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On the Completeness of the System of Airy Functions

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Abstract. Airy functions $Ai\left(x-\lambda_n\right), n=1,2,...$, are considered, where λ_n is the eigenvalue of the one-dimensional Stark operator on the semi-axis with finite potential and Dirichlet boundary condition at zero. The completeness in the space $L_2\left(0,\infty\right)$ of a system of functions $\left\{Ai\left(x-\lambda_n\right)\right\}_{n=1}^{\infty}$ is proved.

Key Words and Phrases: Airy function, Stark operator, eigenvalues, completeness of a system of functions.

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1. Introduction and main result

In the space $L_2\left(0,\infty\right)$ we consider a self-adjoint operator

$$T_0=-\frac{d^2}{dx^2}+x,$$

generated by the left-hand side of the equation

$$-y''+xy=\lambda y,\ 0<x<\infty,\quad \lambda\in C, \tag{1}$$

and boundary condition

$$y\left(0\right)=0. \tag{2}$$

It is well known [1] that the equation (1) has a solution $f\left(x,\lambda\right)$ in the form

$$f\left(x,\lambda\right)=Ai\left(x-\lambda\right), \tag{3}$$

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where $Ai\,(z)$ is the Airy function of the first kind. It is also known that (see [1]) $Ai\,(z)$ is an entire function of order 3/2 and type 2/3. We have (see [1]) the following asymptotic equalities as $|z| \to \infty$:

$$\begin{aligned} &\mathrm{A}i\,(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta} \left[1 + O\left(\zeta^{-1}\right)\right], \\ &Ai'\,(z) \sim -\pi^{-\frac{1}{2}} z^{\frac{1}{4}} e^{-\zeta} \left[1 + O\left(\zeta^{-1}\right)\right], \ |\arg z| < \pi, \end{aligned} \tag{4}$$

where $\zeta = \frac{2}{3} z^{\frac{3}{2}}$. Since $q_0\,(x) = x \to +\infty$ for $x \to +\infty$, the spectrum of the operator T_0 consists of simple real eigenvalues. From (3), (4) it follows that for each fixed λ from the complex plane, the relation $f\,(x, \lambda) \in L_2\,(0, \infty)$ holds. Therefore, the spectrum of the problem (1)-(2), i.e. of the operator T_0, coincide with the zeros of the function $f\,(0, \lambda) = Ai\,(-\lambda)$. The function $Ai\,(-\lambda)$ has [1] zeros $\lambda_n^0, n = 1, 2, ...$ only on the positive semi axis and the following asymptotic equality is valid:

$$\lambda_n^0 = g\left(\frac{3\pi\,(4n-1)}{8}\right), \tag{5}$$

where

$$g\,(z) \sim z^{\frac{2}{3}} \left(1 + \frac{5}{48} z^{-2} - \frac{5}{36} z^{-4} + \frac{77125}{82944} z^{-6} - \frac{108056875}{6967296} z^{-8} + ...\right), z \to \infty.$$

We now consider the self-adjoint operator

$$T = T_0 + q\,(x)$$

in space $L_2\,(0, \infty)$, where the real potential $q\,(x)$ is twice differentiable and finite. Such an operator describes (see [1, 2]) the influence of the electric field potential and is called the Stark operator. Note that in the context of various spectral problems, the one-dimensional Stark operators have been studied by many authors (see [3, 4, 5, 6, 7] and references therein). Many important results were obtained on the resonances of the one-dimensional Stark operators in [8, 9].

In [5] the asymptotic behavior of the eigenvalues of the operator T has been studied. It was proved there that the spectrum of the operator T consists of a sequence of simple real eigenvalues $\lambda_n, n \geq 1$, and the following asymptotic formula is valid:

$$\lambda_n = \left(\frac{3\pi\,(4n-1)}{8}\right)^{\frac{2}{3}} + O\left(n^{-\frac{2}{3}}\right), n \to \infty. \tag{6}$$

Of particular interest is the question of the completeness of the system of functions $\{f\,(x, \lambda_n)\}_{n=1}^{\infty}$. This matter can be very useful in the study of inverse spectral problem for the operator T, since the completeness of this system of functions plays a key role in the unique solvability of the Gelfand-Levitan equation (see [10, 11]).

The main result of this paper is the following theorem.

Theorem 1. *Let the numbers* λ_n, $\lambda_n \neq \lambda_k$, $(n \neq k)$ *be in the form of* (6). *Then the system of functions* $\{f(x, \lambda_n)\}_{n=1}^{\infty}$ *is complete in* $L_2(0, \infty)$.

Proof. Let $h(x) \in L_2(0, \infty)$ be such that

$$\int_0^\infty h(x)\, Ai(x - \lambda_n)\, dx = 0,\ n \geq 0.$$

Consider the Hadamard factorization of the function $Ai(-\lambda)$:

$$Ai(-\lambda) = C_0 e^{p\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n^0}\right) e^{\frac{\lambda}{\lambda_n^0}},$$

where $C_0 = Ai(0) = \frac{3^{-\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)}$. Introduce the function

$$A(\lambda) = C_1 e^{p\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) e^{\frac{\lambda}{\lambda_n^0}},$$

the set of roots of which coincides with the sequence λ_n, where $C_1 = C_0 \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n^0}$. It follows from (5), (6) that $A(\lambda)$ is an entire function of order $\frac{3}{2}$. For $h(x) \in L_2(0, \infty)$, as shown in [7], $\int_0^\infty h(x)\, Ai(x - \lambda)\, dx$ is an entire function of order $\rho \leq \frac{3}{2}$. It follows that $A^{-1}(\lambda) \int_0^\infty h(x)\, Ai(x - \lambda)\, dx$ is an entire function of order $\rho \leq \frac{3}{2}$. Further, when $0 \leq \arg \lambda \leq 2\pi$, the function $Ai^{-1}(-\lambda) \int_0^\infty h(x)\, Ai(x - \lambda)\, dx$ admits (see [7]) the estimate

$$\left| Ai^{-1}(-\lambda) \int_0^\infty h(x)\, Ai(x - \lambda)\, dx \right| \leq M \|h\| R^{\frac{1}{2}},\ R = |\lambda| > R_0.$$

On the other hand, inside the corner $\delta \leq \arg \lambda \leq 2\pi - \delta$, $\delta > 0$, the relation

$$\left| \frac{\lambda_n - \lambda_n^0}{\lambda - \lambda_n} \right| \leq \frac{C n^{-\frac{2}{3}}}{|\lambda_n \sin \delta|} \leq \frac{C_1}{n^{\frac{4}{3}}},$$

holds. Then from the formula

$$\frac{Ai(-\lambda)}{A(\lambda)} = \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - \lambda_n^0}{\lambda - \lambda_n}\right)$$

it follows that

$$\left|\frac{Ai\left(-\lambda\right)}{A\left(\lambda\right)}\right| \leq C_2.$$

Using the last relations, we obtain

$$\left|A^{-1}\left(\lambda\right)\int_0^{\infty} h\left(x\right) Ai\left(x-\lambda\right)dx\right| \leq M_1 \left\|f\right\| R^{\frac{1}{2}}\,, \tag{7}$$

where $R = |\lambda| > R_0$, $\delta \leq \arg\lambda \leq 2\pi - \delta$. Now let $\delta > 0$ be such that the sector angle is smaller than $\frac{2\pi}{3}$. Applying the Phragmen-Lindelof theorem [12] to the function $(1+\lambda)^{-\frac{1}{2}} A^{-1}\left(\lambda\right)\int_0^{\infty} h\left(x\right) Ai\left(x-\lambda\right)dx$, we find that the estimate (7) also holds in the sector $-\delta \leq \arg\lambda \leq \delta$. From this, using the Liouville's theorem [12] we conclude that $A^{-1}\left(\lambda\right)\int_0^{\infty} h\left(x\right) Ai\left(x-\lambda\right)dx \equiv 0$, i.e.

$$H\left(\lambda\right) = \int_0^{\infty} h\left(x\right) Ai\left(x-\lambda\right)dx \equiv 0.$$

On the other hand, as shown in [13] (see Theorem 2.1), for all $h\left(x\right) \in L_2\left(0,\infty\right)$ we have the equality

$$\int_0^{\infty} \left|h\left(x\right)\right|^2 dx = \int_{-\infty}^{\infty} \left|H\left(\lambda\right)\right|^2 d\lambda,$$

and consequently, $h\left(x\right) = 0$. This completes the proof of the theorem. ◄

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