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Cubic Interior Ideal in Semigroups

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Abstract. In this paper we introduce the notion of cubic interior ideal semigroup and we study basic properties of cubic interior ideal. We find necessary and sufficient conditions for cubic interior ideal to be cubic ideal. Characterizations of semisimple semigroups in terms of cubic interior ideal are given. Finally we show that the images or inverse images of a cubic interior ideal of a semigroup become a cubic interior ideal.

Key Words and Phrases: cubic set, cubic ideal, cubic interior ideal, fuzzy set, interval value fuzzy, semigroup.

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1. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S with an associative binary operation [11]. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis.

In 1965, The fundamental concept of fuzzy sets was introduced by Zadeh [12]. At present, it is an important tool in science and engineering (like computer science, control engineering, information sciences, etc). In 1971 [10], Rosenfeld was the first to consider the case when S is a groupoid. He gave the definition of fuzzy subgroups, fuzzy left (right, two-sided) ideals of S. Hong, Jun and Meng investigated some properties and considered the characterization of a fuzzy interior ideal of a semigroup. In 1975 [2] Zadeh made an extension of the concept of fuzzy sets by an interval valued fuzzy sets with the values of the membership function being closed subintervals of the interval $[0, 1]$. The notion of interval valued fuzzy sets have many applications such as approximate reasoning, image processing, decision making, medicine, mobile networks, etc. In 2006, Narayanan and Manikanran [9] initiated the notion of interval valued fuzzy ideal in semigroup.

In 2012, Jun et al [3] introduced a new notion, called a cubic set, investigated several properties of it, and introduced cubic subsemigroups and cubic left (right) ideals of semigroups. Recently, Hadi and Khan [1], defined the cubic generalized bi-ideal in semigroups and studied important properties of those.

In this paper we study the cubic interior ideal and we find necessary and sufficient conditions for cubic interior ideal to be cubic ideal. Characterizations of semisimple semigroups in terms of cubic interior ideal are given. Furthermore we show that the images or inverse images of a cubic interior ideal of a semigroup become a cubic interior ideal.

2. Preliminaries

2.1. Semigroups and fuzzy semigroups

Definition 1. *A* semigroup *is an ordered pair* $(S, \cdot)$*, where* S *is a non-empty set and the dot "$\cdot$" is a binary operation on* S*, i.e., a mapping* $(a, b) \mapsto a \cdot b$ *from* $S \times S$ *to* S *such that for all* $a, b, c \in S, (a \cdot b) \cdot c = a \cdot (b \cdot c)$ *(associative law).*

For convenience, $(S, \cdot)$ is abbreviated to S and $a \cdot b$ to ab. The associative property in a semigroup ensures that the two products $a(bc)$ and $(ab)c$ are same, which can be denoted as abc.

Definition 2. *[2] A semigroup* S *is said to be* regular *if for each element* $a \in S$*, there exists an element* $x \in S$ *such that* $a = axa$.

Definition 3. *[6] A semigroup* S *is called* intra-regular *if for every* $a \in S$ *there exist* $x, y \in S$ *such that* $a = xa^2y$.

Definition 4. *[8] A semigroup* S *is said to be* left regular *if for each element* $a \in S$*, there exists an element* $x \in S$ *such that* $a = xa^2$.

Definition 5. *[8] A semigroup* S *is called* semisimple *if every ideal of* S *is idempotent. It is clear that* S *is semisimple if and only if* $a \in (SaS)(SaS)$ *for every* $a \in S$*, that is there exist* $x, y, z \in S$ *such that* $a = xayaz$.

Definition 6. *[5] A non-empty subset* A *of semigroup* S *is a subsemigroup of* S *if* $A^2 \subseteq A$.

Theorem 1. *Let* A *be a non-empty subset of* S*. Then* A *is a subsemigroup of* S *if and only if* $xy \in A$*. for all* $x, y \in A$.

Definition 7. *[11] A non-empty subset* A *of a semigroup* S *is called a* left *(right) ideal of* S *if* $SA \subseteq A(AS \subseteq A)$*. An ideal of* S *is a nonempty subset which is both a left ideal and a right ideal of* S.

Definition 8. *[5] A subsemigroup A of a semigroup S is called an* interior ideal *of S if $SAS \subseteq A$.*

Definition 9. *[7] Let X be a non-empty set. A mapping $f : X \to [0,1]$ is a* fuzzy subset *of S.*

Definition 10. *[7] Let S be a semigroup. A fuzzy subset f_A of S is said to be a* fuzzy subsemigroup *of S if $f_A(xy) \geq \min\{f_A(x), f_A(y)\}$, for all $x, y \in S$.*

Definition 11. *[7] Let S be a semigroup. A fuzzy subset f_A of S is said to be a* fuzzy left *(right) ideal of S if $f_A(xy) \geq f_A(y)(f_A(xy) \geq f_A(x))$, for all $x, y \in S$. A non-empty fuzzy subset of a semigroup S is a* fuzzy ideal *of S if it is a fuzzy left ideal and fuzzy right ideal of S.*

Definition 12. *[4] A fuzzy subsemigroup f_A of a semigroup S is called a* fuzzy interior *ideal of S if $f_A(xay) \geq f_A(a)$ for all $a, x, y \in S$.*

Definition 13. *[3] For a family $\{f_i \mid \in I\}$ of fuzzy sets in X, we define the join $(\vee)$ and meet $(\wedge)$ operations as follows:*

$$\left(\bigvee_{i\in I} f_i\right)(x) = \sup\{f_i(x) \mid i \in I\} \quad \textit{and} \quad \left(\bigwedge_{i\in I} f_i\right)(x) = \inf\{f_i(x) \mid i \in I\},$$

respectively, for all $x \in X$.

Now we introduce a new concept concerning an interval.

Definition 14. *[2] An interval number on $[0,1]$, say $\overline{a}$, is a closed subinterval of $[0,1]$, that is $\overline{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$, i.e.,*

$$D[0,1] = \{\overline{a} = [a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}.$$

The interval $[a,a]$ is identified with the number $a \in [0.1]$.

Definition 15. *[2] Let $\overline{a}_i = [a_i^-, a_i^+] \in D[0,1]$ for all $i \in I$, where I is an index set. We define*

$$r\inf_{i\in I} \overline{a}_i = \left[\inf_{i\in I} a_i^-, \inf_{i\in I} a_i^+\right] \quad \textit{and} \quad r\sup_{i\in I} \overline{a}_i = \left[\sup_{i\in I} a_i^-, \sup_{i\in I} a_i^+\right].$$

We define the operations "$\succeq$", "$\preceq$", "$=$", "$r\min$" "$r\max$" in case of two element in $D[0,1]$. We consider two interval numbers $\overline{a} := [a^-, a^+]$ and $\overline{b} := [b^-, b^+]$ in $D[0,1]$. Then

(1) $\overline{a} \succeq \overline{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$

(2) $\overline{a} \preceq \overline{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$

(3) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$

(4) $r\min\{\overline{a}, \overline{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$

(5) $r\max\{\overline{a}, \overline{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Definition 16. *[2] Let X be a set. An interval valued fuzzy set A on X is defined as*

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)] \mid x \in X\},$$

where μ_A^- and μ_A^+ are two fuzzy sets of X such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$. Putting $\overline{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$, we see that $A = \{x, \overline{\mu}_A(x) \mid x \in X\}$, where $\overline{\mu}_A : X \to D[0,1]$.

Definition 17. *[2] For a family $\{\overline{\mu}_{A_i} \mid\in I\}$ of interval valued fuzzy sets in X, we define the $(\sqcup_{i\in I}\overline{\mu}_{A_i})$ and $(\sqcap_{i\in I}\overline{\mu}_{A_i})$ as follows:*

$$(\sqcup_{i\in I}\overline{\mu}_{A_i})(x) = r\sup \overline{\mu}_{A_i}(x) \quad \text{and} \quad (\sqcap_{i\in I}\overline{\mu}_{A_i})(x) = r\inf \overline{\mu}_{A_i}(x),$$

respectively, for all $x \in X$, where $\overline{\mu}_A : X \to D[0,1]$.

2.2. Cubic semigroups.

Definition 18. *[2] Let X be a non-empty set. A* cubic set *$\mathcal{A}$ in X is a structure of the form*

$$\mathcal{A} = \{\langle x, \overline{\mu}_A(x), f_A(x)\rangle : x \in X\}$$

and denoted by $\mathcal{A} = \langle \overline{\mu}_A, f_A\rangle$, where $\overline{\mu}_A$ is an interval valued fuzzy set (briefly, IVF) in X and f_A is a fuzzy set in X. In this case we will use

$$\mathcal{A}(x) = \langle \overline{\mu}_A(x), f_A(x)\rangle = \langle [\mu_A^-(x), \mu_A^+(x)], f_A(x)\rangle$$

for all $x \in X$. Note that a cubic set is a generalization of an intuitionistic fuzzy set.

Definition 19. *[2] Let S be a semigroup. Then cubic set characteristic function $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A}\rangle$ is defined as*

$$\overline{\mu}_{\chi_A}(x) = \begin{cases} [1,1], & \text{if } x \in A, \\ [0,0], & \text{if } x \notin A. \end{cases} \quad \text{and} \quad f_{\chi_A}(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x \notin A. \end{cases}$$

Definition 20. *[2] The whole cubic set $\mathcal{S}$ in a semigroup S is defined to be a structure*

$$\overline{\mathcal{S}} = \{\langle x, 1_S(x), 0_S(x)\rangle : x \in S\}$$

with $1_S(x) = [1,1]$ and $0_S(x) = 0$ for all $x \in S$. It will briefly denoted by $\overline{\mathcal{S}} = \langle 1_S, 0_S\rangle$.

Definition 21. *[2] For two cubic sets $\mathcal{A} = \langle \overline{\mu}_A, f_A\rangle$ and $\mathcal{B} = \langle \overline{\mu}_B, f_B\rangle$ in a semigroup S, we define*

$$\mathcal{A} \sqsubseteq \mathcal{B} \Leftrightarrow \overline{\mu}_A \preceq \overline{\mu}_B \quad \textit{and} \quad f_A \geq f_B$$

Definition 22. *[2] Let $\mathcal{A} = \langle \overline{\mu}_A, f_A\rangle$ and $\mathcal{B} = \langle \overline{\mu}_B, f_B\rangle$ be two cubic sets in a semigroup S. Then the cubic product of $\mathcal{A}$ and $\mathcal{B}$ is a structure*

$$\mathcal{A} \boxdot \mathcal{B} = \{\langle x, (\overline{\mu}_A \Box \overline{\mu}_B)(x), (f_A \cdot f_B)(x)\rangle : x \in S\},$$

which is briefly denoted by $\mathcal{A} \boxdot \mathcal{B} = \langle(\overline{\mu}_A \Box \overline{\mu}_B), (f_A \cdot f_B)\rangle$, where $\mu_A \Box \mu_B$ and $f_A \cdot f_B$ are defined as follows, respectively:

$$(\overline{\mu}_A \Box \overline{\mu}_B)(x) = \begin{cases} r\sup_{x=yz}\{r\min\{\overline{\mu}_A(y), \overline{\mu}_B(z)\}\} & \textit{if} \quad F_x \neq \emptyset, \\ [0,0], & \textit{if} \quad F_x = \emptyset, \end{cases}$$

and

$$(f_A \cdot f_B)(x) = \begin{cases} \inf\limits_{x=yz} \max\{f_A(y), f_A(z)\} & \textit{if} \quad F_x \neq \emptyset, \\ 1, & \textit{if} \quad F_x = \emptyset, \end{cases}$$

for all $x \in S$.

Definition 23. *[2] Let $\mathcal{A} = \langle \overline{\mu}_A, f_A\rangle$ and $\mathcal{B} = \langle \overline{\mu}_B, f_B\rangle$ be two cubic sets in a semigroup S. Then the intersection of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \bar{\sqcap} \mathcal{B}$, is the cubic set*

$$\mathcal{A} \bar{\sqcap} \mathcal{B} = \langle \overline{\mu}_A \sqcap \overline{\mu}_B, f_A \vee f_B\rangle,$$

where $(\mu_A \sqcap \mu_B)(x) = r\min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}$ and $(f_A \vee f_B)(x) = \max\{f_A(x), f_B(x)\}$ for all $x \in S$. And the union of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \underline{\sqcup} \mathcal{B}$, is the cubic set

$$\mathcal{A} \underline{\sqcup} \mathcal{B} = \langle \overline{\mu}_A \sqcup \overline{\mu}_B, f_A \wedge f_B\rangle,$$

where $(\mu_A \sqcup \mu_B)(x) = r\max\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}$ and $(f_A \wedge f_B)(x) = \min\{f_A(x), f_B(x)\}$ for all $x \in S$.

Proposition 1. *[2] For any cubic sets $\mathcal{A} = \langle \mu_A, f_A\rangle, \mathcal{B} = \langle \mu_B, f_B\rangle$ and $\mathcal{C} = \langle \mu_C, f_C\rangle$ in a semigroup S, the following statements hold:*

(1) $\mathcal{A}\overline{\sqcup}(\mathcal{B}\overline{\sqcap}C) = (\mathcal{A}\overline{\sqcup}\mathcal{B})\overline{\sqcap}(\mathcal{A}\overline{\sqcup}\mathcal{C})$,

(2) $\mathcal{A}\overline{\sqcap}(\mathcal{B}\overline{\sqcup}C) = (\mathcal{A}\overline{\sqcap}\mathcal{B})\overline{\sqcup}(\mathcal{A}\overline{\sqcap}\mathcal{C})$,

(3) $\mathcal{A} \boxdot (\mathcal{B}\overline{\sqcup}C) = (\mathcal{A} \boxdot \mathcal{B})\overline{\sqcup}(\mathcal{A} \boxdot \mathcal{C})$,

(4) $\mathcal{A} \boxdot (\mathcal{B}\overline{\sqcap}C) = (\mathcal{A} \boxdot \mathcal{B})\overline{\sqcap}(\mathcal{A} \boxdot \mathcal{C})$.

Definition 24. *[2] A cubic set $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ in a semigroup S is called a* cubic subsemigroup *of S if it satisfies:*

(1) $\mu_A(xy) \succeq r\min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$,

(2) $f_A(xy) \leq \max\{f_A(x), f_A(y)\}$, *for all* $x, y \in S$.

Theorem 2. *[2] Let $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ and $\mathcal{B} = \langle \overline{\mu}_B, f_B \rangle$ be cubic subsemigroups of S. Then $A\overline{\sqcap}B = \langle \mu_A \sqcap \mu_B, f_A \vee f_B \rangle$ is a cubic subsemigroup of S.*

Theorem 3. *[2] Let S be a semigroup and let A be a non-empty subset of S. Then A is a subsemigroup of S if and only if the characteristic cubic set $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic subsemigroup of S.*

Definition 25. *[2] Let $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ be a cubic set in X. For any $k \in [0, 1]$ and $[s, t] \in D[0, 1]$, we define $U(A, [s, t], k)$ as follows:*

$$U(\mathcal{A}, [s, t], k) = \{x \in X \mid \overline{\mu}_A(x) \succeq [s, t], f_A(x) \leq k\},$$

and we say it is a cubic level set of $\mathcal{A} = \langle \mu_A, f_A \rangle$.

Theorem 4. *[2] Let S be a semigroup. Then $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S if and only if the level set $U(\mathcal{A}, [s, t], k)$ is a subsemigroup of S.*

Theorem 5. *[2] Let S be a semigroup. Then $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S if and only if $\mathcal{A} \boxdot \mathcal{A} \sqsubseteq \mathcal{A}$.*

Definition 26. *[2] A cubic set $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ in a semigroup S is called a* cubic left *(right) ideal of S if it satisfies:*

(1) $\mu_A(xy) \succeq \overline{\mu}_A(y)$ $(\overline{\mu}_A(xy) \succeq \overline{\mu}_A(x))$,

(2) $f_A(xy) \leq f_A(y)$, $(f_A(xy) \leq f_A(x))$ *for all* $x, y \in S$.

A non-empty cubic set $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ of S is called a cubic ideal *of S if it is a cubic left ideal and a cubic right ideal of S.*

Theorem 6. *[2] Let $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ and $\mathcal{B} = \langle \overline{\mu}_B, f_B \rangle$ be cubic left (right) ideals of S. Then $A\overline{\sqcap}B = \langle \overline{\mu}_A \sqcap \overline{\mu}_B, f_A \vee f_B \rangle$ is a cubic left (right) ideal of S.*

Theorem 7. *[2] Let S be a semigroup and let A be a non-empty subset of S. Then A is a cubic left (right) ideal of S if and only if the characteristic cubic set $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic left (right) ideal of S.*

Proposition 2. *[2] For non-empty subsets G and H of a semigroup S, we have*

1. $\chi_G \boxdot \chi_H = \chi_{GH}$, *i.e.,* $\langle \overline{\mu}_{\chi_G} \Box \overline{\mu}_{\chi_H}, f_{\chi_G} \cdot f_{\chi_H} \rangle$,
2. $\chi_G \overline{\sqcap} \chi_H = \chi_{G \overline{\sqcap} H}$, *i.e.,* $\langle \overline{\mu}_{\chi_G} \sqcap \overline{\mu}_{\chi_H}, f_{\chi_G} \vee f_{\chi_H} \rangle$,
3. $\chi_G \underline{\sqcup} \chi_H = \chi_{G \underline{\sqcup} H}$, *i.e.,* $\langle \overline{\mu}_{\chi_G} \sqcup \overline{\mu}_{\chi_H}, f_{\chi_G} \wedge f_{\chi_H} \rangle$.

Theorem 8. *[2] If $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a right cubic ideal and $\mathcal{B} = \langle \overline{\mu}_B, f_B \rangle$ is a left cubic ideal of a semigorup S, then $\mathcal{A} \boxdot \mathcal{B} \sqsubseteq \mathcal{A} \overline{\sqcap} \mathcal{B}$.*

3. Cubic interior ideal in semigruop

Definition 27. *A cubic subsemigroup $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ in a semigroup S is called a* cubic interior ideal *of S if it satisfies:*

(1) $\overline{\mu}_A(xay) \succeq \overline{\mu}_A(a)$,

(2) $f_A(xay) \leq f_A(a)$, *for all* $x, a, y \in S$.

Example 1. *Consider a semigroup $(S, \cdot)$ defined by the following table:*

$S : \cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	a	a	d

Define a cubic set $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ in S as follows:

$S :$	$\overline{\mu}_A(x)$	$f_A(x)$
a	$[0.9, 1]$	0.3
b	$[0.7, 0.8]$	0.9
c	$[0.5, 0.6]$	0.5
d	$[0.3, 0.4]$	0.7

Then, by routine calculation one can easily verify that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ of S is a cubic interior ideal of S.

Theorem 9. *Let S be a semigroup and $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ be a cubic ideal of S. Then $\mathcal{A}$ is a cubic interior ideal of S.*

Proof. Let $x, y \in S$. Since $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic right ideal of S, we have

$$\overline{\mu}_A(xy) \succeq \overline{\mu}_A(x) \quad \text{and} \quad f_A(xy) \leq f_A(x).$$

Thus
$\overline{\mu}_A(xy) \succeq \overline{\mu}_A(x) \succeq r\min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$ and $f_A(xy) \leq f_A(x) \leq \max\{f_A(x), f_A(y)\}$.

Hence $\mathcal{A} = \langle \mu_A, f_A \rangle$ is a cubic subsemigroup of S. Let $a, x, y \in S$. Then

$$\overline{\mu}_A(xay) \;=\; \overline{\mu}_A(x(ay)) \succeq \overline{\mu}_A(ay) \succeq \overline{\mu}_A(a).$$

And again

$$f_A(xay) \;=\; f_A(x(ay)) \leq f_A(ay) \leq f_A(a).$$

Hence $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S. ◄

Remark 1. *In the example below we can show that the converse of Theorem 9 is not true. Consider* $\overline{\mu}_A(cac) = \overline{\mu}_A(b) = [0.2.04] \not\succeq [0.3, 0.4] = r\min\{\overline{\mu}_A(c), \overline{\mu}_A(c)\}$ *and* $f_A(cac) = f_A(b) = 0.3 \nleq 0.1 = \max\{f_A(c), f_A(c)\}$. *Then* $\mathcal{A} = \langle \mu_A, f_A \rangle$ *is not a cubic ideal of* S.

In the following result we show that cubic interior ideal and cubic ideal coincide.

Theorem 10. *In a regular semigroup the cubic interior ideals and cubic ideals coincide.*

Proof. Suppose that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of a semigroup S and let $x, y \in S$. Since S is a regular semigroup, for any $x \in S$ there exists $s \in S$ such that $x = xsx$. Thus $\overline{\mu}_A(xy) = \overline{\mu}_A((xsx)y) = \overline{\mu}_A((xs)xy) \succeq \overline{\mu}_A(x)$ and $f_A(xy) = f_A((xsx)y) = f_A(xs)xy \leq f_A(x)$. Hence $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic right ideal of S. Similarly, we can prove that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic left ideal of S. Therefore $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic ideal of S. ◄

Theorem 11. *In a left regular semigroup* S*, the cubic interior ideals and cubic ideals coincide.*

Proof. Suppose that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S and let $a, b \in S$. Since S is a left regular, there exists $x \in S$ such that $a = xa^2$. Then

$$\overline{\mu}_A(ab) = \overline{\mu}_A((xa^2)b) = \overline{\mu}_A(xaab) = \overline{\mu}_A((xa)ab) \succeq \overline{\mu}_A(a)$$

and

$$f_A(ab) \;=\; f_A((xa^2)b) = f_A(xaab) = f_A((ax)x(by)) = ((xa)ab) \leq f_A(a).$$

Hence $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic right ideal of S. Similarly, we can prove that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic left ideal of S. Therefore $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic ideal of S. ◄

Theorem 12. *In a right regular semigroup S, the cubic interior ideals and cubic ideals coincide.*

Proof. The proof is similar to the proof of Theorem 11. ◄

Theorem 13. *In an intra-regular semigroup the cubic interior ideals and cubic ideals coincide.*

Proof. Suppose that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of a semigroup S and let $x, y \in S$. Since S is a intra-regular semigroup, for any $x \in S$ there exist $a, b \in S$ such that $x = ax^2b$. Thus

$$\overline{\mu}_A(xy) = \overline{\mu}_A((ax^2b)y) = \overline{\mu}_A((axxb)y) = \overline{\mu}_A((ax)x(by)) \succeq \overline{\mu}_A(x)$$

and

$$f_A(xy) = f_A((ax^2b)y) = f_A((axxb)y) = f_A((ax)x(by)) \leq f_A(x).$$

Hence $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic right ideal of S. Similarly, we can prove that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic left ideal of S. Therefore $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic ideal of S. ◄

Theorem 14. *In a semisimple semigroup S, the cubic interior ideals and cubic ideals coincide.*

Proof. Suppose that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S and let $a, b \in S$. Since S is a semisimple, there exist $x, y, z \in S$ such that $a = xayaz$. Thus

$$\overline{\mu}_A(ab) = \overline{\mu}_A((xayaz)b) = \overline{\mu}_A((xay)a(zb)) \succeq \overline{\mu}_A(a)$$

and

$$f_A(ab) = f_A((xayaz)b) = f_A((xay)a(zb)) \leq f_A(a).$$

Hence $\mathcal{A} = \langle \mu_A, f_A \rangle$ is a cubic right ideal of S. Similarly, we can prove that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic left ideal of S. Therefore $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic ideal of S. ◄

Corollary 1. *In regular, left (right) regular intra-regular and semisimple semigroup, the cubic interior ideals of S and cubic ideals coincide.*

The following theorems establish the basic properties of a cubic interior ideal.

Theorem 15. *Let $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ and $\mathcal{B} = \langle \overline{\mu}_B, f_B \rangle$ be cubic interior ideals of S. Then $A \overline{\sqcap} B = \langle \overline{\mu}_A \sqcap \overline{\mu}_B, f_A \vee f_B \rangle$ is a cubic interior ideal of S.*

Proof. Step 1: Let $x, y \in S$. Since $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ and $\mathcal{B} = \langle \overline{\mu}_B, f_B \rangle$ are cubic interior ideals of S, we have $A \overline{\sqcap} B = \langle \overline{\mu}_A \sqcap \overline{\mu}_B, f_A \vee f_B \rangle$ is a cubic subsemigroup of S, by Theorem 2.

Step 2: Let $x, a, y \in S$. Then

$$(\overline{\mu}_A \sqcap \overline{\mu}_B)(xay) = r\min\{\overline{\mu}_A(xay), \overline{\mu}_B(xay)\} \succeq r\min\{\overline{\mu}_A(a), \overline{\mu}_B(a)\} = (\overline{\mu}_A \sqcap \overline{\mu}_B)(a).$$

And

$$(f_A \vee f_B)(xay) = \max\{f_A(xay), f_B(xay)\}\} \leq \max\{f_A(a), f_B(a)\} = (f_B \vee f_B)(a).$$

Thus
$(\overline{\mu}_A \sqcap \overline{\mu}_B)(xay) \succeq (\overline{\mu}_A \sqcap \overline{\mu}_B)(a)$ and $(f_A \vee f_B)(xay) \leq (f_A \vee f_B)(a)$.
Hence $A \overline{\sqcap} B = \langle \overline{\mu}_A \sqcap \overline{\mu}_B, f_A \vee f_B \rangle$ is a cubic interior ideal of S. ◀

Corollary 2. *The intersection of any family of cubic interior ideals of semigroup S is a cubic interior ideal of a semigroup S.*

Proof. It is straightforward. ◀

Theorem 16. *Let S be a semigroup and let A be a non-empty subset of S. Then A is an interior ideal of S if and only if the characteristic cubic set $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic interior ideal of S.*

Proof. Suppose that A is an interior ideal of S. We will show that $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic interior ideal of S.

Step 1: Let $x, y \in S$. Since A is an interior ideal of S, we have $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic subsemigroup of S, by Theorem 3.

Step 2: Let $x, a, y \in S$. Then we have the following cases:

Case (1): $a \in A$. Then $xay \in A$. Thus $\overline{\mu}_{\chi_A}(a) = \overline{\mu}_{\chi_A}(xay) = [1, 1]$ and $f_{\chi_A}(y) = f_{\chi_A}(xay) = 0$. It follows that

$$\overline{\mu}_{\chi_A}(xay) = [1, 1] \succeq \overline{\mu}_{\chi_A}(a) \quad \text{and} \quad f_{\chi_A}(xay) = 0 \leq f_{\chi_A}(a).$$

Case (2): $a \notin A$. Then

$$\overline{\mu}_{\chi_A}(xay) = [1, 1] \succeq [0, 0] = \overline{\mu}_{\chi_A}(a) \quad \text{and} \quad f_{\chi_A}(xay) = 0 \leq 1 = f_{\chi_A}(a).$$

From cases (1)-(2) we have $\overline{\mu}_{\chi_A}(xay) \succeq \overline{\mu}_{\chi_A}(a)$ and $f_{\chi_A}(xay) \leq f_{\chi_A}(a)$. This implies that $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic interior ideal of S.

Conversely, suppose that $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic interior ideal of S. We will show that A is an interior ideal of S.

Step 1: Let $x, y \in A$. Since $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic interior ideal of S, it follows that A is a subsemigroup of S, by Theorem 3.

Stepv 2: Let $x, a, y \in S$ and $a \in A$. Then $\overline{\mu}_{\chi_A}(a) = [1, 1]$ and $f_{\chi_A}(a) = 0$. Since $\chi_A = \langle \overline{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic interior ideal of S, it follows that $\overline{\mu}_{\chi_A}(xay) \succeq \mu_{\chi_A}(a)$ and $f_{\chi_A}(xay) \leq f_{\chi_A}(a)$. Thus $\mu_{\chi_A}(xay) = [1, 1]$ and $f_{\chi_A}(xay) = 0$. Hence $xay \in A$. Therefore A is an interior ideal of S. ◄

Theorem 17. *Let S be a semigroup. Then $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S if and only if the level set $U(\mathcal{A}, [s, t], k)$ is an interior ideal of S.*

Proof. Suppose that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S. We will show that the cubic level set $U(\mathcal{A}, [s, t], k)$ is an interior ideal of S.

Step 1: Let $x, y \in U(\mathcal{A}, [s, t], k)$. By Theorem 4, it follows that $U(\mathcal{A}, [s, t], k)$ is a subsemigroup of S.

Step 2: Let $a, x, y \in U(\mathcal{A}, [s, t], k)$ and $[s, t] \in D[0, 1], k \in [0, 1]$.
Then $\overline{\mu}_A(a) \succeq [s, t], \overline{\mu}_A(x) \succeq [s, t], \overline{\mu}_A(y) \succeq [s, t]$ and $f_A(a) \leq k, f_A(x) \leq k, f_A(y) \leq k$.

Since $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S, we have

$$\overline{\mu}_A(xay) \succeq \overline{\mu}_A(a) \quad \text{and} \quad f_A(xay) \leq f_A(a)$$

for all $a, x, y \in S$. Thus $\overline{\mu}_A(xay) \succeq [s, t]$ and $f_A(xa(yz)) \leq k$. Hence $(xay) \in U(\mathcal{A}, [s, t], k)$ for all $a, x, y \in U(\mathcal{A}, [s, t], k)$ and $[s, t] \in D[0, 1], k \in [0, 1]$, so the cubic level set $U(\mathcal{A}, [s, t], k)$ is an interior ideal of S.

Conversely, suppose that the cubic level set $U(\mathcal{A}, [s, t], k)$ is an interior ideal of S.
We will show that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S.

Step 1: Let $x, y \in S$. By Theorem 4, $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic subsemigroup of S.

Step 2: Assume that $\mu_A^-(xay) < \mu_A^-(a)$. Then there exist $a, x, y \in S$ such that $\mu_A^-(a) \geq [s, t], \mu_A^-(a) \geq [s, t], \mu_A^-(a) \geq [s, t]$ and $\mu_A^-(xay) < [s, t]$ Thus $a, x, y \in U(\mathcal{A}, [s, t], k)$ and $xay \notin U(\mathcal{A}, [s, t], k)$. Since $U(\mathcal{A}, [s, t], k)$ is an interior ideal of S, we have $xay \in U(\mathcal{A}, [s, t], k)$ for all $a, x, y \in U(\mathcal{A}, [s, t], k)$. Thus $\mu_A^-(xay) \geq [s, t]$. It is a contradiction. Hence $\mu_A^-(xay) \geq \mu_A^-(a)$ for all $a, x, y \in S$.

Similarly we can show that $\mu_A^+(xay) \geq \mu_A^+(a)$. Hence $\overline{\mu}_A(xay) \succeq \overline{\mu}_A(a)$.

Now let us show that $f_A(xay) \leq f_A(a)$. Assume that $f_A(xay) < f_A(a)$. Then there exist $a, x, y \in S$ such that $f_A(a) \leq k, f_A(x) \leq k, f_A(y) \leq k$ and $f_A(xay) > k$. Thus $a, x, y \in U(\mathcal{A}, [s, t], k)$ and $xay \notin U(\mathcal{A}, [s, t], k)$. Since $U(\mathcal{A}, [s, t], k)$ is an interior ideal of S, we have $xay \in U(\mathcal{A}, [s, t], k)$ for all $a, x, y \in U(\mathcal{A}, [s, t], k)$. Thus $f_A(xay) \leq k$. It is a contradiction. Hence $f_A(xay) \leq f_A(a)$ for all $a, x, y \in S$. Therefore $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S. ◄

Theorem 18. *Let S be a semigroup. Then $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S if and only if $\mathcal{A} \boxdot \mathcal{A} \sqsubseteq \mathcal{A}$ and $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} \sqsubseteq \mathcal{A}$.*

Proof. Assume that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S and let $x \in S$. Since $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic subsemigruop of S, we have $\mathcal{A} \boxdot \mathcal{A} \sqsubseteq \mathcal{A}$ by Theorem 5. Now let us show that $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} \sqsubseteq \mathcal{A}$.

Step 1: We will prove that $(1_S \Box \overline{\mu}_A \Box 1_S)(x) \preceq \overline{\mu}_A(x)$.

If $F_x = \emptyset$, then $(1_S \Box \overline{\mu}_A \Box 1_S)(x) = [0,0]$. Thus $(1_S \Box \overline{\mu}_A \Box 1_S)(x) = [0,0] \preceq \overline{\mu}_A(x)$.

If $F_x \neq \emptyset$, then $(1_S \Box \overline{\mu}_A \Box 1_S)(x) = r \sup\limits_{x=yz} \{r \min\{(1_S \Box \overline{\mu}_A)(y), 1_S(z)\}$.

Since $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S, we have $\overline{\mu}_A(pqz) \succeq \overline{\mu}_A(q)$.

$$\begin{aligned}
(1_S \Box \overline{\mu}_A \Box 1_S)(x) &= r \sup\nolimits_{x=yz} \{r \min\{(1_S \Box \overline{\mu}_A)(y), 1_S(z)\} \\
&= r \sup\nolimits_{x=yz} \{r \min\{r \sup\nolimits_{y=pq} \{r \min\{1_S(p), \overline{\mu}_A(q)\}\}, 1_S(z)\}\} \\
&= r \sup\nolimits_{x=yz} \{r \sup\nolimits_{y=pq} \{r \min\{[1,1], \overline{\mu}_A(q)\}\}, [1,1]\} \\
&= r \sup\nolimits_{x=yz} \{r \sup\nolimits_{y=pq} \{\overline{\mu}_A(q)\}\} \\
&\preceq r \sup\nolimits_{x=yz} \{r \sup\nolimits_{y=pq} \{\overline{\mu}_A(pqz)\}\} = \overline{\mu}_A(x).
\end{aligned}$$

Thus $(1_S \Box \overline{\mu}_A \Box 1_S)(x) \preceq \overline{\mu}_A(x)$.

Step 2: We will prove that $(0_S \cdot f_A \cdot 0_S)(x) = f_A(x)$.

If $F_x = \emptyset$, then $(0_S \cdot f_A \cdot 0_S)(x) = 1$. Thus $(0_S \cdot f_A \cdot 0_S)(x) = 1 \geq f_A(x)$.

If $F_x \neq \emptyset$, then $(0_S \cdot f_A \cdot 0_S)(x) = \inf\limits_{a=xy} \{\max\{(f_A \cdot 0_S)(x), f_A(y)\}\}$.

Since $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ is a cubic interior ideal of S, we have $f_A(pqz) \leq f_A(q)$. Thus

$$\begin{aligned}
(0_S \cdot f_A \cdot 0_S)(x) &= \inf_{x=yz} \{\max\{(0_S \cdot f_A)(y), 0_S(z)\} \\
&= \inf_{x=yz} \{\max\{\inf_{y=pq} \{\max\{0_S(p), f_A(q)\}, 0_S(z)\}\} \\
&= \inf_{x=yz} \{\inf_{y=pq} \{\max\{0, f_A(q), 0\}\}\} = \inf_{x=yz} \{\inf_{y=pq} \{f_A(q)\}\} \\
&\geq \inf_{x=yz} \{\inf_{y=pq} \{f_A(pqz)\}\} = f_A(x).
\end{aligned}$$

Thus $(0_S \cdot f_A \cdot 0_S)(x) \geq f_A(x)$. Hence $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} \sqsubseteq \mathcal{A}$.

Conversely, suppose that $\mathcal{A} \boxdot \mathcal{A} \sqsubseteq \mathcal{A}$ and $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} \sqsubseteq \mathcal{A}$ and let $x, y \in S$. Since $\mathcal{A} \boxdot \mathcal{A} \sqsubseteq \mathcal{A}$, we have $\mathcal{A}$ is a cubic subsemigruop of S, by Theorem 5. Let $x, a, y \in S$. Since $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} \sqsubseteq \mathcal{A}$, we have $(1_S \Box \overline{\mu}_A \Box 1_S)(xay) \preceq \mu_A(xay)$ and $(0_S \cdot f_A \cdot 0_S)(xay) \geq f_A(xay)$. Thus

$$\begin{aligned}
\overline{\mu}_A(xay) &\succeq (1_S \Box \overline{\mu}_A \Box 1_S)(xay) = r \sup\nolimits_{xay=ij} \{r \min\{(1_S \Box \overline{\mu}_A)(i), 1_S(j)\} \\
&= r \sup\nolimits_{xay=ij} \{r \min\{r \sup\nolimits_{i=pq} \{r \min\{1_S(p), \overline{\mu}_A(q)\}, 1_S(j)\}\}\} \\
&= r \sup\nolimits_{xay=ij} r \sup\nolimits_{i=pq} \{r \min\{1_S(p), \overline{\mu}_A(q), 1_S(j)\}\} \\
&= r \sup\nolimits_{xay=ij} r \sup\nolimits_{i=pq} \{r \min\{[1,1], \overline{\mu}_A(q), [1,1]\}\} \\
&= r \sup\nolimits_{xay=pqj} \{\overline{\mu}_A(q)\} = \overline{\mu}_A(a)
\end{aligned}$$

and

$$\begin{aligned} f_A(xay) &\leq (0_S \cdot f_A \cdot 0_S)(xay) = \inf_{xay=ij}\{\max\{(0_S \cdot f_A)(i), 0_S(j)\}\} \\ &= \inf_{xay=ij}\{\max\{\inf_{i=pq}\{\max\{0_S(p), f_A(q)\}\}, 0_S(j)\}\} \\ &= \inf_{xay=ij}\ \inf_{i=pq}\{\max\{0_S(p), f_A(q), 0_S(j)\}\} \\ &= \inf_{xay=ij}\ \inf_{i=pq}\{\max\{0, f_A(q), 0\}\} = \inf_{xay=pqj}\{f_A(q)\} = f_A(a). \end{aligned}$$

Hence $\overline{\mu}_A(xay) \succeq \overline{\mu}_A(a)$ and $f_A(xay) \leq f_A(a)$.
Therefore $\mathcal{A} = \langle \mu_A, f_A \rangle$ is a cubic interior ideal of S. ◄

Theorem 19. *Let S be a left regular semigroup and let $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ be a cubic interior ideal of S. Then $\mathcal{A} \boxdot \mathcal{A} = \mathcal{A}$.*

Proof. Let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a = xa^2 = x(aa) = (xa)a = (x(xa^2))a = (x(xaa))a = ((xx)aa)a = ((x^2)aa)(xa^2) = ((x^2)aa)(xaa)$. Thus

$$\begin{aligned} (\overline{\mu}_A \square \mu_A)(a) &= r\sup_{a=yz}\{r\min\{\overline{\mu}_A(y), \overline{\mu}_A(z)\}\} \\ &= r\sup_{((x^2)aa)(xaa)=yz}\{r\min\{\overline{\mu}_A(y), \overline{\mu}_A(z)\}\} \\ &\succeq r\min\{\overline{\mu}_A((x^2)aa), \overline{\mu}_A(xaa)\} \\ &\succeq r\min\{\overline{\mu}_A(a), \overline{\mu}_A(a)\} = \overline{\mu}_A(a) \end{aligned}$$

and

$$\begin{aligned} (f_A \cdot f_A)(a) &= \inf_{a=yz}\{\max\{f_A(y), f_A(z)\}\} \\ &= \inf_{((x^2)aa)(xaa)=yz}\{\max\{f_A(y), f_A(z)\}\} \\ &\leq \max\{f_A((x^2)aa), f_A(xaa)\} \\ &\leq \max\{f_A(a), f_A(a)\} = f_A(a). \end{aligned}$$

Hence $(\overline{\mu}_A \square \overline{\mu}_A)(a) \succeq \overline{\mu}_A(a)$ and $(f_A \cdot f_A)(a) \leq f_A(a)$. Therefore $\mathcal{A} \sqsubseteq \mathcal{A} \boxdot \mathcal{A}$.

Since $\mathcal{A} = \langle \mu_A, f_A \rangle$ is a cubic subsemigruop of S, we have $\mathcal{A} \boxdot \mathcal{A} \sqsubseteq \mathcal{A}$, by Theorem 5. Thus $\mathcal{A} \boxdot \mathcal{A} = \mathcal{A}$. ◄

Theorem 20. *Let S be a left regular semigroup and let $\mathcal{A} = \langle \mu_A, f_A \rangle$ be a cubic interior ideal of S. Then $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} = \mathcal{A}$*

Proof. Let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a = xa^2 = x(aa) = xaa = x(xaa)a$. Thus

$$\begin{aligned} (1_S \square \overline{\mu}_A \square 1_S)(x) &= r\sup_{x=yz}\{r\min\{(1_S \square \overline{\mu}_A)(y), 1_S(z)\}\} \\ &= r\sup_{x(xaa)a=yz}\{r\min\{(1_S \square \overline{\mu}_A)(y), 1_S(z)\}\} \\ &\succeq r\min\{(1_S \square \overline{\mu}_A)(x(xaa)), 1_S(a)\} \\ &= r\min\{(1_S \square \mu_A)(x(xaa)), [1,1]\} = (1_S \square \overline{\mu}_A)(x(xaa)) \\ &= r\sup_{(x(xaa))=pq}\{r\min\{1_S(p), \overline{\mu}_A(q)\}\} \\ &\succeq r\min\{1_S(x), \overline{\mu}_A(xaa)\}\} = r\min\{[1,1], \overline{\mu}_A(xaa)\} \\ &= \overline{\mu}_A(xaa) \succeq \overline{\mu}_A(a) \end{aligned}$$

and

$$\begin{aligned}(0_S \cdot f_A \cdot 0_S)(x) &= \inf_{x=yz}\{\max\{(0_S \cdot \mu_A)(y), 0_S(z)\}\\ &= \inf_{x(xaa)a=yz}\{\max\{(0_S \cdot f_A)(y), 0_S(z)\}\\ &\leq \max\{(0_S \cdot f_A)(x(xaa)), 0_S(a)\}\\ &= \max\{(0_S \cdot f_A)(x(xaa)), 0\} = (0_S \cdot f_A)(x(xaa))\\ &= \inf_{(x(xaa))=pq}\{\max\{0_S(p), f_A(q)\}\}\\ &\leq \max\{0_S(x), f_A(xaa)\}\}\\ &= \max\{0, f_A(xaa)\} = f_A(xaa) \leq f_A(a)\end{aligned}$$

It implies that $(1_S \Box \overline{\mu}_A \Box 1_S)(x) \succeq \overline{\mu}_A(a)$ and $(0_S \cdot f_A \cdot 0_S)(x) \leq f_A(a)$. Hence $\mathcal{A} \sqsubseteq \mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S}$. By Theorem 18 we have $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} = \mathcal{A}$. ◄

Corollary 3. *Let S be a left regular semigroup and let $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ be a cubic interior ideal of S. Then $\mathcal{A} \boxdot \mathcal{A} = \mathcal{A}$ and $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} = \mathcal{A}$.*

Theorem 21. *Let S be an intra-regular semigroup and let $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ be a cubic interior ideal of S. Then $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} = \mathcal{A}$.*

Proof. Let $a \in S$. Since S is intra-regular, there exists $x \in S$ such that $a = xa^2y$. Thus

$$\begin{aligned}(1_S \Box \mu_A \Box 1_S)(x) &= r\sup_{x=yz}\{r\min\{1_S(y), (\overline{\mu}_A \Box 1_S)(z)\}\\ &= r\sup_{xaay=yz}\{r\min\{1_S(y), (\overline{\mu}_A \Box 1_S)(z)\}\\ &\succeq r\min\{1_S(xa), (\overline{\mu}_A \Box 1_S)(ay)\}\\ &= r\min\{[1,1], (\overline{\mu}_A \Box 1_S)(ay)\} = (\mu_A \Box 1_S)(ay)\\ &= r\sup_{ay=pq}\{r\min\{\overline{\mu}_A(p), 1_S(q)\}\}\\ &= r\sup_{xaayy=pq}\{r\min\{\overline{\mu}_A(p), 1_S(q)\}\}\\ &\succeq r\min\{\overline{\mu}_A(xaa), 1_S(y^2)\}\} = r\min\{\overline{\mu}_A(xaa), [1,1]\}\\ &= \overline{\mu}_A(xaa) \succeq \overline{\mu}_A(a)\end{aligned}$$

and

$$\begin{aligned}(0_S \cdot f_A \cdot 0_S)(x) &= \inf_{x=yz}\{\max\{0_S(y), (f_A \cdot 0_S)(z)\}\\ &= \inf_{xaay=yz}\{\max\{0_S(y), (f_A \cdot 0_S)(z)\}\\ &\succeq \max\{0_S(xa), (f_A \cdot 0_S)(ay)\}\\ &= \max\{0, (f_A \cdot 0_S)(ay)\} = (f_A \cdot 0_S)(ay)\\ &= \inf_{ay=pq}\{\max\{f_A(p), 0_S(q)\}\}\\ &= \inf_{xaayy=pq}\{\max\{f_A(p), 0_S(q)\}\}\\ &\succeq \max\{f_A(xaa), 0_S(y^2)\}\} = r\min\{f_A(xaa), 0\}\\ &= f_A(xaa) \succeq f_A(a).\end{aligned}$$

It implies that $(1_S \Box \overline{\mu}_A \Box 1_S)(x) \succeq \overline{\mu}_A(a)$ and $(0_S \cdot f_A \cdot 0_S)(x) \leq f_A(a)$. Hence $\mathcal{A} \sqsubseteq \mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S}$. By Theorem 18 we have $\mathcal{S} \boxdot \mathcal{A} \boxdot \mathcal{S} = \mathcal{A}$. ◄

Lemma 1. *[8] For a semigroup S, the following statements are equivalent.*

1. *S is semisimple,*
2. *Every interior ideal of S is idempotent,*
3. *Every ideal of S is idempotent,*
4. *$A \cap B = AB$ for all ideals A and B of S,*
5. *$A \cap B = AB$ for every ideal A and every interior ideal B of S,*
6. *$A \cap B = AB$ for every interior ideal A and every ideal B of S,*
7. *$A \cap B = AB$ for all interior ideals A and B of S.*

Theorem 22. *Let S be a semigroup. Then the following are equivalent:*

1. *S is semisimple,*
2. *$\mathcal{A} \boxdot \mathcal{A} = \mathcal{A}$, for every cubic interior ideal $\mathcal{A}$ of S,*
3. *$\mathcal{A} \boxdot \mathcal{A} = \mathcal{A}$, for every cubic ideal $\mathcal{A}$ of S,*
4. *$\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \bar{\sqcap} \mathcal{B}$, for every cubic interior ideals $\mathcal{A}$ and $\mathcal{B}$ of S,*
5. *$\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \bar{\sqcap} \mathcal{B}$ for every cubic ideals $\mathcal{A}$ and $\mathcal{B}$ of S,*
6. *$\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \bar{\sqcap} \mathcal{B}$, for every cubic interior ideal $\mathcal{A}$ of S and every cubic ideal $\mathcal{B}$ of S,*
7. *$\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \bar{\sqcap} \mathcal{B}$, for every cubic ideal $\mathcal{A}$ of S and every cubic interior ideal $\mathcal{B}$ of S.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathcal{A}$ is a cubic interior ideal of S and S is a semisimple semigroup and let $a \in S$. Since $\mathcal{A}$ is a cubic subsemigroup of S, we have $\mathcal{A} \boxdot \mathcal{A} \sqsubseteq \mathcal{A}$, by Theorem 5.

Since S is semisimple, there exist $w, x, y, z \in S$ such that $a = (xay)(zaw)$. Thus

$$\begin{array}{rcl}(\overline{\mu}_A \Box \overline{\mu}_A)(a) & = & r\sup_{a=ij}\{r\min\{\overline{\mu}_A(i), \overline{\mu}_A(j)\}\} \\ & = & r\sup_{(xay)(waz)=ij}\{r\min\{\overline{\mu}_A(i), \overline{\mu}_A(j)\}\} \\ & \succeq & r\min\{\overline{\mu}_A(xay), \overline{\mu}_A(waz)\} \\ & \succeq & r\min\{\overline{\mu}_A(a), \overline{\mu}_A(a)\} = \overline{\mu}_A(a)\end{array}$$

and

$$\begin{aligned}(f_A \cdot f_A)(a) &= \inf_{x=ij}\{\max\{f_A(i), f_A(j)\}\} \\ &= \inf_{(xay)(waz)=ij}\{\max\{f_A(i), f_A(j)\}\} \\ &\leq \max\{f_A(xay), f_A(waz)\} \\ &\leq \max\{f_A(a), f_A(a)\} = f_A(a).\end{aligned}$$

Hence $(\overline{\mu}_A \Box \overline{\mu}_A)(a) \succeq \overline{\mu}_A(a)$ and $(f_A \cdot f_A)(a) \leq f_A(a)$, so $\mathcal{A} \sqsubseteq \mathcal{A} \boxdot \mathcal{A}$. Therefore $\mathcal{A} = \mathcal{A} \boxdot \mathcal{A}$.

(2) $\Rightarrow$ (1) Assume that $\mathcal{A} = \mathcal{A} \boxdot \mathcal{A}$ and let A be an interior ideal of S. Then by Theorem 16, χ_A is a cubic interior ideal of S. By hypothesis and Proposition 2,

$$(\overline{\mu}_{\chi_{A^2}})(a) = (\overline{\mu}_{\chi_A} \Box \overline{\mu}_{\chi_A})(a) = \overline{\mu}_{\chi_A}(a) = [1, 1]$$

and

$$(f_{\chi_{A^2}})(a) = (f_{\chi_A} \cdot f_{\chi_A})(a) = f_{\chi_A}(a) = 0.$$

Thus $a \in A^2$. Hence $A^2 = A$. By Lemma 1, it follows that S is semisimple.

(1) $\Rightarrow$ (4) Let $\mathcal{A}$ and $\mathcal{B}$ be cubic interior ideals of S. Assume that S is a semisimple semigroup. Then by Theorem 14, it follows that $\mathcal{A}$ and $\mathcal{B}$ are cubic ideals of S. Thus by Theorem 8, $\mathcal{A} \boxdot \mathcal{B} \sqsubseteq \mathcal{A} \overline{\sqcap} \mathcal{B}$. Now let us show that $\mathcal{A} \overline{\sqcap} \mathcal{B} \sqsubseteq \mathcal{A} \boxdot \mathcal{B}$. Let $a \in S$. Since S is semisimple, there exist $w, x, y, z \in S$ such that $a = (xay)(zaw)$. Thus

$$\begin{aligned}(\overline{\mu}_A \Box \overline{\mu}_B)(a) &= r\sup_{a=ij}\{r\min\{\overline{\mu}_A(i), \overline{\mu}_B(j)\}\} \\ &= r\sup_{(xay)(waz)=ij}\{r\min\{\overline{\mu}_A(i), \overline{\mu}_B(j)\}\} \\ &\succeq r\min\{\overline{\mu}_A(xay), \overline{\mu}_B(waz)\} \\ &\succeq r\min\{\overline{\mu}_A(a), \overline{\mu}_B(a)\} = (\overline{\mu}_A \sqcap \overline{\mu}_B)(a)\end{aligned}$$

and

$$\begin{aligned}(f_A \cdot f_B)(a) &= \inf_{x=ij}\{\max\{f_A(i), f_B(j)\}\} \\ &= \inf_{(xay)(waz)=ij}\{\max\{f_A(i), f_B(j)\}\} \\ &\leq \max\{f_A(xay), f_B(waz)\} \\ &\leq \max\{f_A(a), f_B(a)\} = (f_A \vee f_B)(a).\end{aligned}$$

Hence $(\overline{\mu}_A \Box \overline{\mu}_B)(a) \succeq (\overline{\mu}_A \sqcap \overline{\mu}_B)(a)$ and $(f_A \cdot f_B)(a) \leq (f_A \vee f_B)(a)$, so $\mathcal{A} \overline{\sqcap} \mathcal{B} \sqsubseteq \mathcal{A} \boxdot \mathcal{B}$. Therefore $\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \overline{\sqcap} \mathcal{B}$.

(4) $\Rightarrow$ (1) Suppose that $\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \overline{\sqcap} \mathcal{B}$ and let A, B be interior ideals of S. Then by Theorem 16, χ_A and χ_B are cubic interior ideals of S. By hypothesis and Proposition 2, we have

$$(\overline{\mu}_{\chi_{AB}})(a) = (\overline{\mu}_{\chi_A} \Box \overline{\mu}_{\chi_B})(a) = (\overline{\mu}_{\chi_A} \sqcap \overline{\mu}_{\chi_B})(a) = \mu_{\chi_{A \sqcap B}}(a) = [1, 1]$$

and

$$(f_{\chi_{AB}})(a) = (f_{\chi_A} \cdot f_{\chi_B})(a) = (f_{\chi_A} \vee f_{\chi_B})(a) = f_{\chi_{A \vee B}}(a) = 0.$$

Thus $a \in AB$. Hence $AB = A \cap B$. By Lemma 1, S is semisimple.

(1) $\Rightarrow$ (6) Let $\mathcal{A}$ be a cubic interior ideal of S and let $\mathcal{B}$ be a cubic ideal of S. We will show that $\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \bar{\sqcap} \mathcal{B}$. Since S is a semigroup, $\mathcal{B}$ is a cubic interior ideal of S. Thus by (4), $\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \bar{\sqcap} \mathcal{B}$.

(6) $\Rightarrow$ (1) Suppose that $\mathcal{A} \boxdot \mathcal{B} = \mathcal{A} \bar{\sqcap} \mathcal{B}$ and let A be an interior ideal of S and let B be an ideal of S. Then by Theorems 16 and 7, χ_A is a cubic interior ideal of S and χ_B is a cubic ideal of S. By hypothesis and Proposition 2, we have

$$(\overline{\mu}_{\chi_{AB}})(a) = (\overline{\mu}_{\chi_A} \square \overline{\mu}_{\chi_B})(a) = (\overline{\mu}_{\chi_A} \sqcap \overline{\mu}_{\chi_B})(a) = \overline{\mu}_{\chi_{A \cap B}}(a) = [1, 1]$$

and

$$(f_{\chi_{AB}})(a) = (f_{\chi_A} \cdot f_{\chi_B})(a) = (f_{\chi_A} \vee f_{\chi_B})(a) = f_{\chi_{A \vee B}}(a) = 0.$$

Thus $a \in AB$. Hence $AB = A \cap B$. By Lemma 1, S is semisimple.

So, (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (7) are straightforward. ◄

4. Homomorphic inverse image operation to get cubic set

In this section, we study some properties of homomorphic and inverse image of cubic set.

Definition 28. *[2] Let $\mathcal{C}(X)$ be the family of cubic sets in X. Let X and Y be given classical sets. A mapping $h : X \to Y$ induces two mappings: $\mathcal{C}_h : \mathcal{C}(X) \to \mathcal{C}(Y)$, $\mathcal{A} \mapsto \mathcal{C}_h(\mathcal{A})$ and $\mathcal{C}_h^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$, $\mathcal{B} \mapsto \mathcal{C}_h^{-1}(\mathcal{B})$, where $\mathcal{C}_h(\mathcal{A})$ is given by*

$$\mathcal{C}_h(\overline{\mu}_A)(y) = \begin{cases} r\sup_{y=h(x)} \overline{\mu}_A(x), & \text{if } \quad h^{-1}(y) \neq 0, \\ [0,0], & \text{otherwise} \end{cases}$$

$$\mathcal{C}_h(f_A)(y) = \begin{cases} \inf_{y=h(x)} f_A(x), & \text{if } \quad h^{-1}(y) \neq 0, \\ 1, & \text{otherwise} \end{cases}$$

for all $y \in Y$. The inverse image *$\mathcal{C}_h^{-1}(\mathcal{B})$ is defined by $\mathcal{C}_h^{-1}(\overline{\mu}_B)(x) = \overline{\mu}_B(h(x))$ and $\mathcal{C}_h^{-1}(f_B)(x) = f_B(h(x))$ for all $x \in X$. Then the mapping $\mathcal{C}_h$ ($\mathcal{C}_h^{-1}$) is called a cubic transformation (inverse cubic transformation) induced by h. A cubic set $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle$ in X has the* cubic property *if for any subset T of X there exists $x_0 \in T$ such that $\overline{\mu}_A(x_0) = r\sup_{x \in T} \overline{\mu}_A(x)$ and $f_A(x_0) = \inf_{x \in T} f_A(x)$.*

Theorem 23. *For a homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h : \mathcal{C}(X) \to \mathcal{C}(Y)$ be the cubic transformation induced by h. If $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle \in \mathcal{C}(X)$ is a cubic interior ideal of X which has the cubic property, then $\mathcal{C}_h(A)$ is a cubic interior ideal of Y.*

Proof. Let $\mathcal{A} = \langle \overline{\mu}_A, f_A\rangle \in \mathcal{C}(X)$ be a cubic interior ideal of X and let $x_0 \in h^{-1}(h(x))$, $y_0 \in h^{-1}(h(y))$ be such that $\overline{\mu}_A(x_0) = r\sup_{a\in h^{-1}(h(x))} \overline{\mu}_A(a)$, $f_A(x_0) = \inf_{a\in h^{-1}(h(x))} f_A(a)$ and $\overline{\mu}_A(y_0) = r\sup_{b\in h^{-1}(h(y))} \overline{\mu}_A(a)$, $f_A(y_0) = \inf_{b\in h^{-1}(h(y))} f_A(b)$, respectively. Then

$$\begin{aligned}
\mathcal{C}_h(\overline{\mu}_A)(h(x)h(y)) &= r\sup_{z\in h^{-1}(h(x)h(y))} \overline{\mu}_A(z) \succeq \overline{\mu}_A(x_0y_0)\\
&\succeq r\min\{\overline{\mu}_A(x_0), \overline{\mu}_A(y_0)\}\\
&= r\min\{r\sup_{a\in h^{-1}(h(x))} \overline{\mu}_A(a), r\sup_{b\in h^{-1}(h(y))} \overline{\mu}_A(b)\}\\
&= r\min\{\mathcal{C}_h(\overline{\mu}_A(a))(h(x)), \mathcal{C}_h(\overline{\mu}_A(a))(h(y))\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_h(f_A)(h(x)h(y)) &= \inf_{z\in h^{-1}(h(x)h(y))} f_A(z) \leq f_A(x_0y_0)\\
&\leq \max\{f_A(x_0), f_A(y_0)\}\\
&= \max\{\inf_{a\in h^{-1}(h(x))} f_A(a), \inf_{b\in h^{-1}(h(y))} f_A(b)\}\\
&= \max\{\mathcal{C}_h(h(x)), \mathcal{C}_h(h(y))\}.
\end{aligned}$$

Thus $\mathcal{C}_h(\overline{\mu}_A)(h(x)h(y)) \succeq r\min\{\mathcal{C}_h(\overline{\mu}_A(a))(h(x)), \mathcal{C}_h(\overline{\mu}_A(a))(h(y))\}$ and $\mathcal{C}_h(f_A)(h(x)h(y)) \leq \max\{\mathcal{C}_h(h(x)), \mathcal{C}_h(h(y))\}$. Hence $\mathcal{C}_h(\mathcal{A})$ is a cubic subsemigroup of Y.

Similarly, let $h(a), h(x), h(y) \in h(x)$ and let $x_0 \in h^{-1}(h(a))$, $y_0 \in h^{-1}(h(x))$, $w_0 \in h^{-1}(h(y))$ be such that
$\overline{\mu}_A(x_0) = r\sup_{a\in h^{-1}(h(a))} \overline{\mu}_A(a)$, $f_A(x_0) = \inf_{a\in h^{-1}(h(a))} f_A(a)$,
$\overline{\mu}_A(y_0) = r\sup_{b\in h^{-1}(h(x))} \overline{\mu}_A(b)$, $f_A(y_0) = \inf_{b\in h^{-1}(h(x))} f_A(b)$ and $\overline{\mu}_A(w_0) = r\sup_{c\in h^{-1}(h(y))} \overline{\mu}_A(c)$, $f_A(w_0) = \inf_{c\in h^{-1}(h(y))} f_A(c)$, respectively. Then

$$\begin{aligned}
\mathcal{C}_h(\overline{\mu}_A)(h(a)h(x)h(y)) &= r\sup_{k\in h^{-1}(h(a)h(x)(h(y))} \overline{\mu}_A(k) \succeq \overline{\mu}_A(x_0y_0w_0)\\
&\succeq \overline{\mu}_A(y_0) = r\sup_{c\in h^{-1}(h(x))} \overline{\mu}_A(b) = \mathcal{C}_h(\overline{\mu}_A(b))(h(x))
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_h(f_A)(h(a)h(x)h(y)) &= \inf_{k\in h^{-1}(h(a)h(x)(h(y))} f_A(k) \succeq f_A(x_0y_0w_0)\\
&\leq f_A(y_0) = \inf_{c\in h^{-1}(h(x))} f_A(b) = \mathcal{C}_h((h(x)).
\end{aligned}$$

Thus
$\mathcal{C}_h(\overline{\mu}_A)(h(a)h(x)h(y)) \succeq \mathcal{C}_h(\overline{\mu}_A(b))(h(x))$ and $\mathcal{C}_h(f_A)(h(a)h(x)h(y)) \leq \mathcal{C}_h((h(x))$.

Hence $\mathcal{C}_h(\mathcal{A})$ is a cubic interior ideal of Y. ◀

Theorem 24. *For a homomorphism $h : X \to Y$ of semigroups, let $\mathcal{C}_h^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the inverse cubic transformation, induced by h. If $\mathcal{A} = \langle \overline{\mu}_A, f_A\rangle \in \mathcal{C}(Y)$ is a cubic interior ideal of Y, then $\mathcal{C}_h^{-1}(B)$ is a cubic interior ideal of X.*

Proof. Suppose that $\mathcal{A} = \langle \overline{\mu}_A, f_A \rangle \in \mathcal{C}(Y)$ is a cubic interior ideal of Y and let $x, y \in X$. Then

$$\begin{aligned} \mathcal{C}_h^{-1}(\overline{\mu}_B(xy)) &= \overline{\mu}_B(h((xy)) = \overline{\mu}_B(h(x)h(y)) \\ &\succeq r\min\{\overline{\mu}_B(h(x)), \overline{\mu}_B(h(y))\} \\ &= r\min\{\mathcal{C}_h^{-1}(\overline{\mu}_B(x)), \mathcal{C}_h^{-1}(\overline{\mu}_B(y))\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_h^{-1}(f_B(xy)) &= f_B(h((xy)) = f_B(h(x)h(y)) \\ &\leq r\min\{f_B(h(x)), f_B(h(y))\} \\ &= r\min\{\mathcal{C}_h^{-1}(f_B(x)), \mathcal{C}_h^{-1}(f_B(y))\}. \end{aligned}$$

Thus $\mathcal{C}_h^{-1}(\overline{\mu}_B(xy)) \succeq r\min\{\mathcal{C}_h^{-1}(\overline{\mu}_B(x)), \mathcal{C}_h^{-1}(\overline{\mu}_B(y))\}$ and
$\mathcal{C}_h^{-1}(f_B(xy)) \leq r\min\{\mathcal{C}_h^{-1}(f_B(x)), \mathcal{C}_h^{-1}(f_B(y))\}$. Hence $\mathcal{C}_h^{-1}(\mathcal{A})$ is a cubic subsemigroup of S.

Let $a, x, y, \in X$. Then

$$\begin{aligned} \mathcal{C}_h^{-1}(\overline{\mu}_B(xay) &= \mu_B(h(xay)) = \overline{\mu}_B(h(x)h(a)h(y)) \\ &\succeq \overline{\mu}_B(h(a)) = \mathcal{C}_h^{-1}(\mu_B(a)) \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_h^{-1}(f_B(xay)) &= f_B(h((xay)) = f_B(h(x)h(a)h(y) \\ &\leq f_B(h(a)) = \mathcal{C}_h^{-1}(f_B(a)). \end{aligned}$$

Thus $\mathcal{C}_h^{-1}(\overline{\mu}_B(xay)) \succeq \mathcal{C}_h^{-1}(\overline{\mu}_B(a))$ and $\mathcal{C}_h^{-1}(f_B(xay)) \leq \mathcal{C}_h^{-1}(f_B(a))$.

Therefore $\mathcal{C}_h^{-1}(B)$ is a cubic interior ideal of X. ◄

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