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Estimates of the Approximations by Zygmund Sums in Morrey-Smirnov Classes of Analytic Functions

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Abstract. In the present work, we investigate the approximation of the functions by Zygmund means of Fourier series in Morrey spaces $L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$, $1 < p < \infty$ in the terms of the modulus of smoothness. The obtained results are applied to estimate the approximation of functions by Zygmund sums of Faber series in Morrey-Smirnov classes defined on simply connected domains of the complex plane. In this case, the obtained estimation depends on the sequence of the best approximation in Morrey-Smirnov classes.

Key Words and Phrases: Morrey spaces, Morrey-Smirnov classes, best approximation, trigonometric polynomials, k-th modulus of smoothness, Zygmund sums of order k.

2010 Mathematics Subject Classifications: 30E10, 41A10, 42A10

1. Introduction, some auxiliary results and main results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. Let $L^p(\mathbb{T})$, $1 \leq p < \infty$ be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int\limits_{\mathbb{T}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The *Morrey spaces* $L_0^{p,\lambda}(\mathbb{T})$ for a given $0 \leq \lambda \leq 2$ and $p \geq 1$, are defined as the set of functions $f \in L^p_{loc}(\mathbb{T})$ such that

$$\|f\|_{L_0^{p,\lambda}(\mathbb{T})} := \left\{ \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int\limits_I |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 2\pi]$. Note that $L_0^{p,\lambda}(\mathbb{T})$ is a Banach space, for $\lambda = 2$ it coincides with $L^p(\mathbb{T})$ and for $\lambda = 0$ with $L^\infty(\mathbb{T})$. If $0 \le \lambda_1 \le \lambda_2 \le 2$, then $L_0^{p,\lambda_1}(\mathbb{T}) \subset L_0^{p,\lambda_2}(\mathbb{T})$. Also, if $f \in L_0^{p,\lambda}(\mathbb{T})$, then $f \in L^p(\mathbb{T})$ and hence $f \in L^1(\mathbb{T})$. The Morrey spaces were first introduced by C. B. Morrey in 1938. The properties of these spaces have been investigated by many authors and, together with weighted Lebesgue spaces L_ω^p, they play an important role in the theory of partial differential equations, in the study of local behavior of the solitions of elliptic differential equations and describe local reqularity more precisely than Lebesgue spaces L^p. The detailed information about properties of the Morrey spaces can be found in [7-11, 14-17, 19, 20, 26, 29, 30, 33, 35, 37, 39, 42, 44].

In what follows, by $L^{p,\lambda}(\mathbb{T})$ we denote the closure of the linear subspace of $L_0^{p,\lambda}(\mathbb{T})$ functions, whose shifts are continuous in $L_0^{p,\lambda}(\mathbb{T})$. Suppose that x, h are real, and let's consider the following series:

$$\Delta_h^\alpha f(x) := \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)h), \quad \alpha > 0, \quad f \in L^{p,\lambda}(\mathbb{T}).$$

Then, by [34, Theorem 11, pp.135] the last series converges absolutely almost everywhere (a. e.) on $\mathbb{T}$. Hence the operator Δ_h^α by [25] is bounded in the space $L^{p,\lambda}(\mathbb{T})$. Namely,

$$\begin{aligned} \Delta_h^\alpha f(x) &= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)h) \\ &= \sum_{k=0}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} f(x + kh). \end{aligned}$$

The function

$$\omega_{p,\lambda}^\alpha(f, \delta) := \sup_{|h| \le \delta} \|\Delta_h^\alpha(f, \cdot)\|_{L^{p,\lambda}(\mathbb{T})}, \ \alpha \in Z^+$$

is called *α-th modulus of smoothness* of $f \in L^{p,\lambda}(\mathbb{T})$, $0 \le \lambda \le 2$, $p \ge 1$.

The modulus of smoothness $\omega_{p,\lambda}^\alpha(f, \delta)_M$ has the following properties [24, 25]:

1) $\omega_{p,\lambda}^\alpha(f, \delta)$ is an increasing function;
2) $\lim_{\delta \to 0} \omega_{p,\lambda}^\alpha(f, \delta) = 0$ for every $f \in L^{p,\lambda}(\mathbb{T})$, $0 \le \lambda \le 2$ and $p \ge 1$;
3) $\omega_{p,\lambda}^\alpha(f + g, \delta) \le \omega_{p,\lambda}^\alpha(f, \delta) + \omega_{p,\lambda}^\alpha(g, \delta)$ for $f, g \in L^{p,\lambda}(\mathbb{T})$;
4) $\omega_{p,\lambda}^\alpha(f, n\delta) \le n^\alpha \omega_{p,\lambda}^\alpha(f, \delta), \quad n \in \mathbb{N}$;
5) $\omega_{p,\lambda}^\alpha(f, s\delta) \le (s + 1)^\alpha \omega_{p,\lambda}^\alpha(f, \delta)$;

6) $\omega^{\alpha}_{p,\lambda}(f,\delta) \le [(n+1)\,\delta+1]^{\alpha}\,\omega^{\alpha}_{p,\lambda}(f,\frac{1}{n+1}),\ n\in\mathbb{N}$.

Let
$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k(x,f) \tag{1}$$
be the Fourier series of the function $f\in L_1(\mathbb{T})$, where $A_k(x,f) := (a_k\,(f)\cos kx + b_k\,(f)\sin kx)$, $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f\in L_1(\mathbb{T})$.

The n-th *partial sums*, *Zygmund means of order* k ($k\in\mathbb{N}$) of the series (1) are defined, respectively, as

$$S_n(x,f) = \frac{a_0}{2}+\sum_{\nu=1}^{n}A_\nu(x,f),$$
$$Z_{n,k}(x,f) = \frac{a_0}{2}+\sum_{\nu=1}^{n}(1-\frac{\nu^k}{(n+1)^k})A_\nu(x,f),\ k=1,2,...$$

It is clear that
$$S_0(x,f)=Z_{0,k}(x,f)=\frac{a_0}{2}.$$

The best approximation to $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\le 2$, $1<p<\infty$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n\,(f)_{L^{p,\lambda}(\mathbb{T})} := \inf\left\{\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})} : T_n\in\prod\nolimits_n\right\}.$$

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := ext\Gamma$. Further, let

$$\mathbb{T}:=\{w\in\mathbb{C}:|w|=1\},\ \mathbb{D}:=int\,\mathbb{T}\ \text{and}\ \mathbb{D}^-:=ext\,\mathbb{T}.$$

Let $w=\varphi(z)$ be the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by

$$\varphi(\infty)=\infty,\ \ \lim_{z\to\infty}\frac{\varphi(z)}{z}>0,$$

and let ψ stand for the inverse of φ.

Let $w=\varphi_1(z)$ denote a function that maps the domain G conformally onto the disk $|w|<1$. The inverse mapping of φ_1 will be denoted by ψ_1. Let Γ_r be the image of the circle $|\varphi_1(z)|=r$, $0<r<1$ under the mapping $z=\psi_1(w)$.

Let us denote by E_p, where $p>0$, the class of all functions $f(z)\ne 0$ that are analytic in G and have the property that the integral

$$\int_{\Gamma_r}|f(z)|^p\,|dz|$$

is uniformly bounded for $0 < r < 1$. We shall call the E_p-class the *Smirnov class.* If the function $f(z)$ belongs to E_p, then $f(z)$ has limiting values $f(z')$ almost everywhere on Γ over all nontangential paths, $|f(z')|$ is summable on Γ, and

$$\lim_{r\to 1} \int_{\Gamma_r} |f(z)|^p |dz| = \int_{\Gamma} |f(z')|^p |dz'| .$$

It is known that $\varphi' = E_1(G^-)$ and $\psi' \in E_1(\mathbb{D}^-)$. General information about Smirnov classes can be found in [12, pp. 168-185].

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. The *Morrey spaces* $L^{p,\lambda}(\Gamma)$ for a given $0 \le \lambda \le 2$ and $p \ge 1$, are defined as the set of functions $f \in L^p_{loc}(\Gamma)$ such that

$$\|f\|_{L^{p,\lambda}(\Gamma)} := \left\{ \sup_F \frac{1}{|F \cap \Gamma|^{1-\frac{\lambda}{2}}} \int_F |f(z)|^p \, dz \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all disks F centered on Γ. Let $G := int\Gamma$ and $L^{p,\lambda}(\Gamma), 0 < \lambda \le 2$, $1 < p < \infty$, be a Morrey space defined on Γ. We define the *Morrey-Smirnov classes* $E^{p,\lambda}(G)$ as

$$E^{p,\lambda}(G) := \left\{ f \in E_1(G) : f \in L^{p,\lambda}(\Gamma) \right\}.$$

For $f \in E^{p,\lambda}(G)$, we define the $E^{p,\lambda}(G)$ norm as

$$\|f\|_{E^{p,\lambda}(G)} := \|f\|_{L^{p,\lambda}(\Gamma)} .$$

Note that if $G = \mathbb{D} = \{z : |z| < 1\}$, then we have the space $H^{p,\lambda}(\mathbb{D}) := E^{p,\lambda}(\mathbb{D})$. The space $H^{p,\lambda}(\mathbb{D})$ is called *Morrey-Hardy space* on the unit disk $\mathbb{D}$. For $f \in L^{p,\lambda}(\Gamma)$ we define the function

$$f_0(t) := f(\psi(t)), \ t \in \mathbb{T}.$$

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \le t\}, \ t \ge 0.$$

The curve Γ is called *Dini-smooth* if it has a parametrization

$$\Gamma : \varphi_0(s), \ 0 \le s \le 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega\left(\varphi_0',t\right)}{t}dt<\infty$$

and $\varphi_0'\left(s\right)\neq 0$ [36, p. 48].

If Γ is a Dini-smooth curve, then there exist [43] the constants c_1 and c_2 such that

$$0<c_1\leq\left|\psi'\left(t\right)\right|\leq c_2<\infty,\ \ |t|>1.$$

Note that if Γ is a Dini-smooth curve, then according to the above property we have $f_0\in L^{p,\lambda}\left(\mathbb{T}\right)$ if $f\in L^{p,\lambda}\left(\Gamma\right)$.

Let $\varphi_k(z),\ k=0,1,2,...$ be the Faber polynomials for G. The Faber polynomials $\varphi_k(z)$, associated with $G\cup\Gamma$, are defined through the expansion

$$\frac{\psi'\left(w\right)}{\psi\left(w\right)-z}=\sum_{k=0}^{\infty}\frac{\varphi_k\left(z\right)}{t^{k+1}}\ ,\ \ z\in G,\ t\in D^- \tag{2}$$

and the equalities

$$\varphi_k\left(z\right)=\frac{1}{2\pi i}\int\limits_T\frac{w^k\psi'\left(w\right)}{\psi\left(w\right)-z}dw\ ,\ \ z\in G, \tag{3}$$

$$\varphi_k\left(z\right)=\varphi^k\left(z\right)+\frac{1}{2\pi i}\int\limits_\Gamma\frac{\varphi^k\left(s\right)}{s-z}ds\ ,\ \ z\in G^-,\ k=0,\ 1,\ 2,... \tag{4}$$

[38, p. 33-38].

Let $f\in E^{p,\lambda}\left(G\right)$. Since $f\in E_1\left(G\right)$, we have

$$f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(s)ds}{s-z}=\frac{1}{2\pi i}\int_T\frac{f(\psi(w))\psi'(w)}{\psi(w)-z}dw\ ,$$

for every $z\in G$. Considering this formula and expansion (2), we can associate with f the formal series [25]

$$f(z)\sim\sum_{k=0}^{\infty}a_k(f)\varphi_k(z)\ ,\ z\in G\ , \tag{5}$$

where

$$a_k(f):=\frac{1}{2\pi i}\int_T\frac{f(\psi(w))}{w^{k+1}}dw,\ k=0,\ 1,\ 2,...$$

This series is called the *Faber series* expansion of $f,$ and the coefficients $a_k(f),\ k = 0,\ 1,\ 2,...$ are called *Faber coefficients* of f.

The *n-th partial sums and Zygmund sums of the series* (5) are defined, respectively, as

$$\begin{aligned} S_n(z,f) &= \sum_{k=o}^{n} a_k(f)\varphi_k(z)\ , \\ Z_{n,k}(x,f) &= \frac{a_0}{2} + \sum_{\nu=1}^{n}(1-\frac{\nu^k}{(n+1)^k})a_k(f)\varphi_k(z). \end{aligned}$$

Note that if $f \in E^{p,\lambda}(G)\,,\ 0 < \lambda \leq 2\ ,\ 1 < p < \infty$, then by [24] and [25] the function

$$f_0^+(w) .= \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0(t)\,dt}{t-w},\ w \in D$$

belongs to $H^{p,\lambda}(D)$.

Let Γ be a Dini-smooth curve. We define the k-th modulus of smoothness of $f \in E^{p,\lambda}(G)\,,\ 0 < \lambda \leq 2\ ,\ 1 < p < \infty$, as

$$\Omega^k_{\Gamma,\ p,\lambda}(\ f,\delta) := \omega^k_{p,\lambda}(\ f_0^+,\delta),\ \delta > 0,$$

for $l = 1,2,3...$

Let $\mathcal{P}$:={all polynomials (with no restriction on the degree)}, and let $\mathcal{P}(D)$ be the set of traces of members of $\mathcal{P}$ on D. We define the operator T as follows:

$$\begin{aligned} T &: = \mathcal{P}(D) \longrightarrow E^{p,\lambda}(G)\,, \\ T(P)(z) &: = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w)-z}dt,\ z \in G. \end{aligned}$$

Then taking into account (4) and (5) we have

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \sum_{k=0}^{n} b_k\varphi_k(z),\ z \in G.$$

We use the constants $c, c_1, c_2, ...$ (which can be different in different places) which depend only on the quantities that are not important for the questions of interest

The approximation of the functions by trigonometric polynomials in non-weighted and weighted Morrey spaces have been investigated in [4-6, 18, 24, 25, 31, 32]. In this work, we study the deviation of functions from their Zygmund means in terms of the modulus of smoothness $\omega_{p,\lambda}^{k}(f,\cdot)$ and best approximation $E_n(f)_{L^{p,\lambda}(\mathbb{T})}$ of these functions in the Morrey spaces $L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$, $1<p<\infty$. Also, we investigate the approximation of the functions by Zygmund sums of Faber series in Morrey-Smirnov classes $E^{p,\lambda}(G), 0<\lambda\leq 2$, $1<p<\infty$ defined in the domains with a Dini-smooth boundary of the complex plane in terms of the best approximation $E_n(f)_{E^{p,\lambda}(G)}$. Similar problems in different spaces have been studied in, for example, [1-3, 13, 21-23, 27-28, 40].

Using Bernstein inequality for trigonometric polynomials T_n of degree $\leq n$ in the Morrey spaces $L^{p,\lambda}(0,2\pi)$, $0<\lambda\leq 2$, $1<p<\infty$, and taking into account the above properties of modulus of smoothness $\omega_{p,\lambda}^{k}(f,\cdot)$ and the proof scheme developed in [40] (see also [13], p. 210), we can prove the following theorem.

Theorem 1. *Let* $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ *and* $1<p<\infty$. *Then for* $k\in\mathbb{N}^{+}$ *the estimate*

$$\omega_{p,\lambda}^{k}(f,\frac{1}{n})\leq\frac{c_3}{n^k}\left\{\sum_{\nu=1}^{n}\nu^{\beta k-1}E_{\nu}^{\beta}(f)_{L^{p,\lambda}(\mathbb{T})}\right\}^{\frac{1}{\beta}},\ n\in\mathbb{N}^{+},\ \beta=\min\{p,2\}$$

holds with a constant $c_3=c_3(p,k,\lambda)>0$ *independent of* n, *where* $\beta=\min\{p,2\}$.

Theorem 2. *Let* G *be a finite, simply connected domain with a Dini-smooth boundary* Γ *and* $f\in E^{p,\lambda}(G), 0<\lambda\leq 2,\ p>1$. *Then the inequality*

$$\Omega_{\Gamma,\ p,\lambda}^{k}(\delta,\ f)\leq\frac{c_4}{n^k}\left\{\sum_{\nu=0}^{n}\nu^{\beta k-1}E_{\nu}^{\beta}(f)_{E^{p,\lambda}(G)}\right\}^{\frac{1}{\beta}},n\in\mathbb{N}^{+}$$

holds with a constant $c_4>0$ *independent of* n, *where* $\beta=\min\{p,2\}$.

Proof. Let $f\in E^{p,\lambda}(G), 0<\lambda\leq 2,\ p>1$. According to [25], $T\left(f_0^{+}\right)=f$. The operator $T: H^{p,\lambda}(D)\to E^{p,\lambda}(G)$ is linear, bounded , one-to-one and onto. Then $T^{-1}: E^{p,\lambda}(G)\to H^{p,\lambda}(D)$ is linear and bounded. We take a $p_n^{*}\in\mathcal{P}_n$ as the best approximating algebraic polynomial for f in $E^{p,\lambda}(G)$. That is

$$E_n(f)_{L^{p,\lambda}(G)}=\|f-p_n^{*}\|_{L^{p,\lambda}(\Gamma)}.$$

Then by [25] we have $T^{-1}(p_n^{*})\in\mathcal{P}_n(D)$. Since the operator T^{-1} is bounded, the inequality

$$\begin{aligned}E_n\left(f_0^+,D\right)_{H^{p,\lambda}(D)} &\leq \left\|f_0^+ - T^{-1}\left(p_n^*\right)\right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \left\|-T^{-1}\left(f\right) - T^{-1}\left(p_n^*\right)\right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \left\|-T^{-1}\left(f - p_n^* -\right)\right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\leq \left\|-T^{-1}\right\| \left\|f - p_n^*\right\|_{L^{p,\lambda}(\lneqq)} \\ &= \left\|-T^{-1}\right\| E_n\left(f\right)_{L^{p,\lambda}(G)}\end{aligned}$$

holds. Finally, considering the last inequality and Theorem 1 we obtain

$$\begin{aligned}\Omega^k_{\Gamma,\ p,\lambda}(\ f,\frac{1}{n}) :=\omega^k_{p,\lambda}(\ f_0^+,\frac{1}{n}) &\leq \frac{c_5}{n^k}\left\{\sum_{\nu=1}^{n}\nu^{\beta k-1}E_\nu^\beta(f)_{H^{p,\lambda}(D)}\right\}^{\frac{1}{\beta}} \\ &\leq \frac{c_6}{n^k}\left\|-T^{-1}\right\|\left\{\sum_{\nu=1}^{n}\nu^{\beta k-1}E_\nu^\beta(f)_{E^{p,\lambda}(G)}\right\}^{\frac{1}{\beta}} \\ &\leq \frac{c_7}{n^k}\left\|-T^{-1}\right\|\left\{\sum_{\nu=1}^{n}\nu^{\beta k-1}E_\nu^\beta(f)_{E^{p,\lambda}(G)}\right\}^{\frac{1}{\beta}} \\ &\leq \frac{c_8}{n^k}\left\{\sum_{\nu=1}^{n}\nu^{\beta k-1}E_\nu^\beta(f)_{E^{p,\lambda}(G)}\right\}^{\frac{1}{\beta}},\end{aligned}$$

which completes the proof of Theorem 2. ◄

Our main results are the following.

Theorem 3. *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$ *and* $1 < p < \infty$. *Then the inequality*

$$\|f - Z_{n,k}(\cdot, f)\|_{L^{p,\lambda}(\mathbb{T})} \leq c_9\omega^k_{p,\lambda}(f,\frac{\pi}{n})$$

holds with a constant $c_9 > 0$ *independent of* n.

Theorem 4. *Under the conditions of Theorem 3, there is a constant* $c = c(p,k,\lambda) > 0$ *independent of* n *such that the inequality*

$$\|f - Z_{n,k}(\cdot, f)\|_{L^{p,\lambda}(\mathbb{T})} \leq \frac{c_{10}}{n^k}\left\{\sum_{\nu=0}^{n}\nu^{\beta k-1}E_\nu^\beta(f)_{L^{p,\lambda}(\mathbb{T})}\right\}^{\frac{1}{\beta}} \tag{6}$$

holds, where $\beta = \min\{p, 2\}$.

Theorem 5. *Let Γ be a Dini-smooth curve. Then for $f \in E^{p,\lambda}(G), 0 < \lambda \leq 2, p > 1$ the inequality*

$$\|f - Z_{n,k}(\cdot, f)\|_{L^{p,\lambda}(\Gamma)} \leq \frac{c_{11}}{n^k} \left\{ \sum_{\nu=0}^{n} \nu^{\beta k - 1} E_\nu^\beta (f)_{E^{p,\lambda}(G)} \right\}^{\frac{1}{\beta}}, n \in \mathbb{N}^+$$

holds with a constant $c_{11} > 0$ independent of n, where $\beta = \min\{p, 2\}$.

Note that Theorems 3 and 4 in the Lebesgue spaces $L_p(\mathbb{T})$, $p \geq 1$ have been obtained in [21] and [41], respectively.

2. Proofs of the results

Proof of Theorem 3 Let $f \in L_M(\mathbb{T})$. Then the following inequality holds:

$$\begin{aligned} \|f - Z_{n,k}(\cdot, f)\|_{L^{p,\lambda}(\mathbb{T})} \leq & \|f - S_n(\cdot, f)\|_{L^{p,\lambda}(\mathbb{T})} \\ & + (n+1)^{-k} \left\| \nu^k A_\nu(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ = & U_1 + U_2^{(k)}. \end{aligned} \tag{7}$$

From [24] we get

$$U_1 = \|f - S_n(\cdot, f)\|_{L^{p,\lambda}(\mathbb{T})} \leq c_{12} E_n(f)_{L^{p,\lambda}(\mathbb{T})}. \tag{8}$$

According to [25] the inequality

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})} \leq c_{13}(k, M) \omega_{p,\lambda}^k (f, \frac{1}{n}) \tag{9}$$

holds. Using (8) and (9) we have

$$U_1 = \|f - S_n(\cdot, f)\|_{L^{p,\lambda}(\mathbb{T})} \leq c_{14}(k, M) \omega_{p,\lambda}^k (f, \frac{1}{n}). \tag{10}$$

If k is even

$$\sum_{\nu=1}^{n} \nu^k A_\nu(x, f) = (-1)^{k/2} S_n^{(k)}(x, f),$$

if k is odd

$$\sum_{\nu=1}^{n} \nu^k A_\nu(x, f) = (-1)^{(k+3)/2} \widetilde{S_n}^{(k)}(x, f),$$

where $\widetilde{g(x)}$ is the function that is trigonometricaly conjugate to $g(x)$. Then we can write

$$U_2^{(k)} = \begin{cases} (n+1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})}, & k-\text{even} \\ (n+1)^{-k} \left\| \widetilde{S_n}^{(k)}(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})}, & k-\text{odd}. \end{cases} \tag{11}$$

If k is even, then, according to the definition of modulus of smoothness, Bernstein inequality for the Morrey spaces [25] and (11), we have

$$\begin{aligned} U_2^{(k)} =& (n+1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ \leq& c_{15}(n+1)^{-k} 2^{-k} n^{-k} \left\| \Delta_{\pi/n}^k S_n(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ \leq& 2^{-k} c_{16} \left\| \Delta_{\pi/n}^k S_n(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ =& 2^{-k} c_{16} \left\| \Delta_{\pi/n}^k (S_n(\cdot, f) - f + f) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ \leq& c_{17}(M,k) \left\{ \| f - S_n(\cdot, f) \|_{L^{p,\lambda}(\mathbb{T})} + \left\| \Delta_{\pi/n}^k (f) \right\|_{L^{p,\lambda}(\mathbb{T})} \right\} \\ \leq& c_{18}(M,k) \omega_{p,\lambda}^k (f, \frac{\pi}{n}). \end{aligned} \tag{12}$$

By [25] (see also [24]), we have

$$\left\| \widetilde{S_n}^{(k)}(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})} \leq c_{19} \left\| S_n^{(k)}(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})}. \tag{13}$$

If k is odd, then, combining (11), (13) and (12), we obtain the following inequality:

$$\begin{aligned} U_2^{(k)} =& (n+1)^{-k} \left\| \widetilde{S_n}^{(k)}(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ \leq& c_{20}(n+1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ \leq& c_{21}(M,k) \omega_{p,\lambda}^k (f, \frac{\pi}{n}). \end{aligned} \tag{14}$$

Taking into account (7), (10), (12) and (14), we obtain the inequality in Theorem 3. The proof of Theorem 3 is completed. ◀

Proof of Theorem 4. From Theorem 3 and Theorem 1 we obtain Theorem 4. ◀

Proof of Theorem 5. Let $f \in E^{p,\lambda}(G)$, $0 < \lambda \leq 2$ and $1 < p < \infty$. Since Γ is a Dini smooth curve, by virtue of [25] (see also [24]), the operator T :

$H^{p,\lambda}(D) \longrightarrow E^{p,\lambda}(G),\ 0<\lambda\leq 2\ ,\ \ 1<p<\infty$ is bounded, one-to-one and onto, and $T(f_0^+)=f$. For the function $f\in E^{p,\lambda}(G)$, we have the following Faber series [25]:

$$f(z)\backsim\sum_{k=0}^{\infty}a_k(f)\varphi_k(z),\ z\in G,$$

where

$$a_k\left(f\right):=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0\left(w\right)}{w^{k+1}}dw, k\ \in\mathbb{N}.$$

By [25] we get $f_0^+\in H^{p,\lambda}(D)$. Then for the function f_0^+ we have the following Taylor expansion:

$$f_0^+(w)=\sum_{k=0}^{\infty}a_k(f)w^k.$$

Note that $f_0^+\in E_1(D)$ and boundary function $f_0^+\in L^{p,\lambda}(\mathbb{T})$. Then according to [12, Theorem 3.4, pp.38] the function $f_0^+(w)$ has the following Fourier expansion:

$$f_0^+(t)\backsim\sum_{k=0}^{\infty}a_k(f)e^{itk}.$$

Taking into account the boundedness of the operator T, Theorems 3 and 2, we have

$$\begin{aligned}\parallel f-Z_{n,k}(.,\ f)\parallel_{L^{p,\lambda}(\Gamma)} &= \parallel T(f_0^+)-T(V_{n,m}(.,\ f_0^+))\parallel_{L^{p,\lambda}(\Gamma)}\\ &\leq c_{22}\parallel f_0^+-Z_{n,k}(.,\ f_0^+)\parallel_{L^{p,\lambda}(\mathbb{T})}\\ &\leq c_{23}\omega_{p,\lambda}^k(f_0^+,\frac{1}{n})\\ &= c_{23}\Omega_{\Gamma,\ p,\lambda}^k(\ f,\frac{1}{n})\\ &\leq\frac{c_{24}}{n^k}\left\{\sum_{\nu=1}^{n}\nu^{\beta k-1}E_\nu^\beta(f)_{E^{p,\lambda}(G)}\right\}^{\frac{1}{\beta}}.\end{aligned}$$

which proves Theorem 5. ◄

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