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On Conditions of the Completeness of Some Systems of Bessel Functions in the Space $L^2((0;1);x^{2p}dx)$

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Abstract. We establish the necessary and sufficient conditions for the completeness of the system $(x^{-p-1}\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N})$ in the space $L^2((0;1);x^{2p}dx)$, where J_ν is the Bessel function of the first kind of index $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ is an arbitrary sequence of distinct nonzero complex numbers.

Key Words and Phrases: Bessel function, entire function, completeness, minimality, basis, Jensen formula.

2010 Mathematics Subject Classifications: 42A65, 42C30, 33C10, 30B60, 30D20, 44A15

1. Introduction

Let $L^2((0;1);t^\alpha dt)$, $\alpha \in \mathbb{R}$, denote the weighted Lebesgue space of all measurable functions $f : (0;1) \to \mathbb{C}$, for which

$$\int_0^1 t^\alpha |f(t)|^2\, dt < +\infty.$$

Let

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$$

be the Bessel function of the first kind of index ν. For $\nu > -1$ the function J_ν has an infinite set $\{\rho_k : k \in \mathbb{Z}\}$ of real roots, among which ρ_k, $k \in \mathbb{N}$, are the positive roots and $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, are the negative roots (see [5, p. 94], [8, p. 350]). All roots are simple except, perhaps, the root $\rho_0 = 0$. A system $(e_k : k \in \mathbb{N})$ of elements of the Banach space H is called complete if the closure of the linear span of this system coincides with the whole H (see [6, p. 131], [7, p. 4258]). A system of elements $(e_k : k \in \mathbb{N}) \in H$ is said to be minimal if for each $k \in \mathbb{N}$ the element e_k does not belong to the closure of the linear span of all other elements (see [6, p. 131], [7, p. 4258]).

It is known that the approximation properties of the system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ with $\nu > -1$ depends on the properties of the sequence $(\rho_k : k \in \mathbb{N})$. The classical results relate mainly to the case where $(\rho_k : k \in \mathbb{N})$ is a sequence of positive zeros of the function J_ν (see, for instance, [1], [2], [4], [8]). In particular, it is well known that the system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ is an orthogonal basis for the space $L^2(0; 1)$ when $\nu > -1$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of positive zeros of J_ν (see [1], [4], [8, pp. 355–357]). From this it follows that if $\nu > -1$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of positive zeros of J_ν, then the system $(x^{-\nu}J_\nu(x\rho_k) : k \in \mathbb{N})$ is complete and minimal in $L^2((0; 1); x^{2\nu+1}dx)$. The system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ is also complete in $L^2(0; 1)$ if $(\rho_k : k \in \mathbb{N})$ is a sequence of zeros of the function J'_ν (see [8, pp. 347, 356]). From [2] it follows that if $\nu > -1/2$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of distinct positive numbers such that $\rho_k \le \pi(k+\nu/2)$ for all sufficiently large $k \in \mathbb{N}$, then the system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ is complete in $L^2(0; 1)$.

It is important to study the approximation properties of the above systems of Bessel functions if $(\rho_k : k \in \mathbb{N})$ is an arbitrary sequence of complex numbers. The necessary and sufficient conditions have been obtained in [9] for the completeness and minimality of system $(\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N})$ in the space $L^2(0; 1)$ if $\nu \ge -1/2$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of distinct nonzero complex numbers. In [10], some sufficient conditions for the basis property of this system in $L^2(0; 1)$ have been found. In [11], a criterion for the completeness and minimality of more general systems $(\Theta_{k,\nu,p} : k \in \mathbb{N})$, $\Theta_{k,\nu,p}(x) := x^{-p-1}\sqrt{x\rho_k}J_\nu(x\rho_k)$, in the space $L^2((0; 1); x^{2p}dx)$ has been established, where J_ν is the Bessel function of the first kind of index $\nu \ge 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of distinct nonzero complex numbers. For $\nu = 3/2$ and $p = 1$ such criterion is also available in [9]. Those results are formulated in terms of sequences of zeros of functions from some classes of entire functions.

In this paper, using the methods of [7, §§ 3.3, 4.1], [11], we establish some new necessary and sufficient conditions for the completeness of systems $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ in $L^2((0; 1); x^{2p}dx)$ depending on the properties of the sequence $(\rho_k : k \in \mathbb{N})$ of distinct nonzero complex numbers (see Theorems 1–3). This complements the results of papers [9]–[11].

2. Preliminaries

Let $\log^+ x = \max(0; \log x)$ for $x > 0$. By C_1, C_2, ... we denote arbitrary positive constants. To prove our main results we need the following auxiliary lemmas.

Lemma 1. *(see [7, p. 4263]) Let F be an entire function of exponential type $\sigma \le 1$ for which the integral*

$$\int_{-\infty}^{+\infty} \frac{\log^+ |F(x)|}{1+x^2}\, dx$$

exists and let $(z_k : k \in \mathbb{N})$ be a sequence of nonzero roots of the function $F(z)$. Then

$$\sum_{k\in\mathbb{N}} \left| \operatorname{Im} \frac{1}{z_k} \right| < +\infty.$$

Lemma 2. *(see [7, p. 4304]) Let $(\lambda_n : n \in \mathbb{N})$ be a sequence of distinct nonzero complex numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < +\infty.$$

Then the infinite product

$$G(z) = z^m \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right), \quad m \in \mathbb{Z}_+,$$

defines an entire function of minimal type of order 1 with $|G(x)| \le \exp(\theta(|x|))$, $x \in \mathbb{R}$, where $\theta(x)$ is a positive increasing function satisfying

$$\int_0^{+\infty} \frac{\theta(x)}{x^2}\, dx < +\infty.$$

Lemma 3. *(see [7, p. 4303]) Let $\omega(x)$ be a positive nondecreasing function on $[0;+\infty)$ satisfying $\int_0^{+\infty} x^{-2}\omega(x)\, dx < +\infty$. Then, for all $a > 0$, there exists an entire function*

$$F(z) = \int_{-a}^{a} e^{izt}\, d\sigma(t)$$

satisfying $|F(x)| \le \exp(-\omega(|x|))$, $|x| > x_0$, where $\sigma(t)$ is a function of bounded variation on the segment $[-a;a]$.

Lemma 4. *(see [3], [11]) Let $\nu \ge -1/2$. A function f has the representation*

$$f(z) = z^{-\nu} \int_0^1 \sqrt{t} J_\nu(zt)\gamma(t)\, dt$$

with some function $\gamma \in L^2(0;1)$ if and only if it is an even entire function of exponential type $\sigma \le 1$ such that $z^{\nu+1/2} f(z) \in L^2(0;+\infty)$.

Lemma 5. *(see [11]) Let $\nu \geq 1/2$ and $p \in \mathbb{R}$. An entire function Ω has the representation*

$$\Omega(z) = z^{-\nu} \int_0^1 \sqrt{t} J_\nu(tz) t^{p-1} h(t)\, dt \tag{1}$$

with $h \in L^2((0;1); x^{2p}dx)$ if and only if it is an even entire function of exponential type $\sigma \leq 1$ such that $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0;+\infty)$. In this case,

$$h(t) = t^{-p} \int_0^{+\infty} \sqrt{tz} J_{\nu-1}(tz) z^{-\nu+1/2}(z^{2\nu}\Omega(z))'\, dz.$$

Lemma 6. *(see [11]) Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ if $k \neq n$. For a system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ to be incomplete in the space $L^2((0;1); x^{2p}dx)$ it is necessary and sufficient that the sequence $(\rho_k : k \in \mathbb{Z} \setminus \{0\})$, $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, be a subsequence of zeros of some nonzero even entire function Ω of exponential type $\sigma \leq 1$ satisfying $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0;+\infty)$.*

Lemma 7. *(see [11]) Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and an entire function Ω be defined by the formula (1). Then for all $z = x + iy = re^{i\varphi} \in \mathbb{C}$, we have*

$$|\Omega(z)| \leq C_1(1 + |z|)^{-\nu} \exp(|\operatorname{Im} z|).$$

3. Main results

Theorem 1. *Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of distinct nonzero complex numbers such that $|\operatorname{Im} \rho_k| \geq \delta|\rho_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$. For a system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ to be complete in $L^2((0;1); x^{2p}dx)$ it is necessary and sufficient that*

$$\sum_{k=1}^{\infty} \frac{1}{|\rho_k|} = +\infty. \tag{2}$$

Proof. Necessity. Suppose, to the contrary, that the system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ is not complete in $L^2((0;1); x^{2p}dx)$. Then, by Lemma 6, there exists a nonzero even entire function Ω of exponential type $\sigma \leq 1$ which vanishes at the points ρ_k and satisfies $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0;+\infty)$. By Lemma 5, the function Ω is of the kind (1). Due to Lemma 7, we have $|\Omega(x)| \leq C_1(1 + |x|)^{-\nu}$ for $\nu \geq 1/2$ and all $x \in \mathbb{R}$. This implies

$$\int_{-\infty}^{+\infty} \frac{\log^+ |\Omega(x)|}{1 + x^2}\, dx < +\infty.$$

Therefore, by Lemma 1, we get

$$\sum_{k\in\mathbb{N}}\left|\operatorname{Im}\frac{1}{\rho_k}\right|<+\infty.$$

Since $|\operatorname{Im}\rho_k|\geq\delta|\rho_k|$, $\delta>0$, for all $k\in\mathbb{N}$, and

$$\left|\operatorname{Im}\frac{1}{\rho_k}\right|=\frac{|\operatorname{Im}\rho_k|}{|\rho_k|^2}\geq\frac{\delta}{|\rho_k|},$$

we have $\sum\limits_{k=1}^{\infty}\frac{1}{|\rho_k|}<+\infty$. This contradicts condition (2).

Sufficiency. Let the condition (2) not be fulfilled. Let us prove that the system $(\Theta_{k,\nu,p}:k\in\mathbb{N})$ is not complete in $L^2((0;1);x^{2p}dx)$. Let

$$G(z)=\prod_{k=1}^{\infty}\left(1-\frac{z^2}{\rho_k^2}\right).$$

Since $\sum\limits_{k=1}^{\infty}\frac{1}{|\rho_k|}<+\infty$, then by Lemma 2, the function G is an even entire function of minimal type of order 1 and $|G(x)|\leq\exp(\theta(|x|))$, $x\in\mathbb{R}$, where $\theta:[0;+\infty)\to[0;+\infty)$ is an increasing function satisfying $\int_0^{+\infty}t^{-2}\theta(t)\,dt<+\infty$. Further,

$$\int_0^{+\infty}t^{-2}(1+\nu+1/2)\log(1+t)\,dt<+\infty.$$

Furthermore, according to Lemma 3, there exists an even entire function F of exponential type $\sigma\leq 1$ satisfying

$$|F(x)|\leq\exp(-\theta(|x|)-(1+\nu+1/2)\log(1+|x|)),\quad |x|>x_0.$$

Consider the function $f(z)=G(z)F(z)$. The function f is an even entire function of exponential type $\sigma\leq 1$ vanishing at the points ρ_k and satisfying the estimate

$$|x^{\nu+1/2}f(x)|\leq|x|^{\nu+1/2}\exp(-(1+\nu+1/2)\log(1+|x|))\leq(1+|x|)^{-1},\quad x\in\mathbb{R}.$$

Hence $z^{\nu+1/2}f(z)\in L^2(0;+\infty)$ and by Lemma 4 the function f can be represented in the form $f(z)=z^{-\nu}\int_0^1\sqrt{t}J_\nu(zt)\gamma(t)\,dt$ with some function $\gamma\in L^2(0;1)$. Since $h(t):=t^{1-p}\gamma(t)\in L^2((0;1);t^{2p}dt)$, we have the representation $f(z)=z^{-\nu}\int_0^1\sqrt{t}J_\nu(tz)t^{p-1}h(t)\,dt$ with $h\in L^2((0;1);x^{2p}dx)$. Thus, by Lemmas 5 and 6, the system $(\Theta_{k,\nu,p}:k\in\mathbb{N})$ is incomplete in $L^2((0;1);x^{2p}dx)$. The theorem is proved. ◀

Let $n(t)$ be the number of points of the sequence $(\rho_k : k \in \mathbb{N}) \subset \mathbb{C}$ satisfying the inequality $|\rho_k| \le t$, i.e., $n(t) := \sum_{|\rho_k|\le t} 1$, and let

$$N(r) := \int_0^r \frac{n(t)}{t}\, dt, \quad r > 0.$$

Theorem 2. *Let $\nu \ge 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of distinct nonzero complex numbers. If*

$$\varlimsup_{r\to+\infty} \left(N(r) - \frac{2r}{\pi} + \nu \log(1+r) \right) = +\infty,$$

then the system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ is complete in $L^2((0;1); x^{2p}dx)$.

Proof. It suffices to assume the incompleteness of the system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ and prove that

$$\varlimsup_{r\to+\infty} \left(N(r) - \frac{2r}{\pi} + \nu \log(1+r) \right) < +\infty. \tag{3}$$

By Lemmas 5 and 6, there exists a nonzero even entire function Ω of the form (1) of exponential type $\sigma \le 1$ such that $z^{-\nu+1/2}(z^{2\nu}\Omega(z))' \in L^2(0; +\infty)$ and for which the sequence $(\rho_k : k \in \mathbb{N})$ is a subsequence of zeros. We may suppose $\Omega(0) = 1$. Then, consecutively applying the Jensen formula (see [6], [7, p. 4316]) and Lemma 7, we obtain

$$N(r) \le \frac{1}{2\pi} \int_0^{2\pi} \log|\Omega(re^{i\varphi})|\, d\varphi \le C_2 + \frac{1}{2\pi} \int_0^{2\pi} (r|\sin\varphi| - \nu\log(1+r))\, d\varphi =$$
$$= \frac{2r}{\pi} - \nu\log(1+r) + C_2, \quad r > 0,$$

whence it follows (3). The theorem is proved. ◀

Theorem 3. *Let $\nu \ge 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of distinct nonzero complex numbers. Let $|\rho_k| \le \Delta k + \beta$ for $0 < \Delta \le \pi/2$, $-\Delta < \beta < \Delta(\nu - 1/2)$ and all sufficiently large $k \in \mathbb{N}$. Then the system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ is complete in $L^2((0;1); x^{2p}dx)$.*

Proof. Let $\mu_k = \Delta k + \beta$, $k \in \mathbb{N}$, $n_1(t) = \sum_{\mu_k \le t} 1$ and $N_1(r) = \int_0^r t^{-1} n_1(t)\, dt$. Then $n(t) \ge n_1(t)$, $N(r) \ge N_1(r)$ and $n_1(t) = m$ for $\Delta m + \beta \le t < \Delta(m+1) + \beta$

($n_1(t) = 0$ on $(0;\mu_1)$). Let $r \in [\mu_s;\mu_{s+1})$. Then $s = \frac{r}{\Delta} + O(1)$ as $r \to +\infty$. Therefore, using formula $\log(1+x) = x - \frac{x^2}{2} + O(x^3)$, $x \to 0$, we obtain

$$N_1(r) = \sum_{k=1}^{s-1} \int_{\mu_k}^{\mu_{k+1}} \frac{n_1(t)}{t}\,dt + \int_{\mu_s}^{r} \frac{n_1(t)}{t}\,dt = \sum_{k=1}^{s-1} k \int_{\mu_k}^{\mu_{k+1}} \frac{dt}{t} + \int_{\mu_s}^{r} \frac{s}{t}\,dt =$$

$$= \sum_{k=1}^{s-1} k \log \frac{\mu_{k+1}}{\mu_k} + s \log \frac{r}{\mu_s} = \sum_{k=1}^{s-1} k \log \frac{\Delta(k+1)+\beta}{\Delta k + \beta} + s \log \frac{r}{\Delta s + \beta} =$$

$$= \sum_{k=1}^{s-1} k \log \left(1 + \frac{\Delta}{\Delta k + \beta}\right) + O(1) =$$

$$= \sum_{k=1}^{s-1} ((\Delta k + \beta) - \beta) \left(\frac{1}{\Delta k + \beta} - \frac{\Delta}{2(\Delta k + \beta)^2}\right) + O(1) =$$

$$= s - \left(\frac{\Delta}{2} + \beta\right) \sum_{k=1}^{s-1} \frac{1}{\Delta k + \beta} + O(1) = s - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log(\Delta s + \beta) + O(1) =$$

$$= \frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r + O(1) \quad \text{as} \quad r \to +\infty.$$

Hence, for $0 < \Delta \le \pi/2$ and $-\Delta < \beta < \Delta(\nu - 1/2)$, we get

$$\overline{\lim_{r\to+\infty}} \left(N(r) - \frac{2r}{\pi} + \nu \log(1+r)\right) \ge \overline{\lim_{r\to+\infty}} \left(N_1(r) - \frac{2r}{\pi} + \nu \log(1+r)\right) \ge$$

$$\ge \overline{\lim_{r\to+\infty}} \left(\frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r - \frac{2r}{\pi} + \nu \log r + O(1)\right) = +\infty.$$

Thus, according to Theorem 2, we obtain the required proposition. The proof of theorem is completed. ◀

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