Azerbaijan Journal of Mathematics
V. 10, No 2, 2020, July
ISSN 2218-6816

Quasilinearity of Some Functionals Associated to a Weakened Davis-Choi-Jensen's Inequality for Positive Maps

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Abstract. In this paper we establish some quasilinearity properties of some functionals associated to a weakened Davis-Choi-Jensen's inequality for positive maps and convex (concave) functions. Applications for power function are also provided.

Key Words and Phrases: Jensen's inequality, convex functions, positive maps, superadditivity, subadditivity.

2010 Mathematics Subject Classifications: 47A63, 47A30, 15A60

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [15, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$, then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known [2]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$. Then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \leq \Psi(f(A)). \tag{2}$$

In the recent paper [9] we established the following *weakened version of Davis-Choi-Jensen's inequality* that holds for the larger class of convex functions:

Theorem 2. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ a normalised positive linear map. Then for any selfadjoint operator A whose spectrum $Sp(A)$ is contained in I we have*

$$f(\langle \Phi(A) y, y\rangle) \leq \langle \Phi(f(A)) y, y\rangle \tag{3}$$

for any $y \in K$, $\|y\| = 1$.

For the sake of completeness, we give here a simple proof as follows.

Let $m < M$ and $Sp(A) \subseteq [m, M] \subset I$. Then $m1_H \leq A \leq M1_H$ and since $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$ we have $m1_K \leq \Phi(A) \leq M1_K$ showing that $\langle \Phi(A) y, y \rangle \in [m, M]$ for any $y \in K$, $\|y\| = 1$.

By the gradient inequality for the convex function f we have for $a = \langle \Phi(A) y, y \rangle \in [m, M]$

$$f(t) \geq f(\langle \Phi(A) y, y \rangle) + (t - \langle \Phi(A) y, y \rangle) f'_+(\langle \Phi(A) y, y \rangle)$$

for any $t \in I$, where f'_+ is the lateral derivative of f on I.

Using the continuous functional calculus for the operator A we have for a fixed $y \in K$ with $\|y\| = 1$

$$f(A) \geq f(\langle \Phi(A) y, y \rangle) 1_H + f'_+(\langle \Phi(A) y, y \rangle)(A - \langle \Phi(A) y, y \rangle 1_H). \tag{4}$$

Since $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$, by taking the functional Φ in the inequality (4) we get

$$\Phi(f(A)) \geq f(\langle \Phi(A) y, y \rangle) 1_K + f'_+(\langle \Phi(A) y, y \rangle)(\Phi(A) - \langle \Phi(A) y, y \rangle 1_K) \tag{5}$$

for any $y \in K$ with $\|y\| = 1$.

This inequality is of interest in itself.

Taking the inner product in (5) we have for any $y \in K$ with $\|y\| = 1$

$$\begin{aligned} &\langle \Phi(f(A)) y, y \rangle \\ &\geq f(\langle \Phi(A) y, y \rangle) \|y\|^2 + f'_+(\langle \Phi(A) y, y \rangle) \left(\langle \Phi(A) y, y \rangle - \langle \Phi(A) y, y \rangle \|y\|^2 \right) \\ &= f(\langle \Phi(A) y, y \rangle) \end{aligned}$$

and the inequality (3) is proved.

If the normality condition is dropped, then we have:

Corollary 1. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I and $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$. Then for any selfadjoint operator A whose spectrum $Sp(A)$ is contained in I we have*

$$f\left(\frac{\langle \Psi(A) v, v \rangle}{\langle \Psi(1_H) v, v \rangle} \right) \leq \frac{\langle \Psi(f(A)) v, v \rangle}{\langle \Psi(1_H) v, v \rangle} \tag{6}$$

for any $v \in K$ with $v \neq 0$.

For Jensen's type operator inequalities see [1], [3]-[14] and the references therein.

We define by $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ the convex cone of all linear, positive maps Ψ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, namely $\Psi(1_H)$ is a positive invertible operator in K and define the functional $\triangle_{f,A,v} : \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ by

$$\triangle_{f,A,v}(\Psi) = \langle \Psi(1_H) v, v\rangle f\left(\frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle}\right),$$

where $f : I \to \mathbb{R}$ is a convex (concave) function on the interval I, A is a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \neq 0$.

In this paper we establish some quasilinearity properties of some functionals associated to the weakened Davis-Choi-Jensen's inequality (6) for positive maps and convex (concave) functions. Applications for power function are also provided.

2. The main results

The following result holds:

Theorem 3. *Let $f : I \to \mathbb{R}$ be a convex (concave) function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \neq 0$. If Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and $\lambda \in [0,1]$, then*

$$\triangle_{f,A,v}((1-\lambda)\Psi_1 + \lambda\Psi_2) \le (\ge) (1-\lambda)\triangle_{f,A,v}(\Psi_1) + \lambda\triangle_{f,A,v}(\Psi_2), \tag{7}$$

namely $\triangle_{f,A,v}$ is convex (concave) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

In particular, we have

$$\triangle_{f,A,v}(\Psi_1 + \Psi_2) \le (\ge) \triangle_{f,A,v}(\Psi_1) + \triangle_{f,A,v}(\Psi_2), \tag{8}$$

namely $\triangle_{f,A,v}$ is subadditive (superadditive) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Proof. Assume that $f : I \to \mathbb{R}$ is a convex function on the interval I and $v \in K$, $v \neq 0$.

Let Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} &\triangle_{f,A,v}((1-\lambda)\Psi_1 + \lambda\Psi_2) \\ &= \langle ((1-\lambda)\Psi_1 + \lambda\Psi_2)(1_H) v, v\rangle f\left(\frac{\langle ((1-\lambda)\Psi_1 + \lambda\Psi_2)(A) v, v\rangle}{\langle ((1-\lambda)\Psi_1 + \lambda\Psi_2)(1_H) v, v\rangle}\right) \\ &= [(1-\lambda)\langle \Psi_1(1_H) v, v\rangle + \lambda\langle \Psi_2(1_H) v, v\rangle] \\ &\times f\left(\frac{(1-\lambda)\langle \Psi_1(A) v, v\rangle + \lambda\langle \Psi_2(A) v, v\rangle}{(1-\lambda)\langle \Psi_1(1_H) v, v\rangle + \lambda\langle \Psi_2(1_H) v, v\rangle}\right). \end{aligned} \tag{9}$$

Using the convexity of f we have

$$f\left(\frac{(1-\lambda)\langle\Psi_1(A)v,v\rangle+\lambda\langle\Psi_2(A)v,v\rangle}{(1-\lambda)\langle\Psi_1(1_H)v,v\rangle+\lambda\langle\Psi_2(1_H)v,v\rangle}\right) \tag{10}$$
$$=f\left(\frac{(1-\lambda)\langle\Psi_1(1_H)v,v\rangle\frac{\langle\Psi_1(A)v,v\rangle}{\langle\Psi_1(1_H)v,v\rangle}+\lambda\langle\Psi_2(1_H)v,v\rangle\frac{\langle\Psi_2(A)v,v\rangle}{\langle\Psi_2(1_H)v,v\rangle}}{(1-\lambda)\langle\Psi_1(1_H)v,v\rangle+\lambda\langle\Psi_2(1_H)v,v\rangle}\right)$$
$$\leq\frac{(1-\lambda)\langle\Psi_1(1_H)v,v\rangle f\left(\frac{\langle\Psi_1(A)v,v\rangle}{\langle\Psi_1(1_H)v,v\rangle}\right)+\lambda\langle\Psi_2(1_H)v,v\rangle f\left(\frac{\langle\Psi_2(A)v,v\rangle}{\langle\Psi_2(1_H)v,v\rangle}\right)}{(1-\lambda)\langle\Psi_1(1_H)v,v\rangle+\lambda\langle\Psi_2(1_H)v,v\rangle}$$

and by multiplying (10) by $(1-\lambda)\langle\Psi_1(1_H)v,v\rangle+\lambda\langle\Psi_2(1_H)v,v\rangle>0$ and by using (9), we get

$$\begin{aligned}&\triangle_{f,A,v}((1-\lambda)\Psi_1+\lambda\Psi_2)\\&\leq(1-\lambda)\langle\Psi_1(1_H)v,v\rangle f\left(\frac{\langle\Psi_1(A)v,v\rangle}{\langle\Psi_1(1_H)v,v\rangle}\right)+\lambda\langle\Psi_2(1_H)v,v\rangle f\left(\frac{\langle\Psi_2(A)v,v\rangle}{\langle\Psi_2(1_H)v,v\rangle}\right)\\&=(1-\lambda)\triangle_{f,A,v}(\Psi_1)+\lambda\triangle_{f,A,v}(\Psi_2),\end{aligned}$$

which proves the convexity of $\triangle_{f,A,v}$.

We have by (7)

$$\begin{aligned}\triangle_{f,A,v}(\Psi_1+\Psi_2)&=\triangle_{f,A,v}\left(\frac{2\Psi_1+2\Psi_2}{2}\right)\leq\frac{\triangle_{f,A,v}(2\Psi_1)+\triangle_{f,A,v}(2\Psi_2)}{2}\\&=\frac{2\triangle_{f,A,v}(\Psi_1)+2\triangle_{f,A,v}(\Psi_2)}{2}=\triangle_{f,A,v}(\Psi_1)+\triangle_{f,A,v}(\Psi_2)\end{aligned}$$

for any $\Psi_1,\ \Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, which proves (8). ◄

For $\Psi_1,\ \Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ we denote $\Psi_2\succ_I\Psi_1$ if $\Psi_2-\Psi_1\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, see also [10]. This means that $\Psi_2-\Psi_1$ is a linear positive functional and $\Psi_2(1_H)-\Psi_1(1_H)\in\mathcal{B}^{++}(K)$.

We have:

Corollary 2. *Let $f:I\to[0,\infty)$ be a concave function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v\in K$, $v\neq0$.*

(i) If $\Psi_1,\ \Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2\succ_I\Psi_1$, then

$$\triangle_{f,A,v}(\Psi_2)\geq\triangle_{f,A,v}(\Psi_1), \tag{11}$$

namely $\triangle_{f,A,v}$ is operator monotonic in the order "$\succ_I$" of $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

(ii) If $\Psi,\ \Upsilon\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, $t,\ T>0$ with $T>t$ and $T\Upsilon\succ_I\Psi\succ_I t\Upsilon$, then

$$T\triangle_{f,A,v}(\Upsilon)\geq\triangle_{f,A,v}(\Psi)\geq t\triangle_{f,A,v}(\Upsilon). \tag{12}$$

Proof. (i) Let $\Psi_1,\ \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2 \succ_I \Psi_1$. Then by (8) we have

$$\triangle_{f,A,v}(\Psi_2) = \triangle_{f,A,v}(\Psi_1+\Psi_2-\Psi_1) \geq \triangle_{f,A,v}(\Psi_1) + \triangle_{f,A,v}(\Psi_2-\Psi_1)$$

implying that

$$\triangle_{f,A,v}(\Psi_2) - \triangle_{f,A,v}(\Psi_1) \geq \triangle_{f,A,v}(\Psi_2-\Psi_1).$$

Since f is positive and $\Psi_2 - \Psi_1 \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2(1_H) - \Psi_1(1_H) \in \mathcal{B}^{++}(K)$, it follows that $\triangle_{f,A,v}(\Psi_2-\Psi_1) \geq 0$ and the inequality (11) is proved.

(ii) The proof follows by (11) on taking first $\Psi_2 = T\Upsilon$, $\Psi_1 = \Psi$ and then $\Psi_2 = \Psi$, $\Psi_1 = t\Upsilon$ and by the positive homogeneity of $\triangle_{f,A,v}$. ◄

We consider now the functional $\triangle_{f,A,v} : \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)] \to \mathcal{B}(K)$ defined by

$$\begin{aligned}\square_{f,A,v}(\Psi) &:= \langle \Psi(f(A))v,v\rangle - \triangle_{f,A,v}(\Psi) \\ &= \langle \Psi(f(A))v,v\rangle - \langle \Psi(1_H)v,v\rangle f\left(\frac{\langle \Psi(A)v,v\rangle}{\langle \Psi(1_H)v,v\rangle}\right),\end{aligned} \tag{13}$$

where $f : I \to \mathbb{R}$ is a convex (concave) function on the interval I, A is a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \neq 0$.

We can state the following result:

Theorem 4. *Let $f : I \to \mathbb{R}$ be a convex (concave) function on the interval I and A a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \neq 0$. Then the functional $\square_{f,A,v}$ is positive (negative) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, and it is positive homogeneous and concave (convex) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. $\square_{f,A,v}$ is also superadditive (subadditive) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.*

Proof. We consider only the convex case. The positivity of $\square_{f,A,v}$ on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ is equivalent to the inequality for general positive linear maps (6). The positive homogeneity follows by the same property of $\triangle_{f,A,v}$ and the definition of $\triangle_{f,A,v}$.

If $\Psi_1,\ \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, $\lambda \in [0,1]$ and $v \in K$, $v \neq 0$, then by Theorem 3 we have

$$\begin{aligned}&\square_{f,A,v}((1-\lambda)\Psi_1 + \lambda\Psi_2) \\ &= \langle ((1-\lambda)\Psi_1 + \lambda\Psi_2)(f(A))v,v\rangle - \triangle_{f,A,v}((1-\lambda)\Psi_1 + \lambda\Psi_2) \\ &\geq (1-\lambda)\langle \Psi_1(f(A))v,v\rangle + \lambda\langle (\Psi_2 f(A))v,v\rangle \\ &- (1-\lambda)\triangle_{f,A,v}(\Psi_1) - \lambda\triangle_{f,A,v}(\Psi_2)\end{aligned}$$

$$= (1-\lambda)\left[\langle \Psi_1 (f(A)) v, v\rangle - \triangle_{f,A,v}(\Psi_1)\right]$$
$$+ \lambda \left[\langle (\Psi_2 f(A)) v, v\rangle - \triangle_{f,A,v}(\Psi_2)\right]$$
$$= (1-\lambda)\,\square_{f,A,v}(\Psi_1) + \lambda \square_{f,A,v}(\Psi_2)$$

that proves the operator concavity of $\square_{f,A,v}$.

The operator superadditivity follows in a similar way and we omit the details. ◀

Corollary 3. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \neq 0$. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$, then*

$$T\square_{f,A,v}(\Upsilon) \geq \square_{f,A,v}(\Psi) \geq t\square_{f,A,v}(\Upsilon) \tag{14}$$

or, equivalently,

$$\begin{aligned} T\left(\langle \Upsilon (f(A)) v, v\rangle - \triangle_{f,A,v}(\Upsilon)\right) &\geq \langle \Psi (f(A)) v, v\rangle - \triangle_{f,A,v}(\Psi) \\ &\geq t\left(\langle \Upsilon (f(A)) v, v\rangle - \triangle_{f,A,v}(\Upsilon)\right) \geq 0. \end{aligned} \tag{15}$$

Now, assume that A is a selfadjoint operator whose spectrum is contained in $[m, M]$ for some real constants $M > m$. If f is convex, then for any $t \in [m, M]$ we have

$$f(t) \leq \frac{(M-t) f(m) + (t-m) f(M)}{M-m}. \tag{16}$$

If A is a selfadjoint operator whose spectrum is contained in $[m, M]$, then $m1_H \leq A \leq M1_H$ and by taking the map Ψ we get $m\Psi(1_H) \leq \Psi(A) \leq M\Psi(1_H)$ for $\Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. This is equivalent to

$$m \leq \frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle} \leq M$$

for any $v \in K$, $v \neq 0$.

If we take $t = \frac{\langle \Psi(A)v,v\rangle}{\langle \Psi(1_H)v,v\rangle}$, $v \in K$, $v \neq 0$ in (16), then we get

$$f\left(\frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle}\right) \leq \frac{(M - \frac{\langle \Psi(A)v,v\rangle}{\langle \Psi(1_H)v,v\rangle}) f(m) + \left(\frac{\langle \Psi(A)v,v\rangle}{\langle \Psi(1_H)v,v\rangle} - m\right) f(M)}{M-m}$$

that is equivalent to

$$\triangle_{f,A,v}(\Psi) \leq \lozenge_{f,A,v}(\Psi)$$

where

$$\lozenge_{f,A,v}(\Psi) := \frac{\langle (M\Psi(1_H) - \Psi(A)) v, v\rangle f(m) + \langle (\Psi(A) - m\Psi(1_H)) v, v\rangle f(M)}{M-m}$$

for $\Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, is a trapezoidal type functional. We observe that $\Diamond_{f,A,v}$ is *additive* and *positive homogeneous* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

We define the functional $\dagger_{f,A,v} : \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ by

$$\begin{aligned}\dagger_{f,A,v}(\Psi) &:= \Diamond_{f,A,v}(\Psi) - \triangle_{f,A,v}(\Psi)\\ &= \frac{\langle (M\Psi(1_H) - \Psi(A))v, v\rangle f(m) + \langle (\Psi(A) - m\Psi(1_H))v, v\rangle f(M)}{M-m}\\ &- \langle \Psi(1_H)v, v\rangle f\left(\frac{\langle \Psi(A)v, v\rangle}{\langle \Psi(1_H)v, v\rangle}\right).\end{aligned}$$

We observe that if f is convex (concave) on $[m, M]$ and $m1_H \le A \le M1_H$, then

$$\dagger_{f,A,v}(\Psi) \ge (\le)\, 0 \text{ for any } \Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]. \tag{17}$$

Theorem 5. *Let $f : I \to \mathbb{R}$ be a convex (concave) function on the interval I and A a selfadjoint operator whose spectrum is contained in $[m, M]$ and $v \in K$, $v \ne 0$. Then the functional $\dagger_{f,A,v}$ is positive (negative) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, and it is positive homogeneous and concave (convex) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. $\dagger_{f,A,v}$ is also superadditive (subadditive) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.*

The proof is similar to the one of Theorem 4 and we omit the details.

Corollary 4. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \ne 0$. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$, then*

$$T\, \dagger_{f,A,v}(\Upsilon) \ge \dagger_{f,A,v}(\Psi) \ge t\, \dagger_{f,A,v}(\Upsilon) \tag{18}$$

or, equivalently,

$$\begin{aligned} & T\left[\frac{\langle (M\Upsilon(1_H) - \Upsilon(A))v, v\rangle f(m) + \langle (\Upsilon(A) - m\Upsilon(1_H))v, v\rangle f(M)}{M-m}\right.\\ &\left. - \langle \Upsilon(1_H)v, v\rangle f\left(\frac{\langle \Upsilon(A)v, v\rangle}{\langle \Upsilon(1_H)v, v\rangle}\right)\right]\\ &\ge \frac{\langle (M\Psi(1_H) - \Psi(A))v, v\rangle f(m) + \langle (\Psi(A) - m\Psi(1_H))v, v\rangle f(M)}{M-m}\\ &- \langle \Psi(1_H)v, v\rangle f\left(\frac{\langle \Psi(A)v, v\rangle}{\langle \Psi(1_H)v, v\rangle}\right)\\ &\ge t\left[\frac{\langle (M\Upsilon(1_H) - \Upsilon(A))v, v\rangle f(m) + \langle (\Upsilon(A) - m\Upsilon(1_H))v, v\rangle f(M)}{M-m}\right.\\ &\left. - \langle \Upsilon(1_H)v, v\rangle f\left(\frac{\langle \Upsilon(A)v, v\rangle}{\langle \Upsilon(1_H)v, v\rangle}\right)\right]\\ &\ge 0. \end{aligned} \tag{19}$$

3. Some examples

Let A_i be selfadjoint operators on H with $Sp(A_i) \subset I$, $i \in \{1, ..., n\}$ and $p = (p_1, ..., p_n)$ an n-tuple of nonnegative weights with $P_n := \sum_{i=1}^{n} p_i > 0$. We write $p \in \mathbb{R}^n_{++}$. Consider also the n-tuple of normalised positive maps $\Phi = (\phi_1, ..., \phi_n)$ with $\phi_i \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(H)]$ for $i \in \{1, ..., n\}$.

If we put

$$\tilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix},$$

then we have $Sp\left(\tilde{A}\right) \subset I$. We can define the positive map

$$\Psi_{p,\Phi} : \mathcal{B}(H) \oplus ... \oplus \mathcal{B}(H) \to \mathcal{B}(H)$$

by

$$\Psi_{p,\Phi}(A_1 \oplus ... \oplus A_n) = \sum_{i=1}^{n} p_i \phi_i (A_i).$$

Using the functional calculus for continuous functions f on I we have

$$\Psi_{p,\Phi}\left(f\left(\tilde{A}\right)\right) = \sum_{i=1}^{n} p_i \phi_i (f(A_i)) \text{ and } f\left(\Psi_{p,\Phi}\left(\tilde{A}\right)\right) = f\left(\sum_{i=1}^{n} p_i \phi_i (A_i)\right).$$

Since

$$\Psi_{p,\Phi}(1_H \oplus ... \oplus 1_H) = \sum_{i=1}^{n} p_i \phi_i (1_H) = P_n 1_H$$

and $P_n > 0$ it follows that $\Psi_{p,\Phi} \in \mathfrak{P}_I [\mathcal{B}(H) \oplus ... \oplus \mathcal{B}(H), \mathcal{B}(H)]$.

If p, $q \in \mathbb{R}^n_{++}$ with $p \geq q$, namely $p_i \geq q_i$ for $i \in \{1, ..., n\}$ and $P_n > Q_n$, then

$$\Psi_{p,\Phi} \succ_I \Psi_{q,\Phi}.$$

Assume also that $r = \min_{i \in \{1,...,n\}} \left\{\frac{p_i}{q_i}\right\}$, $R = \max_{i \in \{1,...,n\}} \left\{\frac{p_i}{q_i}\right\}$ and $r < \frac{P_n}{Q_n} < R$. Then

$$\Psi_{p,\Phi}\left(\tilde{A}\right) - r\Psi_{q,\Phi}\left(\tilde{A}\right) = \sum_{i=1}^{n} (p_i - rq_i) \phi_i (A_i) \geq 0$$

for $\tilde{A} \geq \tilde{0}$,

$$\Psi_{p,\Phi}\left(\tilde{1}_H\right) - r\Psi_{q,\Phi}\left(\tilde{1}_H\right) = \sum_{i=1}^{n} (p_i - rq_i) \phi_i (1_H) = (P_n - rQ_n) 1_H$$

and

$$R\Psi_{q,\Phi}\left(\tilde{1}_H\right) - \Psi_{p,\Phi}\left(\tilde{1}_H\right) = \sum_{i=1}^{n}\left(Rq_i - p_i\right)\phi_i\left(1_H\right) = \left(RQ_n - P_n\right)1_H$$

showing that

$$R\Psi_{q,\Phi} \succ_I \Psi_{p,\Phi} \succ_I r\Psi_{q,\Phi}. \tag{20}$$

Now, observe that for $v \in H$, $\|v\| = 1$ we have

$$\triangle_{f,\tilde{A},v}\left(\Psi_{p,\Phi}\right) = P_n f\left(\frac{\left\langle \sum_{i=1}^{n} p_i\phi_i\left(A_i\right)v, v\right\rangle}{P_n}\right),$$

where $p \in \mathbb{R}^n_{++}$.

Let $f : I \to \mathbb{R}$ be a convex (concave) function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in H$, $\|v\| = 1$. If $p, q \in \mathbb{R}^n_{++}$, then we have by Theorem 3 that

$$\triangle_{f,\tilde{A},v}\left(\Psi_{(1-\lambda)p+\lambda q,\Phi}\right) \le (\ge)\left(1-\lambda\right)\triangle_{f,\tilde{A},v}\left(\Psi_{p,\Phi}\right) + \lambda\triangle_{f,\tilde{A},v}\left(\Psi_{q,\Phi}\right) \tag{21}$$

for any $\lambda \in [0,1]$ and, in particular,

$$\triangle_{f,\tilde{A},v}\left(\Psi_{p+q,\Phi}\right) \le (\ge)\triangle_{f,\tilde{A},v}\left(\Psi_{p,\Phi}\right) + \triangle_{f,\tilde{A},v}\left(\Psi_{q,\Phi}\right). \tag{22}$$

By using (12) for $p, q \in \mathbb{R}^n_{++}$ with $r = \min_{i\in\{1,\dots,n\}}\left\{\frac{p_i}{q_i}\right\}$, $R = \max_{i\in\{1,\dots,n\}}\left\{\frac{p_i}{q_i}\right\}$ and $r < \frac{P_n}{Q_n} < R$ we have

$$\begin{aligned} RQ_n f\left(\frac{\left\langle \sum_{i=1}^{n} q_i\phi_i\left(A_i\right)v, v\right\rangle}{Q_n}\right) &\ge P_n f\left(\frac{\left\langle \sum_{i=1}^{n} p_i\phi_i\left(A_i\right)v, v\right\rangle}{P_n}\right) \\ &\ge rQ_n f\left(\frac{\left\langle \sum_{i=1}^{n} q_i\phi_i\left(A_i\right)v, v\right\rangle}{Q_n}\right), \end{aligned} \tag{23}$$

provided $f : I \to [0,\infty)$ is a concave function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in H$, $\|v\| = 1$.

If we take $f(t) = t^s$, $s \in (0,1)$ and assume that $A_i \ge 0$, $i \in \{1, ..., n\}$, then by (23) we have the power inequality

$$\begin{aligned} R^{1/s}Q_n^{1/s-1}\left\langle \sum_{i=1}^{n} q_i\phi_i\left(A_i\right)v, v\right\rangle &\ge P_n^{1/s-1}\left\langle \sum_{i=1}^{n} p_i\phi_i\left(A_i\right)v, v\right\rangle \\ &\ge r^{1/s}Q_n^{1/s-1}\left\langle \sum_{i=1}^{n} q_i\phi_i\left(A_i\right)v, v\right\rangle, \end{aligned} \tag{24}$$

for $v \in H$, $\|v\| = 1$.

By taking the supremum in this inequality over $v \in H$, $\|v\| = 1$, we get the norm inequality

$$R^{1/s} Q_n^{1/s-1} \left\| \sum_{i=1}^{n} q_i \phi_i (A_i) \right\| \geq P_n^{1/s-1} \left\| \sum_{i=1}^{n} p_i \phi_i (A_i) \right\| \geq r^{1/s} Q_n^{1/s-1} \left\| \sum_{i=1}^{n} q_i \phi_i (A_i) \right\| . \tag{25}$$

We also have

$$\square_{f,\tilde{A},v} (\Psi_{p,\Phi}) := \sum_{i=1}^{n} p_i \langle \phi_i (f (A_i)) v, v \rangle - P_n f \left(\frac{\langle \sum_{i=1}^{n} p_i \phi_i (A_i) v, v \rangle}{P_n} \right), \tag{26}$$

where $p \in \mathbb{R}_{++}^n$.

By utilising (15) we can state that

$$\begin{aligned} &R \left[\sum_{i=1}^{n} q_i \langle \phi_i (f (A_i)) v, v \rangle - Q_n f \left(\frac{\langle \sum_{i=1}^{n} q_i \phi_i (A_i) v, v \rangle}{Q_n} \right) \right] \\ &\geq \sum_{i=1}^{n} p_i \langle \phi_i (f (A_i)) v, v \rangle - P_n f \left(\frac{\langle \sum_{i=1}^{n} p_i \phi_i (A_i) v, v \rangle}{P_n} \right) \\ &\geq r \left[\sum_{i=1}^{n} q_i \langle \phi_i (f (A_i)) v, v \rangle - Q_n f \left(\frac{\langle \sum_{i=1}^{n} q_i \phi_i (A_i) v, v \rangle}{Q_n} \right) \right] \end{aligned} \tag{27}$$

for $p, q \in \mathbb{R}_{++}^n$ with $r = \min_{i \in \{1,\dots,n\}} \left\{ \frac{p_i}{q_i} \right\}$, $R = \max_{i \in \{1,\dots,n\}} \left\{ \frac{p_i}{q_i} \right\}$ and $r < \frac{P_n}{Q_n} < R$.

If we take $f (t) = |t|^{\alpha}$, $t \in \mathbb{R}$ with $\alpha \geq 1$, then for any selfadjoint operators A_i, $i \in \{1, ..., n\}$ we have

$$\begin{aligned} &R \left[\sum_{i=1}^{n} q_i \langle \phi_i (|A_i|^{\alpha}) v, v \rangle - Q_n^{1-\alpha} \left| \left\langle \sum_{i=1}^{n} q_i \phi_i (A_i) v, v \right\rangle \right|^{\alpha} \right] \\ &\geq \sum_{i=1}^{n} p_i \langle \phi_i (|A_i|^{\alpha}) v, v \rangle - P_n^{1-\alpha} \left| \left\langle \sum_{i=1}^{n} p_i \phi_i (A_i) v, v \right\rangle \right|^{\alpha} \\ &\geq r \left[\sum_{i=1}^{n} q_i \langle \phi_i (|A_i|^{\alpha}) v, v \rangle - Q_n^{1-\alpha} \left| \left\langle \sum_{i=1}^{n} q_i \phi_i (A_i) v, v \right\rangle \right|^{\alpha} \right] . \end{aligned} \tag{28}$$

Finally, since

$$\dagger_{f,\tilde{A},v}(\Psi_{p,\Phi}) = \frac{1}{M-m}\left[\left\langle\left(MP_n 1_H - \sum_{i=1}^{n} p_i\phi_i(A_i)\right)v,v\right\rangle f(m)\right.$$
$$\left.+\left\langle\left(\sum_{i=1}^{n} p_i\phi_i(A_i) - mP_n 1_H\right)v,v\right\rangle f(M)\right]$$
$$- P_n f\left(\frac{\left\langle\sum_{i=1}^{n} p_i\phi_i(A_i)v,v\right\rangle}{P_n}\right),$$

by (19) we have

$$R\left\{\frac{1}{M-m}\left[\left\langle\left(MQ_n 1_H - \sum_{i=1}^{n} q_i\phi_i(A_i)\right)v,v\right\rangle f(m)\right.\right.$$
$$\left.+\left\langle\left(\sum_{i=1}^{n} q_i\phi_i(A_i) - mQ_n 1_H\right)v,v\right\rangle f(M)\right]$$
$$\left.-Q_n f\left(\frac{\left\langle\sum_{i=1}^{n} q_i\phi_i(A_i)v,v\right\rangle}{Q_n}\right)\right\}$$
$$\geq \frac{1}{M-m}\left[\left\langle\left(MP_n 1_H - \sum_{i=1}^{n} p_i\phi_i(A_i)\right)v,v\right\rangle f(m)\right.$$
$$\left.+\left\langle\left(\sum_{i=1}^{n} p_i\phi_i(A_i) - mP_n 1_H\right)v,v\right\rangle f(M)\right]$$
$$- P_n f\left(\frac{\left\langle\sum_{i=1}^{n} p_i\phi_i(A_i)v,v\right\rangle}{P_n}\right)$$
$$\geq r\left\{\frac{1}{M-m}\left[\left\langle\left(MQ_n 1_H - \sum_{i=1}^{n} q_i\phi_i(A_i)\right)v,v\right\rangle f(m)\right.\right.$$
$$\left.+\left\langle\left(\sum_{i=1}^{n} q_i\phi_i(A_i) - mQ_n 1_H\right)v,v\right\rangle f(M)\right]$$
$$\left.-Q_n f\left(\frac{\left\langle\sum_{i=1}^{n} q_i\phi_i(A_i)v,v\right\rangle}{Q_n}\right)\right\}$$

for $p,q\in\mathbb{R}_{++}^{n}$ with $r=\min_{i\in\{1,\dots,n\}}\left\{\frac{p_i}{q_i}\right\}$, $R=\max_{i\in\{1,\dots,n\}}\left\{\frac{p_i}{q_i}\right\}$ and $r<\frac{P_n}{Q_n}<R$.

Several other inequalities may be obtained if one chooses the convex functions $f(t)=-\ln t$, $t\ln t$, t^{β}, where $t>0$ and $\beta\in(-\infty,0)\cup[1,\infty)$ or $f(t)=\exp(\gamma t)$, t, $\gamma\in\mathbb{R}$ and $\gamma\neq 0$. The details are omitted.

References

[1] R.P. Agarwal, S.S. Dragomir, *A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces*, Comput. Math. Appl., **59(12)**, 2010, 3785–3812.

[2] M.D. Choi, *Positive linear maps on C*-algebras*, Canad. J. Math., **24**, 1972, 520–529.

[3] S.S. Dragomir, *Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces*, J. Inequal. Appl., **2010**, Art. ID 496821, 2010, 15 pp.

[4] S.S. Dragomir, *A survey of Jensen type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces*, Commun. Math. Anal., **10(1)**, 2011, 82–104.

[5] S.S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type*, Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[6] S.S. Dragomir, *Jensen type weighted inequalities for functions of selfadjoint and unitary operators*, Ital. J. Pure Appl. Math., **32**, 2014, 247–264.

[7] S.S. Dragomir, *Some inequalities of Jensen type for Arg-square convex functions of unitary operators in Hilbert spaces*, Bull. Aust. Math. Soc., **90(1)**, 2014, 65–73.

[8] S.S. Dragomir, *Some Jensen type inequalities for square-convex functions of selfadjoint operators in Hilbert spaces*, Commun. Math. Anal., **14(1)**, 2013, 42–58.

[9] S.S. Dragomir, *A weakened version of Davis-Choi-Jensen's inequality for normalised positive maps*, Proyecciones Journal of Mathematics, **36(1)**, 2017, 81-94.

[10] S.S. Dragomir, *Operator quasilinearity of some functionals associated to Davis-Choi-Jensen's inequality for positive maps*, Bull. Austral. Math Soc., **95(2)**, 2017, 322–332.

[11] M. Kian, *Operator Jensen inequality for superquadratic functions*, Linear Algebra Appl., **456**, 2014, 82–87.

[12] M. Kian, M.S. Moslehian, *Refinements of the operator Jensen-Mercer inequality*, Electron. J. Linear Algebra, **26**, 2013, 742–753.

[13] S.M. Malamud, *Operator inequalities that are inverse to Jensen's inequality, Mat. Zametki*, **69(4)**, 2001, 633–637 (in Russian); translation in *Math. Notes*, **69(3-4)**, 2001, 582–586.

[14] A. Matković, J. Pečarić, *On a variant of the Jensen-Mercer inequality for operators*, J. Math. Inequal., **2(3)**, 2008, 299–307.

[15] J. Pečarić, T. Furuta, J. Mićić Hot, Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

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Received 15 August 2019
Accepted 30 January 2020