha} m(B(0,1))$.

Proof. For $x\in\mathbb{R}^n$ and $\delta>0$ we use Lemma 1. Then we obtain

$$\int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy = (n-\alpha)\int_0^\infty \left(\int_{B(x,r)\cap B(x,\delta)} |f(y)|dy\right)\frac{dr}{r^{n-\alpha+1}}$$

$$\le (n-\alpha)\left(m(B(0,1))\int_0^\delta Mf(x)r^n\frac{dr}{r^{n-\alpha+1}} + m(B(0,1))\int_\delta^\infty Mf(x)\delta^n\frac{dr}{r^{n-\alpha+1}}\right)$$

$$= \frac{n}{\alpha} m(B(0,1))Mf(x)\delta^\alpha,$$

which ends the proof. ◀

Theorem 2 (Hedberg inequality)**.** *Let $0<\alpha<n$ and $f\in L_p(\mathbb{R}^n)$. Then for $1\le p<\frac{n}{\alpha}$ we have the following pointwise inequality:*

$$|I_\alpha f(x)| \le \|f\|_p^{\frac{p\alpha}{n}} (Mf(x))^{1-\frac{p\alpha}{n}}.$$

Proof. For $x\in\mathbb{R}^n$ and $\delta>0$ we have

$$|I_\alpha f(x)| \le \int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy + \int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy.$$

By Theorem 1, we obtain

$$\begin{aligned}
\int\limits_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy &= (n-\alpha)\int\limits_0^\infty \left(\int\limits_{B(x,r)\cap B(x,\delta)} |f(y)|dy \right) \frac{dr}{r^{n-\alpha+1}} \\
&\leq (n-\alpha)\left(m(B(0,1))\int\limits_0^\delta Mf(x)r^n \frac{dr}{r^{n-\alpha+1}} \right. \\
&\quad \left. + m(B(0,1))\int\limits_\delta^\infty Mf(x)\delta^n \frac{dr}{r^{n-\alpha+1}} \right) \\
&\leq \frac{n}{\alpha} m(B(0,1))Mf(x)\delta^\alpha. \qquad (1)
\end{aligned}$$

Now, for $\delta > 0$ the Hölder inequality implies that

$$\begin{aligned}
\int\limits_{\mathbb{R}^n\setminus B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}dy &\leq \|f\|_p \left(\int\limits_{\mathbb{R}^n\setminus B(x,\delta)} |x-y|^{(\alpha-n)q}dy \right)^{\frac{1}{q}} \\
&= \|f\|_p \left(m(B(0,1))\int\limits_\delta^\infty r^{n-1-q(n-\alpha)}dr \right)^{\frac{1}{q}} \\
&\leq \frac{m(B(0,1))}{n-q(n-\alpha)} \|f\|_p \delta^{\alpha-\frac{n}{p}}. \qquad (2)
\end{aligned}$$

Finally, from (1) and (2), we get

$$\left| \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}dxy \right| \leq \frac{m(B(0,1))}{n-q(n-\alpha)} \left(\delta^\alpha Mf(x) + \|f\|_p \delta^{\alpha-\frac{n}{p}} \right). \qquad (3)$$

If we choose $\delta = \left(\frac{Mf(x)}{\|f\|_p}\right)^{-\frac{p}{n}}$, then (3) transforms into

$$|I_\alpha f(x)| \leq \frac{m(B(0,1))}{n-q(n-\alpha)} (Mf(x))^{1-\frac{\alpha p}{n}} \|f\|_p^{\frac{\alpha p}{n}}. \quad ◀$$

3. The Poisson equation and Fefferman's inequality

Let us consider the following problem (Dirichlet problem):

$$\begin{cases} -\Delta z = V & \text{on } B \\ z = 0 & \text{on } \partial B. \end{cases} \tag{4}$$

The equation $-\Delta z = V$ is known as the Poisson equation. It is well-known that the solution of the above problem is given by the convolution

$$z(x) = \int_{\mathbb{R}^n} \phi(x-y)V(y)dy,$$

where ϕ is the fundamental solution of the Laplace equation, and if $V \in C_c^2(\mathbb{R}^n)$, it is clear that $z \in C_c^2(\mathbb{R}^n)$ (see [9] for details).

Furthermore, z can be written as

$$z(x) = \frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla z(y) \cdot (x-y)}{|x-y|^n} dy,$$

where w_{n-1} represents the $(n-1)$-dimensional measure of the sphere S^{n-1}. Then

$$\nabla z(x) = \frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla^2 z(y) \cdot (x-y)}{|x-y|^n} dy.$$

Thus

$$|\nabla z(x)| \leq \frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla^2 z(y)|}{|x-y|^{n-1}} dy.$$

(It is known that $\nabla^2 z = \Delta z$.)

Definition 1. *A function $w(x) \geq 0$ is said to be of A_1-class if*

$$Mw(x) \leq C_1 w(x)$$

for almost all $x \in \mathbb{R}^n$ and for some constant C_1.

Our next task is going to be to state the Fefferman inequality, assuming that V belongs to

$$A_1 \cap L_{\frac{n}{p}}(\mathbb{R}^n) \cap C_c^2(\mathbb{R}^n) \quad \text{with } 1 \leq p < \frac{n}{p}.$$

Note that, since $\Omega \subset \mathbb{R}^n$ is a bounded subset, we have $m(\Omega) < \infty$, implying that $L_{\frac{n}{p}}(\Omega) \subset L_p(\Omega)$ $(1 \leq p < \frac{n}{p})$.

Zamboni proved in [15] the Fefferman inequality allowing V to be in a generalized Kato class, see also [5]. On the other hand, Castillo, Ramos and Rojas in [7] proved the Fefferman inequality allowing V to belong to the Kato class with $p = 2$. For definitions and details see [4, 5, 6, 7, 15].

Based on the ideas given in [8], we prove the following result.

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set and let $V \in A_1 \cap L_{\frac{n}{p}}(\Omega) \cap C_c^2(\Omega)$, $1 \leq p < \frac{n}{p}$. Then*

$$\int_\Omega |u(x)|^{\frac{n}{p}} V(x)dx \leq \|V\|_{L_p} \int_\Omega |\nabla u(x)|^{\frac{n}{p}} dx.$$

Proof. For any $u \in C_c^\infty(\mathbb{R}^n)$, let us consider a ball B such that $u \in C_c^\infty(B)$ and consider the solution z of the problem (4). Next, using

$$I_1 V(x) = \int_B \frac{|V(y)|}{|x-y|^{n-1}} dy,$$

and invoking the Hölder inequality, the Green formula and Theorem 2, one can see that

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{p}} V(x)dx &= -\int_B |u(x)|^{\frac{n}{p}} \Delta z(x)dx = \int_B \nabla |u(x)|^{\frac{n}{p}} \nabla z(x)dx \\
&= \frac{n}{p} \int_B |u(x)|^{\frac{n}{p}-1} \nabla |u(x)| \nabla z(x)dx \\
&\leq \frac{n}{p} \int_B |u(x)|^{\frac{n}{p}-1} |\nabla u(x)| |\nabla z(x)|dx \\
&\leq \frac{n}{p} \int_B |u(x)|^{\frac{n}{p}-1} |\nabla u(x)| \frac{1}{w_{n-1}} \int_B \frac{|\nabla^2 z(y)|}{|x-y|^{n-1}} dydx \\
&= \frac{n}{p} \int_B |u(x)|^{\frac{n}{p}-1} |\nabla u(x)| \frac{1}{w_{n-1}} \int_B \frac{|\Delta z(y)|}{|x-y|^{n-1}} dydx \\
&= \frac{n}{p} \int_B |u(x)|^{\frac{n}{p}-1} |\nabla u(x)| \frac{1}{w_{n-1}} \int_B \frac{|V(y)|}{|x-y|^{n-1}} dydx \\
&= \frac{n}{p} \int_B |u(x)|^{\frac{n}{p}-1} |\nabla u(x)| I_1 V(x)dx
\end{aligned}$$

$$\leq \frac{C(p,n)}{w_{n-1}} \int\limits_B |u(x)|^{\frac{n}{p}-1} |\nabla u(x)| \left(MV(x)\right)^{1-\frac{p}{n}} \|V\|_{L_p}^{\frac{p}{n}} dx$$

$$= \frac{C(p,n)}{w_{n-1}} \|V\|_{L_p}^{\frac{p}{n}} \int\limits_B |u(x)|^{\frac{n}{p}-1} \left(MV(x)\right)^{1-\frac{p}{n}} |\nabla u(x)| dx$$

$$\leq \frac{C(p,n)}{w_{n-1}} \|V\|_{L_p}^{\frac{p}{n}} \left(\int\limits_B |u(x)|^{\frac{n}{p}} MV(x) dx\right)^{1-\frac{p}{n}} \left(\int\limits_B |\nabla u(x)|^p dx\right)^{\frac{p}{n}}$$

$$\leq \frac{C(p,n)}{w_{n-1}} \|V\|_{L_p}^{\frac{p}{n}} \left(\int\limits_B |u(x)|^{\frac{n}{p}} V(x) dx\right)^{1-\frac{p}{n}} \left(\int\limits_B |\nabla u(x)|^p dx\right)^{\frac{p}{n}}.$$

Finally

$$\left(\int\limits_B |u(x)|^{\frac{n}{p}} V(x) dx\right)^{\frac{p}{n}} \leq \frac{C(p,n)}{w_{n-1}} \|V\|_{L_p}^{\frac{p}{n}} \left(\int\limits_B |\nabla u(x)|^p\right)^{\frac{p}{n}}.$$

Therefore,

$$\int\limits_B |u(x)|^{\frac{n}{p}} V(x) dx \leq \frac{C(p,n)}{w_{n-1}} \|V\|_{L_p} \int\limits_B |\nabla u(x)|^{\frac{n}{p}} dx. \quad \blacktriangleleft$$

4. Space of functions of bounded mean oscillation (BMO)

In the same sense that the Hardy space $H^1(\mathbb{R}^n)$ is a substitute for $L^1(\mathbb{R}^n)$, it will turn out that the space BMO$(\mathbb{R}^n)$ (the space of "bounded mean oscillation") is the corresponding natural substitute for the space $L^\infty(\mathbb{R}^n)$ of bounded functions on $\mathbb{R}^n$.

A locally integrable function f belongs to BMO if

$$\frac{1}{m(B_r)} \int\limits_{B_r} |f(x) - f_{B_r}| dm \leq A \tag{5}$$

holds for all balls $B_r = B(x,r)$, where

$$f_{B_r} = \frac{1}{m(B_r)} \int\limits_{B_r} f dm = \fint\limits_{B_r} f dm$$

denotes the mean value of f over the ball and m stands for the Lebesgue measure on $\mathbb{R}^n$. The inequality (5) asserts that over any ball B, the average oscillation of f is bounded. The smallest bound A for which (5) is satisfied is then taken to be the norm of f in this space, and is denoted by $\|f\|_{\mathrm{BMO}}$. Let us begin by making some remarks about functions that are in BMO.

The following result is due to John-Niremberg. If $f \in \mathrm{BMO}$, then there exist positive constants C_1 and C_2 such that, for every $r > 0$ and every ball B_r

$$m(\{x \in B_r : |f(x) - f_{B_r}| > \lambda\}) \leq Ce^{-C_2\lambda/\|f\|_{\mathrm{BMO}}} m(B_r).$$

One consequence of the above result is the following corollary.

Corollary 1. *If $f \in BMO$, then there exist positive constants C_1 and C_2 such that*

$$\int\limits_{B_r} e^{C|f(x)-f_{B_r}|} dm \leq \Big(\frac{C_1 C}{C_2 - C} + 1\Big) m(B_r)$$

for every ball B_r and $0 < C < C_2$.

Proof. Let us define $\varphi(x) = e^x - 1$. Notice that $\varphi(0) = 0$, and hence

$$\begin{aligned}\int\limits_{B_r} (e^{C|f(x)-f_{B_r}|} - 1) =& C \int\limits_0^\infty e^{C\lambda} m(\{x \in B_r : |f(x) - f_{B_r}| > \lambda\}) d\lambda \\ \leq& CC_1 \Big[\int\limits_0^\infty e^{-(C_2-C)\lambda} d\lambda\Big] m(B_r).\end{aligned}$$

From the above inequality we have

$$\int\limits_{B_r} e^{C|f(x)-f_{B_r}|} dm \leq \Big(\frac{CC_1}{C_2 - C} + 1\Big) m(B_r). \quad \blacktriangleleft$$

5. Useful results

The following tools are the spinal cord of our main results.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $u \in W^{1,p}(\Omega)$. Then we have for any measurable set $B \subset \Omega$ with $m(B) > 0$*

$$\left(\int\limits_\Omega |u(x) - u_B|^{\frac{p}{n}} dx\right)^{\frac{p}{n}} \leq w_n \left[\frac{m(\Omega)}{w_n}\right]^{1/n} \frac{(\dim(\Omega))^n}{m(B)} \|\nabla u\|_{L_{\frac{n}{p}}}.$$

Theorem 4. *Let $B_R \subset \mathbb{R}^n$, $u \in W^{1,p}(B_R)$ and assume that for all $B_r \subset B_R$, there exists a constant $k > 0$ such that*

$$\left(\int_{B_r} |\nabla u(x)|^{\frac{n}{p}} dx\right)^{\frac{p}{n}} \le k r^{p-1}.$$

Then there exist two constants β and c depending only on k, p and n such that

$$\left(\int_{B_R} e^{\beta u(x)} dx\right)\left(\int_{B_R} e^{-\beta u(x)} dx\right) \le c[m(B_R)]^2.$$

Proof. It is enough to take $u \in C_c^\infty(\mathbb{R}^n)$ supported in $B_r = B(x_0, r)$. From Lemma 2 we know that

$$\left(\frac{1}{m(B_r)}\int_{B_r} |u(x) - u_{B_r}|^{\frac{n}{p}} dx\right)^{\frac{p}{n}} \le n w_n \frac{(m(B_R))^n}{m(B_r)} \|\nabla u\|_{L_{\frac{n}{p}}}.$$

Thus by Hölder's inequality we have

$$\begin{aligned}
\frac{1}{m(B_r)}\int_{B_r} |u(x) - u_{B_r}| dx \le & \left(\frac{1}{m(B_r)}\int_{B_r} |u(x) - u_{B_r}|^{\frac{n}{p}} dx\right)^{\frac{p}{n}} \\
\le & n w_n \frac{(m(B_R))^n}{m(B_r)} \|\nabla u\|_{L_{\frac{n}{p}}} \\
= & n w_n \frac{(m(B_R))^n}{m(B_r)} \left(\int_{B_r} |\nabla u(y)|^{\frac{n}{p}} dy\right)^{\frac{p}{n}} \\
\le & n w_n \frac{(m(B_R))^n}{m(B_r)} r^{p-1} \\
\le & n w_n \frac{(m(B_R))^n}{m(B_r)} k r^{p-1},
\end{aligned}$$

which implies $u \in \mathrm{BMO}$. Then by Corollary 1 we obtain

$$\int_{B_R} e^{\beta |u(x) - u_{B_R}|} dx \le c m(B_R), \tag{6}$$

where c is the constant that appeared in Corollary 1. Next, (6) implies that

$$\int\limits_{B_R} e^{-\beta(u(x)-u_{B_R})}dx \leq \int\limits_{B_R} e^{\beta|u(x)-u_{B_R}|}dx \leq cm(B_R)$$

and

$$\int\limits_{B_R} e^{\beta(u(x)-u_{B_R})}dx \leq \int\limits_{B_R} e^{\beta|u(x)-u_{B_R}|}dx \leq cm(B_R).$$

Thus

$$\begin{aligned}\left(\int\limits_{B_R} e^{\beta u(x)}dx\right)\left(\int\limits_{B_R} e^{-\beta u(x)}dx\right) &= \left(\int\limits_{B_R} e^{\beta(u(x)-u_{B_R})}dx\right)\left(\int\limits_{B_R} e^{-\beta(u(x)-u_{B_R})}dx\right)\\ &\leq (cm(B_R))^2.\end{aligned}$$

The theorem is proved. ◄

Corollary 2. *If $u \in W^{1,p}(B_R)$ is such that for all $B_r \subset B_R$ there exits a constant $k > 0$ which satisfy*

$$\left(\int\limits_{B_r} |\nabla \log u(x)|^{\frac{n}{p}}dx\right)^{\frac{p}{n}} \leq kr^{p-1}, \tag{7}$$

then there exist two positive constants β and c depending only on k, p and n such that

$$\left(\int\limits_{B_R} |u|^{\beta}dx\right)\left(\int\limits_{B_R} |u|^{-\beta}dx\right) \leq c(m(B_R))^2.$$

Proof. By Theorem 4 there exist two positive constants β and c depending only on k, p and n such that

$$\left(\int\limits_{B_R} e^{\beta \log u(x)}dx\right)\left(\int\limits_{B_R} e^{-\beta \log u(x)}dx\right) \leq c[m(B_R)]^2,$$

which implies

$$\left(\int\limits_{B_R} |u|^{\beta}dx\right)\left(\int\limits_{B_R} |u|^{-\beta}dx\right) \leq c(m(B_R))^2,$$

as was announced. ◄

Corollary 3. *Suppose that* (7) *holds. Then there exists a positive constant* c *such that*

$$\int_{B_{2r}} |u|^\beta dx \le c \int_{B_r} |u|^\beta dx,$$

where $B_{2r} \subset \Omega$. *The above inequality is known as a doubling condition.*

Proof. If (7) holds, then we have

$$\left(\int_{B_r} |u|^\beta dx\right)^{1/2} \left(\int_{B_r} |u|^{-\beta} dx\right)^{1/2} \le c^{1/2}(m(B_r)).$$

From this we obtain

$$\left(\int_{B_r} |u|^{-\beta} dx\right)^{1/2} \le c^{1/2}(m(B_r)) \left(\int_{B_r} |u|^\beta dx\right)^{-1/2}. \tag{8}$$

On the other hand, by the Schwartz inequality we have

$$\begin{aligned} m(B_r) &= \int_{B_r} |u|^{\frac{\beta}{2}} |u|^{-\frac{\beta}{2}} dx \\ &\le \left(\int_{B_r} |u|^\beta dx\right)^{1/2} \left(\int_{B_r} |u|^{-\beta} dx\right)^{1/2} \\ &\le \left(\int_{B_r} |u|^\beta dx\right)^{1/2} \left(\int_{B_{2r}} |u|^{-\beta} dx\right)^{1/2} \\ &\le c^{1/2}(m(B_r)) \left(\int_{B_r} |u|^\beta dx\right)^{1/2} \left(\int_{B_{2r}} |u|^\beta dx\right)^{-1/2}. \end{aligned}$$

In the last inequality we use (8). Thus

$$m(B_r) \le c^{1/2}(m(B_r)) \left(\frac{\int_{B_r} |u|^\beta dx}{\int_{B_{2r}} |u|^\beta dx} \right)^{1/2}.$$

Finally

$$\int\limits_{B_{2r}} |u|^\beta dx \leq c \int\limits_{B_r} |u|^\beta dx. \quad ◀$$

6. The unique continuation principle for elliptic partial differential equations

This principle which states that any solution of an elliptic equation vanishing in a small ball must be identically zero (see, (c) in page 137), is a fundamental property that has various applications e.g. in solvability questions, inverse problems, and control theory.

Definition 2. *Assume $w \in L^1_{\text{loc}}(\Omega)$, $w(x) \neq 0$ for all $x \in \Omega$. We say that w has a zero of infinite order at $x_0 \in \Omega$ if*

$$\lim_{R \to 0} \frac{\int\limits_{B(x_0,R)} w(x)dx}{(m(B(x_0,R)))^k} = 0, \quad \forall k > 0.$$

Which is equivalent to say that a function $u \in L^p_{\text{loc}}(\Omega)$ vanishes of infinite order at point x_0 if for any natural number N there exists a constant C_N such that

$$\int\limits_{B(x_0,r)} |u(x)|^p \, dx \leq C_N r^N$$

for all $N \in \mathbb{N}$ and for a small positive number r. Here

$$B(x_0, r) = \{y \in \mathbb{R}^n : |y - x_0| < r\}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. It is not hard to check that the function

$$w(x) = \frac{1}{|x|^{n+1}} e^{\frac{1}{|x|}} \quad (x \in \mathbb{R}^n)$$

vanishes of infinite order at $x_0 = 0$.

Lemma 3. *Assume $w \in L^1_{\text{loc}}(\Omega)$ and $w(x) \neq 0$ for all $x \in \Omega$. If there exists a positive constant C such that*

$$\int\limits_{B(x_0,2R)} w(x)dx \leq C \int\limits_{B(x_0,R)} w(x)dx, \quad \forall R > 0, \tag{9}$$

where $B(x_0, 2R) \subset \Omega$, then $w(x)$ has no zero of infinite order in Ω.

Proof. Suppose that

$$\lim_{R\to 0}\frac{\displaystyle\int_{B(x_0,R)} w(x)dx}{(m(B(x_0,R)))^\lambda}=0,\quad \lambda>0.$$

Next, applying (9) several times, we have

$$\begin{aligned}\int_{B(x_0,R)} w(x)dx \leq & C\int_{B\left(x_0,\frac{R}{2}\right)} w(x)dx \leq C\left(C\int_{B\left(x_0,\frac{R}{2^2}\right)} w(x)dx\right)\\ \leq & C^2\left(C\int_{B\left(x_0,\frac{R}{2^3}\right)} w(x)dx\right)\\ & \vdots\\ \leq & C^k\left(C\int_{B\left(x_0,\frac{R}{2^k}\right)} w(x)dx\right).\end{aligned}$$

Thus

$$\int_{B(x_0,R)} w(x)dx \leq C^k\int_{B\left(x_0,\frac{R}{2^k}\right)} w(x)dx.$$

Then

$$\begin{aligned}\int_{B(x_0,R)} w(x)dx \leq & C^k\left(m\left(B(x_0,\frac{R}{2^k})\right)\right)^\lambda\frac{1}{\left(m(B\left(x_0,\frac{R}{2^k}\right))\right)^\lambda}\int_{B\left(x_0,\frac{R}{2^k}\right)} w(x)dx\\ = & C^k(2^{-k}R)^{n\lambda}\left(m(B(0,1))\right)^\lambda\frac{1}{\left(m\left(B\left(x_0,\frac{R}{2^k}\right)\right)\right)^\lambda}\int_{B\left(x_0,\frac{R}{2^k}\right)} w(x)dx.\end{aligned}$$

Now, let us choose λ in such a way that $C2^{-\lambda}=1$. Thus

$$\int_{B(x_0,R)} w(x)dx \leq c_1R^{n\lambda}\left(\frac{1}{\left(m\left(B\left(x,\frac{R}{2^k}\right)\right)\right)^\lambda}\right)\int_{B\left(x_0,\frac{R}{2^k}\right)} w(x)dx \to 0$$

if $k \to \infty$, where $c_1 = (m(B(0,1)))^\lambda$. Therefore $w = 0$, which contradict the fact that $w(x) \neq 0$ for all $x \in \Omega$. ◄

Let Ω be a bounded open set in $\mathbb{R}^n$. The equation we consider is of the form

$$\mathrm{div}A(x,u,\nabla u) = B(x,u,\nabla u), \tag{10}$$

where

$$A(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and

$$B(x,u,\xi) : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

are two continuous functions satisfying the conditions

$$\begin{cases} |A(x,u,\xi)| & \leq a|\xi|^{\frac{n}{p}-1} + b(x)|u|^{\frac{n}{p}-1} \\ |B(x,u,\xi)| & \leq c(x)|\xi|^{\frac{n}{p}-1} + d(x)|u|^{\frac{n}{p}-1} \\ A(x,u,\xi) & \geq |\xi|^{\frac{n}{p}} - d(x)|u|^{\frac{n}{p}} \end{cases} \tag{11}$$

for almost all $x \in \Omega$, $\forall u \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$. We assume that p is a fixed point in $(1,n)$, a is a positive constant and b, c and d are measurable functions in Ω whose extensions with zero values outside Ω are such that

$$b^{\frac{n}{n-p}},\ c^{\frac{n}{p}},\ d \in L_{\frac{n}{p}}(\mathbb{R}^n). \tag{12}$$

For more applications such as some existence results for singular elliptic problems see Mâagli and Zribi 2001, Bachar et al 2002, Bachar et al 2003, Zeddini 2003, Bachar and Mâagli 2005, Mâagli and Zribi 2005 [1, 2, 3, 12, 16, 13]. For the study of the unique continuation property on different framework see, for instance, the works of Tao and Zhang 2007 and Granlund and Marola 2012 [10, 14].

Definition 3. *We say that a function $u \in H_0^{1,p}(\Omega)$ is a local weak solution of* (10) *in Ω if*

$$\int_\Omega \{A(x,u(x),\nabla u(x))\nabla\phi(x) + B(x,u(x),\nabla u(x))\phi(x)\}dx = 0 \tag{13}$$

for every $\phi \in C_c^\infty(\Omega)$.

Theorem 5. *Let $u \in H^1(\Omega)$, $u \geq 0$, $u \neq 0$, be a solution of* (10) *satisfying* (11) *and* (12). *Then u has zero of infinite order in Ω.*

Proof. Let $x_0 \in \Omega$, and let $B(x_0, R)$ be a ball such that $B(x_0, 2R)$ is contained in Ω. Consider any B_h contained in $B(x_0, R)$. Let η be a non negative smooth function with support in B_{2h}. Using $\phi = \eta u^{1-p}$ as a test function in (13) we get

$$\int_\Omega |\nabla \log u(x)|^{\frac{n}{p}} \eta^{\frac{n}{p}}(x) dx \leq C_1(p, a) \left\{ \int_\Omega |\nabla \eta(x)|^{\frac{n}{p}} dx + \int_\Omega V(x) \eta^{\frac{n}{p}}(x) dx \right\}, \quad (14)$$

where V is defined by

$$V = b^{\frac{n}{n-p}} + c^{\frac{n}{p}} + d.$$

By Theorem 3, we have

$$\int_\Omega V(x) \eta^{\frac{n}{p}}(x) dx \leq C_2 \int_\Omega |\nabla \eta(x)|^{\frac{n}{p}} dx.$$

Inserting this in inequality (14), we obtain

$$\int_\Omega \eta^{\frac{n}{p}}(x) |\nabla \log u(x)|^{\frac{n}{p}} dx \leq C_3 \int_\Omega |\nabla \eta(x)|^{\frac{n}{p}} dx. \quad (15)$$

Choosing η so that $\eta = 1$ in B_h and $|\nabla \eta| \leq 3/h$, by (15) we have

$$\int_{B_h} |\nabla \log u(x)|^{\frac{n}{p}} dx \leq C_4 h^{n(p-1)}.$$

Therefore, by Corollary 2, we have

$$\int_{B_h} u^\delta(x) dx \int_{B_h} u^{-\delta}(x) \leq C m(B_h)^2.$$

Therefore, Corollary 3 implies

$$\int_{B_{2r}} |u|^\delta dx \leq c \int_{B_r} |u|^\delta dx.$$

That is, the assumption of Lemma 3 is satisfied, hence the conclusion follows for u^δ and thus for u. ◄

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